# The Riemann–Hilbert Method; a Swiss Army knife

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**Abstract** Random Matrix models, nonlinear integrable waves, Painleve' transcendents, determinantal random point processes seem very unrelated topics.

They have, however, a common point in that they can be formulated or related to a Riemann-Hilbert problem, which then enters prominently as a very versatile tool. Its importance is not only in providing a common framework, but also in that it opens the way to rigorous asymptotic analysis using the nonlinear steepest descent method. I will briefly sketch and review some results in the above mentioned areas.

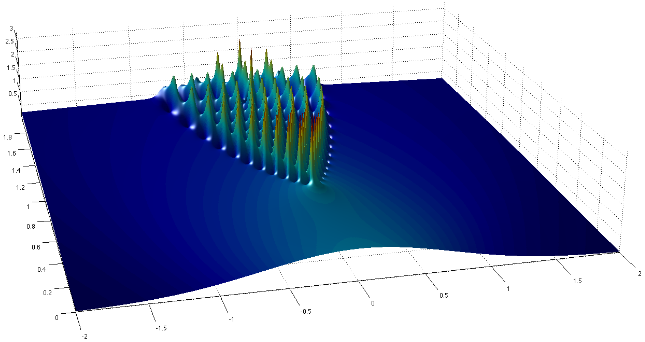
## OUTLINE

- Nonlinear Schrödinger and KdV
- Orthogonal polynomials
- Hermitean Random Matrices
- Determinantal Random Point Fields and Gap probabilities;
- Painlevé equations (Example: Painlevé II)
- Tying it all together: Riemann–Hilbert problems.

The focusing Nonlinear Schrödinger (NLS) equation,

$$i\hbar\partial_t q \ = -\hbar^2\partial_x^2 q - 2|q|^2 q \tag{1}$$

models self-focusing and self-modulation (*optical fibers*). It is **integrable** by inverse scattering methods (Zakharov–Shabat). It exhibits interesting behaviour as $\hbar \to 0$ (**modulational instability**); in different regions of spacetime, there are different asymptotic behaviors (*phases*) separated by **breaking curves** (or **nonlinear caustics**).



The KdV equation

$$u_t = uu_x + \epsilon^2 u_{xxx} \ , \qquad u(x,0) = u_0(x) \qquad \text{rapidly decaying} \tag{2}$$

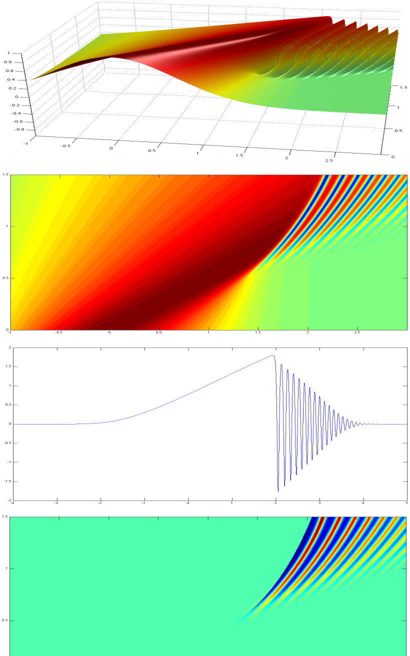
For $\epsilon = 0$ we have Burger's equation $u_t = uu_x$, solved by the hodograph method (characteristics), locally

$$f(u) = x + u\,t \qquad f(u) = u_0^{-1} \tag{3}$$

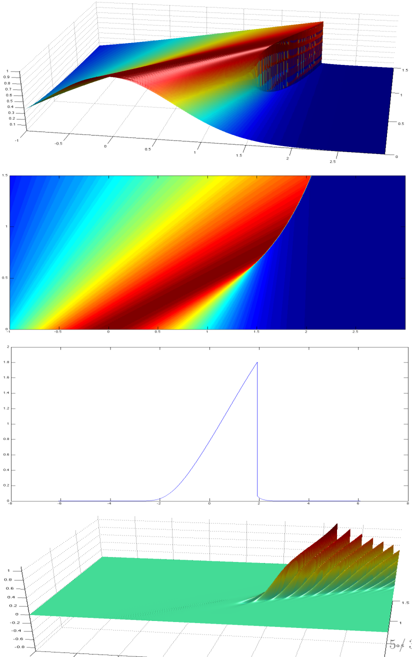
It shocks at $t_0 = \frac{1}{\max u_0'(x)}$.
The small-dispersion also exhibits interesting behavior:

- Near the point of gradient catastrophe $(x_0, t_0)$ its behavior is described in terms of a generalization of the Painlevé I equation with critical scale $\hbar^{\frac{6}{7}}$;
- Near the trailing edge (after the time $t_0$) it is described by the Hastings-McLeod solution of the Painlevé II equation $y''(s) = sy(s) + 2y^3(s)$ with critical scale $\hbar^{\frac{2}{3}}$;
- Near the leading edge the behavior is described in terms of elementary function (superposition of soliton solutions) with scale $\hbar \ln \hbar$.

KdV-small dispersion



KdV-zero dispersion = Burgers



Let $V(z) : \mathbb{R} \to \mathbb{R}$ be a real analytic (or smooth) function (**potential**) growing s.t. $\liminf_{|z|\to\infty} \frac{V(z)}{\ln|z|} = +\infty$. Define the Orthgonal Polynomials as the polynomial basis for $L^2(\mathbb{R}, \mathrm{e}^{-NV(z)}\,\mathrm{d}z)$

$$\int_{\mathbb{R}} p_n(z)p_m(z)\mathrm{e}^{-NV(z)}\,\mathrm{d}z = h_n\delta_{nm}, \quad h_n = \|p_n\|^2, \; p_n(x) = x^n + \dots \tag{4}$$

They satisfy the following three term recurrence relation

$$zp_n(z) = p_{n+1}(z) + \alpha_n p_n(z) + \lambda_n p_{n-1}(z) \tag{5}$$

If $V(z) = V_0(z) + xz$ then the recurrence coefficients solve the **Toda lattice equations**

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mu_n(x) = \mathrm{e}^{\mu_n - \mu_{n+1}} - \mathrm{e}^{\mu_{n-1} - \mu_n}\,, \quad \mathrm{e}^{\mu_n(x)} := h_n(x) \tag{6}$$

Here $h_n(x)$ plays the rôle of $q(x,t)$.

The typical setup: $\mathcal{H}_N := \{M$ Hermitean $N \times N$ matrix $(M = M^\dagger)\}$.

$$\mathrm{d}\mu := \mathrm{d}M \mathrm{e}^{-\mathrm{tr} V(M)} \tag{7}$$

$$\mathrm{d}M = \prod_{i<j} \mathrm{d}\Re(M_{ij})\, \mathrm{d}\Im(M_{ij}) \prod_k \mathrm{d}M_{kk} \tag{8}$$

$$Z_N^{1MM}[V] := \int \mathrm{d}\mu = \text{ Partition function.} \tag{9}$$

Questions of interest (among others)

- Characterize the statistical properties of the eigenvalues of $M$ using the probability measure $(Z_N^{1MM})^{-1}\,\mathrm{d}\mu$.
- Study their limits as the size $N \to \infty$ (and $V$ is suitably scaled).

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It can be shown that

$$Z_N[V] = \int_{U(N)} \mathrm{d}U \int_{\mathbb{R}^N} \prod_{i=1}^{N} \mathrm{d}x_i \Delta(X)^2 \mathrm{e}^{-N\sum_{i=1}^{N} V(x_i)}$$

$$\Delta(X) := \prod_{i<j} (x_i - x_j) = \det \begin{pmatrix} 1 & x_1 & \dots & x_1^{N-1} \\ 1 & x_2 & \dots & x_2^{N-1} \\ \vdots & \dots & & \vdots \\ 1 & x_N & \dots & x_N^{N-1} \end{pmatrix}$$

Up to the volume of the unitary group (which can by computed) the partition function shows that the eigenvalues behave like a random **Coulomb gas** with (unnormalized) density

$$\rho_N(x_1, \dots, x_N) = \frac{1}{Z_N} \exp -N^2 \left[ \frac{1}{N} \sum_{j=1}^{N} V(x_j) - \frac{2}{N^2} \sum_{j \neq k} \ln |x_j - x_k| \right] \qquad (10)$$

# Connection to OPs: Dyson's theorem and Determinantal Random Point Fields

One can show that the **correlation functions**

$$\rho_k(x_1,\ldots,x_k) := \frac{N!}{(N-k)!k!}\int_{\mathbb{R}^{N-k}} \mathrm{d}x_{k+1}\cdots \mathrm{d}x_N \rho_N(x_1,\ldots,x_N) \tag{11}$$

define a **random point process** of **determinantal** form

$$\rho_k(x_1,\ldots,x_k) = \det\left[K_N(x_j,x_\ell)\right]_{j,\ell\leqslant k} \tag{12}$$

$$K_N(x,y) = \mathrm{e}^{-\frac{N}{2}(V(x)+V(y))}\sum_{j=0}^{N-1}\frac{1}{h_j}p_j(x)p_j(y) \tag{13}$$

where $p_j$ are the orthogonal polynomials of $\mathrm{e}^{-\Lambda V(x)}\,\mathrm{d}x$.

This last formula shows that the statistics of the eigenvalues of the RM is an example of **determinantal random point field (process)**; all correlation functions are expressed in terms of determinants of a single **kernel** $K(x,y)$.

This is the simplest model, with $V(x) = x^2$

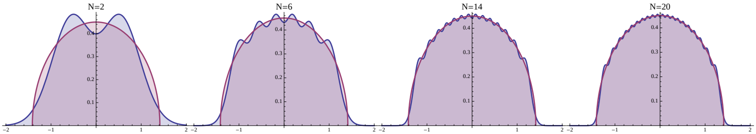
$$\mathrm{d}\mu \propto \mathrm{d}M \mathrm{e}^{-N \mathrm{Tr} M^2} = \mathrm{d}M \mathrm{e}^{-\frac{N}{2}\sum_{i<j}|M_{ij}|^2 - N\sum_j |M_{jj}|^2} \tag{14}$$

The entries are **independent and normal**. The eigenvalues $x_j$ are **not independent**:

$$\mathrm{d}\mu(x_1, \ldots, x_N) \propto \mathrm{e}^{-\frac{N}{2}\sum_i x_i^2} \prod_{i<j} |x_i - x_j|^2 \tag{15}$$

The **density** of eigennvalues can be computed in closed form and has a limit as $N \to \infty$ given by the **Wigner semicircle law**

$$\rho_W(x) = \frac{1}{\pi}\sqrt{2 - x^2} \tag{16}$$



If we zoom in to the **edge of the spectrum** at $x = \sqrt{2}$

$$N^{\frac{1}{3}} \frac{\sqrt{2}}{2} \rho_N \left( \sqrt{2} + \frac{\sqrt{2}}{2\mathbf{N^{\frac{2}{3}}}} \xi \right) \underset{N \to \infty}{\longrightarrow} \left( Ai'(\xi) \right)^2 - \xi Ai^2(\xi) \qquad (17)$$

were $Ai(\xi)$ is the **Airy function**, (special) solution of the Airy equation

$$f''(\xi) = \xi f(\xi) \qquad (18)$$

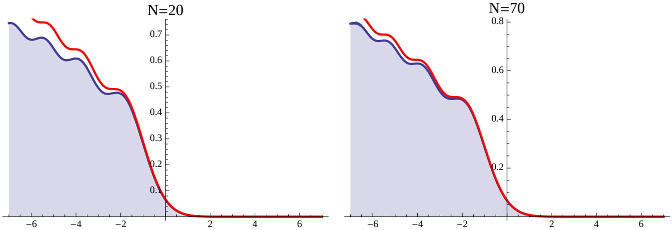



FIGURE: Comparison between the actual density and the Airy density (in red )

The behavior is **universal**: the constants may change but the scaling and the limit is independent of the matrix model.
Tracy and Widom were studying the **gap probability** of eigenvalues of the Gaussian random matrix model in a certain scaling regime near the edge of the spectrum

$$F_N(x) := \mathbb{P}(\lambda_{max} < x) \tag{19}$$

with $\lambda_{max}$ the largest eigenvalue.
Then

$$F(x) := \lim_{N\to\infty} F_N\left(\sqrt{2} + \frac{\sqrt{2}\xi}{2N^{\frac{2}{3}}}\right) = \exp\left(-\int_\xi^\infty (s-\xi)y(s)^2\,\mathrm{d}s\right) \tag{20}$$

where $y(x)$ is the **Hastings–McLeod** solution of the Painlevé II equation, namely the unique solution of

$$y'' = \xi y + 2y^3 \tag{21}$$

that has the asymptotics $y(\xi) \sim Ai(\xi)$ as $\xi \to \infty$. The same distribution appears in other areas: if $\ell_N(\pi)$ is the length of the **longest increasing subsequence** of the random permutation $\pi \in \mathfrak{S}_N$ then

$$\lim_{N\to\infty} Prob\left(\frac{\ell_N - 2\sqrt{N}}{N^{\frac{1}{6}}} \leqslant \xi\right) = \mathrm{e}^{-\int_\xi^\infty (s-\xi)y(s)^2\,\mathrm{d}s} = TW\ distro \tag{22}$$

We refer to the excellent review of A. Soshnikov ['00].

### Definition

A Random Point Process is a probability on the space of configuration of $N \leqslant \infty$ points in a configuration measure space $(X, \mathrm{d}x)$ (e.g. $\mathbb{R}$). It is determined by the **correlation functions**

$$\rho_k(x_1, x_2, \dots, x_k) \prod \mathrm{d}x_j = \mathbb{E}\left(\text{Number of particles in each } [x_j, x_j + \mathrm{d}x_j]\right) \quad (23)$$

It may depend on parameters (time $\Rightarrow$ nonstationary RPP)

If $B_j$ are (Borel) subsets of $X$ and $\#_j$ = number of points in $B_j$ (an **integer-valued random variable**) then the above reads

$$\left\langle \prod_{j=1}^{m} \binom{\#_j}{k_j} \right\rangle = \frac{1}{\prod_{j=1}^{m} k_j!} \int_{B_1^{k_1} \times \dots B_m^{k_m}} \rho_k(x_1, \dots, x_{k_1}, x_{k_1+1} \dots) d^k x \quad (24)$$

where $k = \sum_{j=1}^{m} k_j$.

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**Question**

What is the probability of finding zero ($\ell$) particles in a subset $B \subset X$?

$$\left\langle \binom{\sharp_B}{k} \right\rangle = \frac{1}{k!}\int_{B^k} \rho_k(x_1,\dots x_k)\,\mathrm{d}^k x = \sum_{n=k}^{\infty} \binom{n}{k}\,\mathbb{P}\{\text{ there are } n \text{ particles in } B\} \tag{25}$$

Now multiply by $(-1)^k$ and sum over $k \geqslant 1$:

$$\sum_{k=1}^{\infty}(-1)^k \frac{1}{k!}\int_{B^k}\rho_k(x_1,\dots x_k)\,\mathrm{d}^k x = \sum_{k=1}^{\infty}(-1)^k \sum_{n=k}^{\infty}\binom{n}{k}\,\mathbb{P}\{n \text{ particles in } B\} = \tag{26}$$

$$= \sum_{n=1}^{\infty}\mathbb{P}\{n \text{ particles in } B\}\sum_{k=1}^{n}\binom{n}{k}(-1)^k = -\sum_{n=1}^{\infty}\mathbb{P}\{n \text{ particles in } B\} = \tag{27}$$

$$= -1 + \mathbb{P}\{0 \text{ particles in } B\} \tag{28}$$

We now interchange the summations...

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The sum over $k$ is equal to $(1-1)^n - 1 \equiv -1$:

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The sum over the probabities of having $n \geqslant 0$ particles must be one! So we have

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$$\mathbb{P}\{0 \text{ particles in } B\} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{B^k} \rho_k(x_1, \dots x_k)\, \mathrm{d}^k x$$

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We can repeat all by multiplying by $(-z)^k$ ($z$ an indeterminate) to get the generating function

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$$= \sum_{n=1}^{\infty} \mathbb{P}\left\{ n \text{ particles in } B \right\} \sum_{k=1}^{n} \binom{n}{k} (-z)^k = \sum_{n=1}^{\infty} (1-z)^n \mathbb{P}\left\{ n \text{ particles in } B \right\} \tag{27}$$

**Generation function of occupation numbers**

$$F_B(z) = 1 + \sum_{k=1}^{\infty} \frac{(-z)^k}{k!} \int_{B^k} \rho_k(x_1, \dots x_k)\, \mathrm{d}^k x = \left\langle (1-z)^{\sharp_B} \right\rangle$$

In general

### Definition

The generating functions of the occupation numbers in the sets $B_j$

$$F_{\vec{B}}(z_1, \dots, z_m) := \left\langle \prod_{j=1}^{m} (1 - z_j)^{\#_j} \right\rangle = \sum_{\ell_1, \dots, \ell_m = 0}^{\infty} \left\langle \prod_{j=1}^{m} \binom{\#_j}{k_j} (-z_j)^{\ell_j} \right\rangle \tag{28}$$

We take the simplest case of one set (as before), for simplicity:

$$F_B(z) := \left\langle (1 - z)^{\#_B} \right\rangle = \sum_{k=0}^{\infty} \left\langle \binom{\#_B}{k} (-z)^k \right\rangle \tag{29}$$

We now introduce a special class of Random Point Fields, called **Determinantal**

**Definition**

The RPP is **determinantal** (DRPP) if all corr. functions are determinants of a **Kernel**

$$K(x,y) : X^2 \to \mathbb{R} \tag{30}$$

$$\rho_k(x_1,\dots,x_k) = \det \begin{bmatrix} K(x_1,x_1) & K(x_1,x_2) & \dots & K(x_1,x_k) \\ K(x_2,x_1) & \dots & & \\ \vdots & & & \\ K(x_k,x_1) & \dots & & K(x_k,x_k) \end{bmatrix} \tag{31}$$

It is clear that a necessary condition for the well-definiteness is that the above determinants are all positive (**Total Positivity (TP) of the kernel**).

One then has

**Lemma**

The generating function $F_{\vec{B}}(\vec{z})$ admits the following representation

$$F_{\vec{B}}(z_1, \ldots, z_m) := \left\langle \prod_{j=1}^{m} (1 - z_j)^{\#_j} \right\rangle = \det \left[ \mathrm{Id} - \sum_{j=1}^{m} z_j K \Big|_{B_j} \right] \tag{32}$$

Thus the probability of observing no particles in a subset $B_j \subset \mathbb{R}$ ($z_j = 1$) is given by a **Fredholm determinant**. If the configuration space $X$ is of the form $X_0 \times \{1, \ldots, r\}$ then the scalar kernel on $X$ is the same as a $r \times r$ matrix valued kernel on $X_0$. There is a condition that $K$ as an integral operator on $L^2(X, dx)$ must satisfy so that the process is well-defined and this is that its (operator) norm is $\leqslant 1$. We now prove this lemma for the simplest case; Skip Proof

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**Proof**. We have seen before that

$$F_B(z) = 1 + \sum_{k=1}^{\infty} \frac{(-z)^k}{k!} \int_{B^k} \rho_k(x_1, \ldots x_k) \, \mathrm{d}^k x = \left\langle (1-z)^{\sharp_B} \right\rangle \tag{33}$$

But the correlations are determinants! Hence

$$= 1 + \sum_{k=1}^{\infty} \frac{(-z)^k}{k!} \int_{B^k} \det \left[ K(x_i, x_j) \right]_{i,j \leqslant k} \mathrm{d}^k x \tag{34}$$

...and this is the **definition** of Fredholm determinant, to be seen now.

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$$F_B(z) = 1 + \sum_{k=1}^{\infty} \frac{(-z)^k}{k!} \int_{B^k} \rho_k(x_1, \dots x_k)\, \mathrm{d}^k x = \left\langle (1-z)^{\sharp_B} \right\rangle \tag{33}$$

But the correlations are determinants! Hence

$$= 1 + \sum_{k=1}^{\infty} \frac{(-z)^k}{k!} \int_{B^k} \det\left[K(x_i, x_j)\right]_{i,j \leqslant k} \mathrm{d}^k x \tag{34}$$

...and this is the **definition** of Fredholm determinant, to be seen now.

Given an integral operator $\mathcal{K} : L^2(X,\, \mathrm{d}x) \to L^2(X,\, \mathrm{d}x)$ then

$$(\mathcal{K}f)(x) = \int_X K(x,y) f(y)\,\mathrm{d}y \tag{35}$$

$$\det(\mathrm{Id} - z\mathcal{K}) = 1 + \sum_{n=1}^{\infty} \frac{(-z)^n}{n!} \int_{X^n} \det\left[K(x_j, x_k)\right]_{j,k\leqslant n} \mathrm{d}x_1 \ldots \mathrm{d}x_n. \tag{36}$$

The series defines an entire function of $z$ as long as $\mathcal{K}$ is **trace-class**. For sufficiently small $z$ (less than the spectral radius of $\mathcal{K}$) then the following can be used equivalently

$$\ln\det(Id - z\mathcal{K}) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{Tr}\mathcal{K}^n \tag{37}$$

**Remark**

The definition of Fredholm determinant **coincides with the usual determinant** for finite-dimensional matrices (if the measure $\mathrm{d}x$ is finitely supported, i.e.).

I will now show how these topics, NLS, KdV, OPs, RM, RPP, Fredholm dets, can be addressed using a very versatile tool which has appeared in many facies also as Lax pairs and it is pervasive in all integrable systems. The tool is often referred to as a **Riemann–Hilbert Problem (RHP)**; the name refers (with a stretch) to one of Hilbert problems, namely, the reconstruction of a matrix ODE with Fuchsian singularities given its monodromy representation. However nowadays it takes a wider scope, as we shall see momentarily.

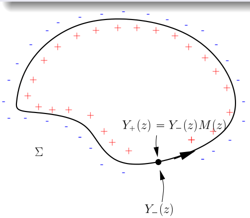
OPs, NLS, KdV, Gap probatilities, Painlevé equations, etc. are related to a particular type of boundary value problem in the complex plane. A **Riemann–Hilbert problem** is a **boundary–value problem** for a matrix–valued, piecewise analytic function $\Gamma(z)$. We will not enter in the details of smoothness. Everything is assumed smooth enough.

### PROBLEM

Let $\Sigma$ be an **oriented** (union of) curve(s) and $M(z)$ a (sufficiently smooth) matrix function defined on $\Sigma$. Find a *matrix-valued* function $Y(z)$ with the properties that

- $Y(z)$ is **analytic** on $\mathbb{C}\backslash\Sigma$;
- $\lim_{z\to\infty} Y(z) = \mathbf{1}$ (or some other normalization);
- for all $z \in \Sigma$, denoting by $Y(z)_\pm$ the (nontangential) boundary values of $Y(z)$ from the left/right of $\Sigma$, we have

$$Y_+(z) = Y_-(z)M(z)\ . \tag{38}$$



In the scalar case, a RHP is reducible to the Sokhotsky-Plemelji formula and a solution can be written explicitly as a Cauchy transform;

**Theorem (Sokhotsky-Plemelji formula)**

*Let $h(w)$ be $\alpha$–Hölder on $\Sigma$ and*

$$f(z) := \frac{1}{2i\pi} \int_{\Sigma} \frac{h(w)\,\mathrm{d}w}{w - z} \tag{39}$$

*Then $f_+(w) - f_-(w) = h(w)$ and $f_+(w) + f_-(w) =: H(h)(w)$ exists (the Cauchy principal value).*

In the matrix case -however- the solution **cannot** be written explicitly (at best an integral equation can be derived) and hence the problem is genuinely transcendental.

We will now parade the Riemann–Hilbert problems that are associated to each of the objects introduced earlier.
We will then conclude with some remarks on their practical use.

**Theorem (Fokas-Its-Kitaev '92)**

*The solution of the following RHP determines the OPs*

$$\Gamma_+(z) = \Gamma_-(z) \begin{bmatrix} 1 & e^{-V(z)} \\ 0 & 1 \end{bmatrix}, \quad z \in \mathbb{R} \tag{40}$$

$$\Gamma(z) = \left(\mathbf{1} + \mathcal{O}(z^{-1})\right) \begin{bmatrix} z^n & 0 \\ 0 & z^{-n} \end{bmatrix} \tag{41}$$

*where $p_n(x) = \Gamma_{11}(z)$ is the $n$-th orthogonal polynomial. Moreover*

$$h_n = -2i\pi \lim_{z\to\infty} z^{n+1}\Gamma_{12}(z) \tag{42}$$

**Theorem**

*The previous Problem admits a unique solution of the form*

$$\Gamma_n(z) := \begin{bmatrix} p_n(z) & \frac{1}{2i\pi}\int_{\mathbb{R}} \frac{p_n(x)e^{-V(x)}\,dx}{x-z} \\ \frac{-2i\pi}{h_{n-1}} p_{n-1}(z) & \frac{-1}{h_{n-1}}\int_{\mathbb{R}} \frac{p_{n-1}(x)e^{-V(x)}\,dx}{x-z} \end{bmatrix} \tag{43}$$

*where $p_n, p_{n-1}$ are the* monic *orthogonal polynomials for the measure* $e^{-V(x)}\,dx$

The nonlinear Schrödinger equation (in $1$ spatial dimension)

$$i\hbar q_t(x,t) = -\hbar^2 q_{xx}(x,t) \pm 2|q(x,t)|^2 q(x,t) \tag{44}$$

The version with the $+$ is called **defocusing** while the other is called **focusing**.

**Theorem (Zakharov)**

*Let $\Gamma(z;x,t)$ be a $2\times 2$ matrix, analytic in $z \in \mathbb{C}\backslash\mathbb{R}$, admitting (nontangential) boundary values on $\mathbb{R}$ from the top/bottom, denoted $\Gamma_\pm(z;x,t)$ and such that*

$$\Gamma_+(z;x,t) = \Gamma_-(z;x,t)\begin{bmatrix} 1-|r(z)|^2 & -\overline{r}(z)\mathrm{e}^{-\frac{2i}{\hbar}(2tz^2+xz)} \\ r(z)\mathrm{e}^{\frac{2i}{\hbar}(2tz^2+xz)} & 1 \end{bmatrix} \tag{45}$$

$$\Gamma(z;x,t) = \mathbf{1} + \mathcal{O}(z^{-1}) \ , \quad |z|\to\infty \tag{46}$$

**Then** *the function of $x,t$*

$$q(x,t) := 2i \lim_{z\to\infty} z\Gamma_{12}(z;x,t) \tag{47}$$

*is a solution of the defocusing NLS, with initial data given by the data that was associated to the scattering transform.*

The **advantage** of the formulation of the Theorem is that the $x,t$ dependence is in plain sight; the **disadvantage** is that it is not possible (in general) to obtain a closed formula for the solution of the advocated Riemann–Hilbert problem.

**Theorem (Solitonless case)**

*Let $\vec{m}(z;x,t)$ be a $1 \times 2$ vector, analytic on $\mathbb{C}\backslash\mathbb{R}$, admitting nontangential boundary values $\vec{m}_\pm$ and such that*

$$\vec{m}_+(z;x,t) = \vec{m}_-(z;x,t) \begin{bmatrix} 1 - |r(z)|^2 & -\overline{r}(z)\mathrm{e}^{-\frac{2i}{\epsilon}(4tz^3+xz)} \\ r(z)\mathrm{e}^{\frac{2i}{\epsilon}(4tz^3+xz)} & 1 \end{bmatrix} \tag{48}$$

$$\vec{m}(z;x,t) \to (1,1) \quad |z| \to \infty \tag{49}$$

**Then** *the function*

$$u(x,t) = -2i\frac{\partial}{\partial x}\lim_{z\to\infty}\vec{m}_1(z;x,t) \tag{50}$$

*solves the KdV equation, with initial datum encoded in the scattering data $r(z)$.*

# Fredholm determinants and RHPs: Its-Izergin-Korepin-Slavnov (IIKS) theory

This theory links certain types of integral operators to Riemann–Hilbert problems: Let $\Sigma \subset \mathbb{C}$ be a collection of contours and

$$K(l,\mu) := \frac{f^T(l)\cdot g(\mu)}{l-\mu} \ , \qquad f,g \in Mat(r\times p,\mathbb{C}) \ , \quad f^T(l)\cdot g(l) \equiv 0 \tag{51}$$

The integral operator with kernel $K(l,\mu)$ acts on $L^2(\Sigma,\mathbb{C}^p)$.

$$\begin{aligned} \mathcal{K} : L^2(\Sigma,\mathbb{C}^p) &\to L^2(\Sigma,\mathbb{C}^p) \\ \varphi(\mu) &\mapsto (\mathcal{K}\varphi)(\lambda) = \int_\Sigma \frac{f^T(l)\cdot g(\mu)}{l-\mu}\varphi(l)\,\mathrm{d}l \end{aligned} \tag{52}$$

## Remark

The Airy kernel is of this type:

$$K_{\mathrm{Ai}}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y} \tag{53}$$

and hence the Tracy-Widom distribution can be derived using the methods of RHPs (this is not the way it was originally derived) [B-Cafasso 2010]

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We can get informations on the Fredholm determinant of $K$ by using the

## Jacobi variational formula

$$\partial \ln\det(\mathrm{Id}-K) = \mathrm{Tr}_{L^2(\Sigma)}\left((\mathrm{Id}+R)\circ\partial K\right) \tag{53}$$

where $R$ is the **resolvent operator**:

$$R := -K\circ(Id-K)^{-1} \tag{54}$$

## THE **resolvent** OPERATOR

$$R(l,\mu) := -K \circ (\mathrm{Id} - K)^{-1}(l,\mu) = \frac{f^T(l)\Gamma^T(l)\Gamma^{-T}(\mu)g(\mu)}{l-\mu} \tag{55}$$

where $\Gamma(l)$ solves the RHP

$$\Gamma_+(l) = \Gamma_-(l)\left(\mathbf{1}_r - 2i\pi f(l)g^T(l)\right) \ , \quad l \in \Sigma \tag{56}$$

$$\Gamma(l) = \mathbf{1}_r + \mathcal{O}(l^{-1}) \ , \quad l \to \infty \tag{57}$$

Paul Painlevé studied (1900) and classified all **second order** ODEs

$$y'' = R(y', y, x) \tag{58}$$

with $R$ a rational function, such that the only moveable singularities of the solutions are poles (i.e. not essential singularities or branchpoint). This is highly nontrivial since the equations are nontlinear. Of all the $50$ canonical form, all but $6$ are reducible to previously known ODEs and special functions. The six extra are known ever since as **Painlevé equations**.

**P-I**

$$y'' = 6y^2 + x \tag{59}$$

**P-II**

$$y'' = 2y^3 + xy + \alpha \tag{60}$$

**P-III**

$$x\,y\,y'' = x(y')^2 - yy' + \delta x + \beta y + \alpha y^3 + \gamma x y^4 \tag{61}$$

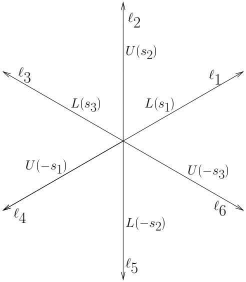
Etc.

All the Painlevé equations are related to a Riemann–Hilbert problem. For example P-II

$$L(s) := \begin{bmatrix} 1 & 0 \\ s\,\mathrm{e}^{\frac{i4}{3}z^3+ixz} & 1 \end{bmatrix},$$

$$U(s) := \begin{bmatrix} 1 & s\,\mathrm{e}^{-\frac{i4}{3}z^3-ixz} \\ 0 & 1 \end{bmatrix}$$

$$s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0$$




$$\Gamma(z) \sim \mathbf{1} + \mathcal{O}(z^{-1})$$

$$u = u(x;\vec{s}) = 2 \lim_{z\to\infty} z\,\Gamma_{12}(z;x,\vec{s})$$

The reformulation of integrable systems in terms of RHPs produces (or is produced, depending on the point of view) a Lax representation. However, this connection is not only of pure theoretical interest: it actually **helps** in studying asymptotic behaviors and has been used in the proof of

- small dispersion of KdV;
- semiclassical asymptotics of NLS;
- strong asymptotic of general orthogonal polynomials in the complex plane for large degrees;
- first proofs of universality of scaling regimes in random matrices.

The method of analysis is the **Deift-Zhou** nonlinear steepest descent method; it is a "matrix analogue" of the classical steepest descent method for oscillatory integrals depending on a (large) parameter.
In this respect there are still many (more or less technical) questions that require continuing improvements of the method.

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