Non-commutative Hermite-Pade approximation and integrability

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Outline

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1. Essentials of Padé approximants

2. Non-commutative Hermite–Padé approximation problem

THE PADÉ PROBLEM

Q: Given power series $f(x) = \sum_{i=0}^{\infty} f_i x^i$ with $f_0 \neq 0$, and given $(m, n) \in \mathbb{N}_0^2$

find polynomials $P_{m,n}(x)$ and $Q_{m,n}(x)$ with $\deg P_{m,n} \leq m$, $\deg Q_{m,n} \leq n$ such that their ratio approximates $f(x)$ to the highest order possible

$$f(x) = \frac{P_{m,n}(x)}{Q_{m,n}(x)} + O(x^{m+n+1})$$

A: Rewrite the above equation in the form

$$Q_{m,n}(x) \left(\sum_{i=0}^{\infty} f_i x^i \right) - P_{m,n}(x) = O(x^{m+n+1})$$

and collect coefficient at different powers of x to get a system of *linear* homogeneous equations, which leads to (Jacobi, 1846)

$$P_{m,n}(x) = \begin{vmatrix} F_m(x) & xF_{m-1}(x) & \cdots & x^n F_{m-n}(x) \\ f_{m+1} & f_m & \cdots & f_{m-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m+n} & f_{m+n-1} & \cdots & f_m \end{vmatrix}, \quad Q_{m,n}(x) = \begin{vmatrix} 1 & x & \cdots & x^n \\ f_{m+1} & f_m & \cdots & f_{m-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m+n} & f_{m+n-1} & \cdots & f_m \end{vmatrix}$$

where $F_k(x) = \sum_{i=0}^{k} f_i x^i$, and $F_k(x) = 0$ for $k < 0$

THE FROBENIUS IDENTITIES (1881) AND DISCRETE-TIME TODA LATTICE EQUATION (HIROTA, 1977)

$$W_{m,n}(x) = a(x)P_{m,n}(x) + b(x)Q_{m,n}(x), \qquad a(x), b(x) \quad \text{arbitrary}$$

$$\Delta_{m,n} = \begin{vmatrix} f_m & f_{m-1} & \cdots & f_{m-n+1} \\ f_{m+1} & f_m & \cdots & f_{m-n+2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m+n-1} & f_{m+n-2} & \cdots & f_m \end{vmatrix}$$

$$\Delta_{m+1,n} W_{m,n}(x) - \Delta_{m,n} W_{m+1,n}(x) = x \Delta_{m+1,n+1} W_{m,n-1}(x)$$

$$\Delta_{m,n} W_{m,n+1}(x) - \Delta_{m,n+1} W_{m,n}(x) = x \Delta_{m+1,n+1} W_{m-1,n}(x)$$

$$\Delta_{m+1,n} W_{m,n+1}(x) - \Delta_{m,n+1} W_{m+1,n}(x) = x \Delta_{m+1,n+1} W_{m,n}(x) \tag{L1}$$

$$\Delta_{m+1,n} W_{m-1,n}(x) + \Delta_{m,n+1} W_{m,n-1}(x) = \Delta_{m,n} W_{m,n}(x) \tag{L2}$$

$$\Delta_{m+1,n}\Delta_{m-1,n} + \Delta_{m,n+1}\Delta_{m,n-1} = \Delta_{m,n}^2 \tag{DTTL}$$

$$W_{m+1,n}(x) W_{m-1,n}(x) + W_{m,n+1}(x) W_{m,n-1}(x) = W_{m,n}^2(x)$$

$M_{[i;j]}$ — the square matrix M with i-th row and j-th column removed

SYLVESTER'S IDENTITY

$$\det M \cdot \det M_{[i_1 i_2; j_1 j_2]} = \det M_{[i_1;j_1]} \cdot \det M_{[i_2;j_2]} - \det M_{[i_1;j_2]} \cdot \det M_{[i_2;j_1]}, \qquad i_1 < i_2, \quad j_1 < j_2$$

Relation to orthogonal polynomials

$$\int_{\mathbb{R}} P_r(x)P_s(x)d\mu(x) = 0, \qquad r \neq s, \qquad P_r(x) = x^r + \dots, \qquad \mu > 0$$

$$f(z) = \sum_{k=0}^{\infty} \frac{c_k}{z^{k+1}} = \int_{\mathbb{R}} \frac{d\mu(x)}{z-x}, \qquad c_k = \int_{\mathbb{R}} x^k d\mu(x),$$

$$P_n(x) = \frac{1}{D_n}\begin{vmatrix} c_0 & c_1 & c_2 & \dots & c_n \\ c_1 & c_2 & c_3 & \dots & c_{n+1} \\ \vdots & \vdots & \ddots & & \vdots \\ c_{n-1} & c_n & c_{n+1} & \dots & c_{2n-1} \\ 1 & x & x^2 & \dots & x^n \end{vmatrix}, \qquad D_n = \begin{vmatrix} c_0 & c_1 & \dots & c_{n-1} \\ c_1 & c_2 & \dots & c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \dots & c_{2n-2} \end{vmatrix}$$

Conditions $\int x^k P_n(x)d\mu(x) = 0$, $k = 0, \dots, n-1$, imply the three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n P_{n-1}(x)$$

Variation of the measure

$$d\mu(t,x) = e^{-xt}d\mu(x) \quad \Rightarrow \quad \begin{aligned} \dot{a}_n(t) &= a_n(t)(b_{n-1}(t) - b_n(t)) \\ \dot{b}_n(t) &= a_n(t) - a_{n-1}(t) \end{aligned}$$

which are the Toda lattice equations in the Flaschka form

Remark: Variation $d\mu(m,x) = x^m d\mu(x)$ gives rise to a system equivalent to the DTTL equation

OUTLINE

1. ESSENTIALS OF PADÉ APPROXIMANTS

2. NON-COMMUTATIVE HERMITE–PADÉ APPROXIMATION PROBLEM

HERMITE–PADÉ PROBLEM

Consider m formal series $(f_1(x), \ldots, f_m(x))$ in variable x

$$f_i(x) = \sum_{j=0}^{\infty} f_j^i x^j = f_i^0 + f_i^1 x + f_i^2 x^2 + \ldots,$$

with non-commuting coefficients f_j^i, where the parameter x commutes with all the coefficients. Given $n = (n_1, \ldots, n_m) \in \mathbb{Z}^m_{\geq -1}$, denote $|n| = n_1 + \cdots + n_m$.

HERMITE–PADÉ PROBLEM

Find a system of polynomials $(Y_1(x), \ldots, Y_m(x))$, not all equal to zero, with corresponding degrees $\deg Y_i(x) \leq n_i$, $i = 1, \ldots, m$ (degree of the zero polynomial equals -1), such that

$$f_1(x) Y_1(x) + \cdots + f_m(x) Y_m(x) = x^{|n|+m-1} \Gamma(x)$$

for a series $\Gamma(x) = \sum_{j=0}^{\infty} \Gamma^j x^j$.

When $m = 2$ and $f_2(x) \equiv -1$ then the rational function $Y_2(x) Y_1(x)^{-1}$ is the Padé approximation to the series $f_1(x)$

QUASIDETERMINANTS

DEFINITION *[Gelfand, Retakh 1992]*

Given square matrix $X = (x_{ij})_{i,j=1,\ldots,n}$ with formal entries x_{ij}. In the free division ring generated by the set $\{x_{ij}\}_{i,j=1,\ldots,n}$ consider the formal inverse matrix $Y = X^{-1} = (y_{ij})_{i,j=1,\ldots,n}$ to X. The (i,j)th quasideterminant $|X|_{ij}$ of X is the inverse $(y_{ji})^{-1}$ of the (j,i)th element of Y, and is often written explicitly as

$$|X|_{ij} = \begin{vmatrix} x_{11} & \cdots & x_{1j} & \cdots & x_{1n} \\ \vdots & & \vdots & & \vdots \\ x_{i1} & \cdots & \boxed{x_{ij}} & \cdots & x_{in} \\ \vdots & & \vdots & & \vdots \\ x_{n1} & \cdots & x_{nj} & \cdots & x_{nn} \end{vmatrix}.$$

In the simplest case of $n = 2$ we have

$$\begin{vmatrix} \boxed{x_{11}} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = x_{11} - x_{12} x_{22}^{-1} x_{21}.$$

When the elements of the matrix X commute between themselves, what we denote by placing the letter c over the equality sign, then the familiar matrix inversion formula gives

$$|X|_{ij} \overset{c}{=} (-1)^{i+j} \frac{\det X}{\det X^{ij}}.$$

RECURRENCE RELATION

For $n \geq 2$ let X^{ij} be the square matrix obtained from X by deleting the ith row and the jth column (with index i/j skipped from the row/column enumeration), then

$$|X|_{ij} = x_{ij} - \sum_{\substack{i' \neq i \\ j' \neq j}} x_{ij'} (|X^{ij}|_{i'j'})^{-1} x_{i'j}$$

provided all terms in the right-hand side are defined.

HOMOLOGICAL RELATIONS

- Row homological relations:

$$-|X|_{ij} \cdot |X^{ik}|_{sj}^{-1} = |X|_{ik} \cdot |X^{ij}|_{sk}^{-1}, \qquad s \neq i.$$

- Column homological relations:

$$-|X^{kj}|_{is}^{-1} \cdot |X|_{ij} = |X^{ij}|_{ks}^{-1} \cdot |X|_{kj}, \qquad s \neq j.$$

Let $X_0 = (x_{ij})$, $i, j = 1, \ldots, k$, be a submatrix of X that is invertible. For $p, q = k+1, \ldots, n$, set

$$c_{pq} = \begin{vmatrix} & X_0 & & x_{1q} \\ & & & \vdots \\ & & & x_{kq} \\ x_{p1} & \ldots & x_{pk} & \boxed{x_{pq}} \end{vmatrix},$$

and consider the $(n-k)\times(n-k)$ matrix $C = (c_{pq})$, $p, q = k+1, \ldots, n$. Then for $i, j = k+1, \ldots, n$,

$$|X|_{ij} = |C|_{ij}\ .$$

In applications Sylvester's identity is usually used in conjunction with row/column permutations.

EXAMPLE

Let us take $n = 3$ and $k = 1$

$$|X|_{33} = \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & \boxed{x_{33}} \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & \boxed{x_{22}} \end{vmatrix} & \begin{vmatrix} x_{11} & x_{13} \\ x_{21} & \boxed{x_{23}} \end{vmatrix} \\ \begin{vmatrix} x_{11} & x_{12} \\ x_{31} & \boxed{x_{32}} \end{vmatrix} & \boxed{\begin{vmatrix} x_{11} & x_{13} \\ x_{31} & \boxed{x_{33}} \end{vmatrix}} \end{vmatrix} =$$

$$\begin{vmatrix} x_{11} & x_{13} \\ x_{31} & \boxed{x_{33}} \end{vmatrix} - \begin{vmatrix} x_{11} & x_{12} \\ x_{31} & \boxed{x_{32}} \end{vmatrix} \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & \boxed{x_{22}} \end{vmatrix}^{-1} \begin{vmatrix} x_{11} & x_{13} \\ x_{21} & \boxed{x_{23}} \end{vmatrix}.$$

DEFINITIONS

Define matrix $\mathcal{M}(n)$ of $(|n| + m - 1)$ rows and $(|n| + m)$ columns

$$\mathcal{M}(n) = \begin{pmatrix} f_0^1 & & 0 & & & f_0^m & & 0 \\ f_1^1 & \ddots & & \cdots & \cdots & f_1^m & \ddots & \\ \vdots & & f_0^1 & & & \vdots & & f_0^m \\ \vdots & & \vdots & \cdots & \cdots & \vdots & & \vdots \\ f_{|n|+m-2}^1 & \cdots & f_{|n|+m-n_1-2}^1 & & & f_{|n|+m-2}^m & \cdots & f_{|n|+m-n_m-2}^m \end{pmatrix};$$

its columns are divided into m groups, the ith group is composed out of $n_i + 1$ columns depending on $f_i(x)$ only.

Supplement the matrix $\mathcal{M}(n)$ by the line

$$\left(f_{|n|+m-1}^1, f_{|n|+m-2}^1, \dots, f_{|n|+m-n_1-1}^1, \cdots, \cdots, f_{|n|+m-1}^m, \dots, f_{|n|+m-n_m-1}^m\right),$$

as the last row. By $\rho^i(n)$ denote the quasideterminant of such extended matrix, with respect to the element in the last row and the last column of the ith block.

Supplement the matrix $\mathcal{M}(n)$ at the bottom by the line

$$X_k = (0, \dots, 0, \dots \; \dots, 1, x, \dots, x^{n_k}, \dots \; \dots, 0, \dots, 0),$$

consisting of zeros except for the kth block of the form $1, x, \dots, x^{n_k}$. By $Z_k^{(i)}(n, x)$ denote the quasideterminant of such matrix with respect to the element in the last row and the last column of the ith block.

Solution of the non-commutative Hermite–Padé problem

Lemma

By the row homological relations the functions $\rho^i(n)$, and $Z_k^{(i)}(n,x)$, when exist, satisfy equations

$$\rho^i(n)[\rho^i(n-e_j)]^{-1} = -\rho^j(n)[\rho^j(n-e_i)]^{-1}, \qquad i \neq j$$

$$Z_k^{(i)}(n,x)[\rho^i(n-e_j)]^{-1} = -Z_k^{(j)}(n,x)[\rho^j(n-e_i)]^{-1}, \qquad i \neq j, k = 1, \dots, m$$

When it exists, $Z_k^{(i)}(n,x)$ is a polynomial in x of degree n_k with the leading order term

$$Z_k^{(i)}(n,x) = [\rho^k(n)]^{-1}\rho^i(n)x^{n_k} + \text{lower order terms}, \qquad i,k = 1,\dots,m$$

Proposition

The system $(Z_1^{(i)}(n,x),\dots,Z_m^{(i)}(n,x))$ where $i = 1,\dots,m$, provides solutions of the non-commutative Hermite–Padé problem with asymptotics

$$f_1(x)Z_1^{(i)}(n,x) + \dots + f_m(x)Z_m^{(i)}(n,x) = x^{|n|+m-1}\rho^i(n) + \text{higher order terms.}$$

The product

$$\psi_k(n,x) = Z_k^{(i)}(n,x)[\rho^i(n)]^{-1}, \qquad i,k = 1,\dots,m,$$

is independent of the index i and the polynomials $(\psi_1(n,x),\dots,\psi_m(n,x))$ provide solution of the non-commutative Hermite–Padé problem with the asymptotic

$$f_1(x)\psi_1(n,x) + \dots + f_m(x)\psi_m(n,x) = x^{|n|+m-1} + \text{higher order terms.}$$

PROPOSITION

The polynomials $\psi_k(n, x)$, $k = 1, \dots, m$, satisfy the linear problem of the non-Abelian Hirota–Miwa system

$$\psi_k(n - e_i, x) - \psi_k(n - e_j, x) = \psi_k(n, x) U_{ij}(n)$$

where $U_{ij}(n) = \rho^j(n)[\rho^j(n - e_i)]^{-1}$, $i \neq j$, satisfy the corresponding non-linear system

$$\begin{aligned} U_{ij}(n) + U_{ji}(n) = 0, \quad U_{ij}(n) + U_{jk}(n) + U_{ki}(n) = 0, \\ U_{ij}(n) U_{ik}(n - e_j) = U_{ik}(n) U_{ij}(n - e_k), \end{aligned} \qquad i, j, k \quad \text{distinct},$$

[Nimmo 2006]

In the commutative commutative case it is possible to introduce the single potential function $\tau(n)$ such that

$$U_{ij}(n) = \frac{\tau(n)\tau(n - e_i - e_j)}{\tau(n - e_i)\tau(n - e_j)}, \qquad i < j,$$

then the remaining (second) part of the above system reduces to Hirota's discrete KP equation

$$\tau(n - e_i)\tau(n - e_j - e_k) - \tau(n - e_j)\tau(n - e_i - e_k) + \tau(n - e_k)\tau(n - e_i - e_j) = 0,$$

where $1 \leq i < j < k \leq N$.

$$Z_k(n,x) \stackrel{c}{=} \begin{vmatrix} f_0^1 & & 0 & & f_0^k & & 0 & & f_0^m & & 0 \\ f_1^1 & \ddots & & \cdots\;\cdots & f_1^k & \ddots & & \cdots\;\cdots & f_1^m & \ddots & \\ \vdots & & f_0^1 & & \vdots & & f_0^k & & \vdots & & f_0^m \\ \vdots & & \vdots & \cdots\;\cdots & & & & \cdots\;\cdots & \vdots & & \vdots \\ f^1_{|n|+m-2} & \cdots & f^1_{|n|+m-n_1-2} & & f^k_{|n|+m-2} & \cdots & f^k_{|n|+m-n_k-2} & & f^m_{|n|+m-2} & \cdots & f^m_{|n|+m-n_m-2} \\ 0 & \cdots & 0 & \cdots\;\cdots & 1 & \cdots & x^{n_k} & \cdots\;\cdots & 0 & \cdots & 0 \end{vmatrix}$$

$$\tau(n) \stackrel{c}{=} \begin{vmatrix} f_0^1 & & 0 & & f_0^m & & 0 \\ f_1^1 & \ddots & & \cdots\;\cdots & f_1^m & \ddots & \\ \vdots & & f_0^1 & & \vdots & & f_0^m \\ \vdots & & \vdots & \cdots\;\cdots & \vdots & & \vdots \\ f^1_{|n|+m-2} & \cdots & & & & \cdots & f^m_{|n|+m-n_m-2} \\ f^1_{|n|+m-1} & \cdots & f^1_{|n|+m-n_1-1} & & f^m_{|n|+m-1} & \cdots & f^m_{|n|+m-n_m-1} \end{vmatrix}$$

Such polynomials give the so called canonical solution of the Hermite–Padé problem

$$f_1(t)Z_1(n,t) + \cdots + f_m(t)Z_m(n,t) = x^{|n|+m-1}\tau(n) + \text{higher order terms}$$

[Della Dora & Di Crescenzo, 1984], [Paszkowski 1982]

moreover

$$\rho^i(n) \stackrel{c}{=} (-1)^{n_{i+1}+\cdots+n_m+m-i} \frac{\tau(n)}{\tau(n-e_i)},$$

$$\psi_k(n,x) \stackrel{c}{=} \frac{Z_k(n,x)}{\tau(n)}$$

In the commutative case the polynomials $Z_k(n,x)$ satisfy the additional constraint

$$xZ_k(n,x)\tau(n) = Z_k(n+e_1,x)\tau(n-e_1) + \cdots + Z_k(n+e_m,x)\tau(n-e_m),$$

which supplements Hirota's discrete KP system with the following equation

$$[\tau(n)]^2 = \tau(n+e_1)\tau(n-e_1) + \cdots + \tau(n+e_m)\tau(n-e_m)$$

[Paszkowski, 1982]

THEOREM

The polynomials $\boldsymbol{\psi}(n,x) = (\psi_1(n,x), \dots, \psi_m(n,x))$, which provide solution of the non-commutative Hermite–Padé problem, satisfy also the constraint

$$x\boldsymbol{\psi}(n,x) = \boldsymbol{\psi}(n+e_1,x)\rho^1(n+e_1)[\rho^1(n)]^{-1} + \cdots + \boldsymbol{\psi}(n+e_m,x)\rho^m(n+e_m)[\rho^m(n)]^{-1},$$

and the quasideterminants $\rho^j(n)$ satisfy the following equation

$$1 = \rho^1(n+e_1)[\rho^1(n)]^{-1} + \cdots + \rho^m(n+e_m)[\rho^m(n)]^{-1}$$

COROLLARY

The above constraint provides addmisible/integrable reduction of the non-Abelian Hirota–Miwa system

In the case $m=2$ we obtain the non-commutative discrete-time Toda chain equation

$$\rho^2(n+e_1)\left([\rho^2(n-e_2)]^{-1} - [\rho^2(n)]^{-1}\right)\rho^2(n-e_1) = \rho^2(n+e_2) - \rho^2(n)$$

obtained recently with the non-commutative Padé approximation theory

[A. D. & A. Siemaszko, 2021]

Conclusion and open problems

- Although the Hermite–Padé approximants had been introduced by Hermite in his proof of transcendency of the Euler constant *e* they have found applications not only in the number theory, but also in the theory of multiple orthogonal polynomials, random matrices or numerical analysis
- Our approach provides new class of solutions to the non-Abelian Hirota–Miwa systems. The solutions are subject to an additional integrable constraint
- The question of additional reductions to non-commutative Painlevé equations and application to multiple orthogonal matrix polynomials are still open
- Extension of the relation of the Hermite–Padé approximation to INTERPOLATION in both commutative and non-commutative case leads to NON-AUTONOMOUS version of the above multidimensional fully discrete Toda lattice equations (work in progress)

A. Doliwa, A. Siemaszko, *Hermite-Padé approximation and integrability*, `arXiv:2201.06829` [to appear in:] Numerical Algorithms

A. Doliwa, *Non-commutative Hermite-Padé approximation and integrability*, `arXiv:2202.00782`