Hirota equation and the quantum plane

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Outline

1. Desargues maps and Hirota's discrete KP equation
2. The normalization map
   - Pentagonal property of the normalization map
   - Ultralocality and quantum – the normalization map case
   - The bialgebra structure of the quantum plane
3. The Veblen flip map

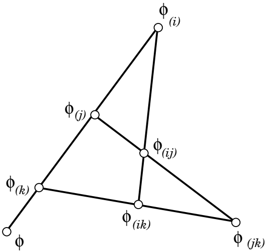
1. A. Doliwa, Desargues maps and the Hirota-Miwa equation, Proc. R. Soc. A 466 (2010) 1177-1200
2. A. Doliwa, S. M. Sergeev, The pentagon relation and incidence geometry, arXiv:1108.0944
3. A. Doliwa, Hirota equation and the quantum plane

Outline

Desargues maps

Maps $\phi : \mathbb{Z}^K \to \mathbb{P}^M(\mathbb{D})$, $K, M \geq 2$, such that the points $\phi(n)$, $\phi_{(i)}(n)$ and $\phi_{(j)}(n)$ are collinear, for all $n \in \mathbb{Z}^K$, $i \neq j$

Notation: $\phi_{(i)}(n_1, \ldots, n_i, \ldots, n_K) = \phi(n_1, \ldots, n_i + 1, \ldots, n_K)$



Remark 1 By the Menelaus theorem this configuration gives geometric interpretation of the discrete S-KP equation *[Konopelchenko & Schief '02]*

Remark 2 Four dimensional consistency of discrete S-KP has combinatorics of the Desargues theorem *[Schief '09]*

The Desargues map system

In homogeneous coordinates $\phi : \mathbb{Z}^K \to \mathbb{D}^{M+1}$

$$\phi + \phi_{(i)} A_{ij} + \phi_{(j)} A_{ji} = 0, \qquad i \neq j,$$

where $A_{ij} : \mathbb{Z}^K \to \mathbb{D}^{\times}$.
The compatibility condition of the above linear system reads

$$A_{ij}^{-1} A_{ik} + A_{kj}^{-1} A_{ki} = 1,$$
$$A_{ik(j)} A_{jk} = A_{jk(i)} A_{ik},$$

where i, j, k are distinct *[AD '10]*

Gauge transformations: $\phi = \tilde{\phi} F$, where $F : \mathbb{Z}^K \to \mathbb{D}^{\times}$ - gauge function results in $\tilde{A}_{ij} = F_{(i)} A_{ij} F^{-1}$

The Hirota system gauge

One can find homogeneous coordinates such that $A_{ji} = -A_{ij} = U_{ij}^{-1}$

$$\phi_{(i)} - \phi_{(j)} = \phi U_{ij}, \qquad 1 \leq i \neq j \leq K,$$

When $\mathbb{D} = \mathbb{F}$ is comutative then the functions U_{ij} can be parametrized in terms of a single potential $\tau : \mathbb{Z}^K \to \mathbb{F}$

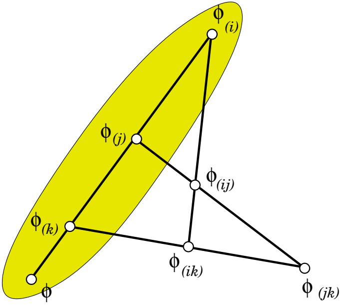
$$U_{ij} = \frac{\tau \tau_{(ij)}}{\tau_{(i)} \tau_{(j)}}, \qquad 1 \leq i < j \leq K$$

The nonlinear system reads *[Hirota '81], [Miwa '82]*

$$\tau_{(i)}\tau_{(jk)} - \tau_{(j)}\tau_{(ik)} + \tau_{(k)}\tau_{(ij)} = 0, \qquad 1 \leq i < j < k \leq K$$

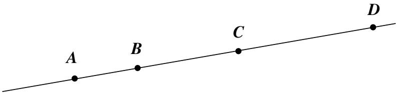
Outline

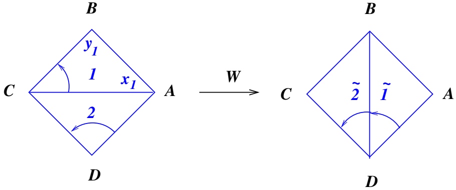
The first part of the Desargues map equations



$$A_{ij}^{-1}A_{ik} + A_{kj}^{-1}A_{ki} = 1, \qquad i,j,k \quad \text{distinct}$$

The normalization map in the simplex representation



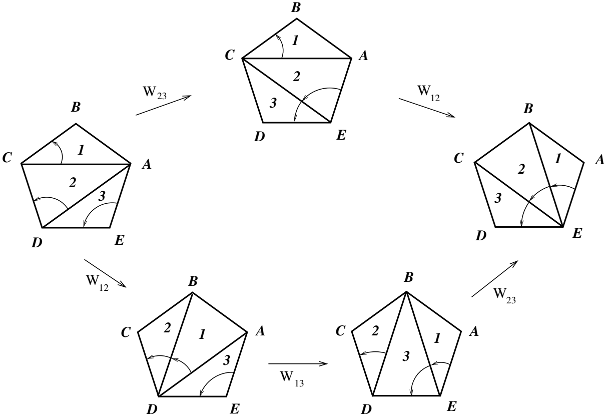


$$\phi_C = \phi_B y_1 + \phi_A x_1, \qquad \phi_D = \phi_B \tilde{y}_1 + \phi_A \tilde{x}_1,$$
$$\phi_D = \phi_C y_2 + \phi_A x_2, \qquad \phi_D = \phi_C \tilde{y}_2 + \phi_B \tilde{x}_2,$$

$$\tilde{x}_1 = x_2 + x_1 y_2, \qquad \tilde{y}_1 = y_1 y_2$$
$$\tilde{x}_2 = -y_1 x_1^{-1} x_2, \qquad \tilde{y}_2 = y_2 + x_1^{-1} x_2$$

The pentagon property of the normalization map

What happens when we add a fifth point?



$$W_{23} \circ W_{13} \circ W_{12} = W_{12} \circ W_{23}$$

$$W_{12} \circ W_{23} \begin{pmatrix} x_1, y_1 \\ x_2, y_2 \\ x_3, y_3 \end{pmatrix} =$$

$$= \left(\begin{array}{rl} x_3 + x_2 y_3 + x_1 y_2 y_3, & y_1 y_2 y_3 \\ -y_1 x_1^{-1}(x_3 + x_2 y_3), & y_2 y_3 + x_1^{-1}(x_3 + x_2 y_3) \\ -y_2 x_2^{-1} x_3, & y_3 + x_2^{-1} x_3 \end{array} \right) =$$

$$= W_{23} \circ W_{13} \circ W_{12} \begin{pmatrix} x_1, y_1 \\ x_2, y_2 \\ x_3, y_3 \end{pmatrix}$$

Ultra-locality and quantum

$\Bbbk \subset \mathcal{Z}(\mathbb{D})$ – fixed subfield of the division ring $\mathbb{D}$
$\mathbb{D}_i$ – division $\Bbbk$-subalgebra generated by x_i, y_i, $i = 1, 2$

General position assumptions

- x_1, y_1, x_2, y_2 do not belong to $\Bbbk$ and are linearly independent
- $\mathbb{D}_1 \cap \mathbb{D}_2 = \Bbbk$

Ultra-locality – "variables with different indices commute"

Proposition

Assume that the map W preserves the ultra-locality condition

$$x_i x_j = x_j x_i, \qquad y_i y_j = y_j y_i, \qquad x_i y_j = y_j x_i, \qquad i \neq j,$$

then under general position assumptions there exists $q \in \Bbbk^{\times}$ such that

$$x_i y_i = q y_i x_i, \qquad \tilde{x}_i \tilde{y}_i = q \tilde{y}_i \tilde{x}_i, \qquad i = 1, 2$$

The Poisson property of the normalization map

Corollary

The normalization map provides automorphism of the division ring $\mathbb{k}_q(x_1, y_1, x_2, y_2)$ of quantum rational functions

The classical limit $q \to 1$

Observation

The normalization map provides a Poisson automorphism of the commutative field of rational functions $\mathbb{k}(x_1, y_1, x_2, y_2)$ equipped with the Poisson structure

$$\{x_i, y_i\} = x_i y_i, \qquad \{x_i, x_j\} = \{y_i, y_j\} = \{x_i, y_j\} = 0, \qquad i \neq j$$

Cartier–Foata property

Cartier–Foata matrix – "elements in different rows commute"

Proposition

Given 2 by 2 Cartier–Foata matrix

$$M = \begin{pmatrix} y_1 & x_1 \\ x_2 & y_2 \end{pmatrix},$$

with elements satisfying the generic position conditions. If its inverse

$$M^{-1} = \begin{pmatrix} \left(y_1 - x_1 y_2^{-1} x_2\right)^{-1} & \left(x_2 - y_2 x_1^{-1} y_1\right)^{-1} \\ \left(x_1 - y_1 x_2^{-1} y_2\right)^{-1} & \left(y_2 - x_2 y_1^{-1} x_1\right)^{-1} \end{pmatrix}$$

is Cartier–Foata matrix as well, then there exists $q \in \Bbbk^{\times}$ such that

$$x_i y_i = q y_i x_i, \qquad i = 1, 2.$$

Quantum plane $\Bbbk_q[x, y] = \Bbbk\{x, y \mid xy - qyx = 0\}$

The normalization map W defines unital morfism of algebras (coproduct)

$$\Delta : \Bbbk_q[x, y] \to \Bbbk_q[x, y] \otimes \Bbbk_q[x, y] = \Bbbk_q[x_1, y_1, x_2, y_2]$$

given on generators by

$$\Delta(x) = 1 \otimes x + x \otimes y = \tilde{x}_1, \qquad \Delta(y) = y \otimes y = \tilde{y}_1$$

which due to pentagonal property of W satisfies the coassociativity condition

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$$

$$\Delta^2(x) = 1 \otimes 1 \otimes x + 1 \otimes x \otimes y + x \otimes y \otimes y, \qquad \Delta^2(y) = y \otimes y \otimes y$$

Quantum plane: counit and antipode

The unital morphism $\epsilon : \mathbb{k}_q[x, y] \to \mathbb{k}$ (counit) satisfying

$$(\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id} = (\epsilon \otimes \mathrm{id}) \circ \Delta$$

is given by

$$\epsilon(x) = 0, \qquad \epsilon(y) = 1$$

To construct the coresponding Hopf algebra enlarge the quantum plane by the inverse of y and find on $\mathbb{k}_q[x, y, y^{-1}]$ a linear map (antipode) by requiring

$$\sum_{(a)} S(a_{(1)})a_{(2)} = \sum_{(a)} a_{(1)}S(a_{(2)}) = \epsilon(a)1, \qquad \Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$$

We have

$$S(x) = -xy^{-1}, \qquad S(y) = y^{-1}$$

Quantum pentagon equation

Using quantum dilogarithm function (Faddeev, Woronowicz) we can find

$$\mathbf{W} \in \overline{\Bbbk_q(x_1, y_1, x_2, y_2)}$$

such that for any rational function $F \in \Bbbk_q(x_1, y_1, x_2, y_2)$

$$\mathbf{W} F(x_1, y_1, x_2, y_2) \mathbf{W}^{-1} = F(\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2)$$

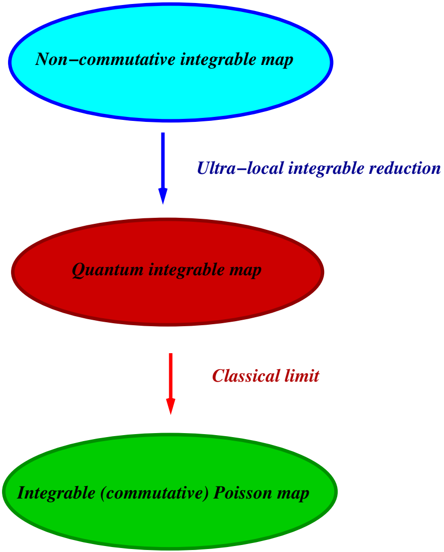
in particular

$$\Delta(a) = \mathbf{W}(a \otimes 1)\mathbf{W}^{-1}, \qquad a \in \Bbbk_q[x, y, y^{-1}]$$

After eventual rescaling such a $\mathbf{W}$ satisfies the quantum pentagon equation

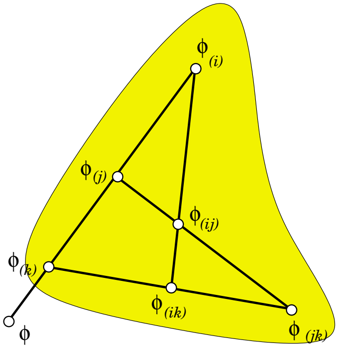
$$\mathbf{W}_{12}\mathbf{W}_{13}\mathbf{W}_{23} = \mathbf{W}_{23}\mathbf{W}_{12}$$

Quantization as reduction



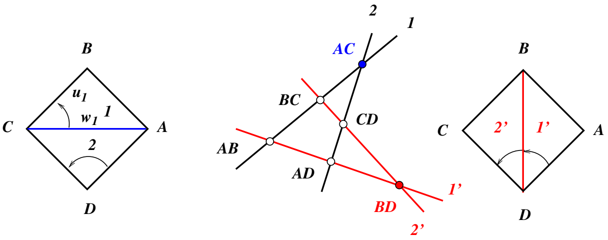
Outline

The Veblen configuration and the second part of the Desargues map equations



$$A_{ik(j)}A_{jk} = A_{jk(i)}A_{ik}, \qquad i, j, k \quad \text{distinct}$$

The Veblen flip in $\mathbb{P}^M(\mathbb{D})$, $\mathbb{D}$ – division ring



The Veblen flip and its simplex representation

$$\phi_{AB} = \phi_{AC} w_1 + \phi_{BC} u_1$$
$$\phi_{AC} = \phi_{AD} w_2 + \phi_{CD} u_2$$

$\phi \in \mathbb{D}^{M+1}$, and $u, w \in \mathbb{D}^\times$

$$\phi_{AB} = \phi_{AD} w_1' + \phi_{BD} u_1'$$
$$\phi_{BC} = \phi_{BD} w_2' + \phi_{CD} u_2'$$

The Veblen flip map

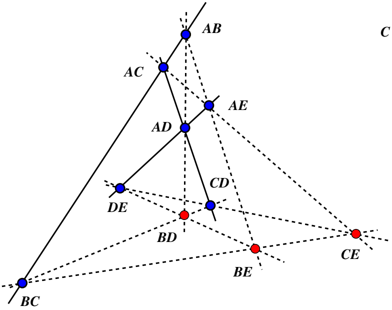
$W^G : \mathbb{D}^2 \times \mathbb{D}^2 \dashrightarrow \mathbb{D}^2 \times \mathbb{D}^2$

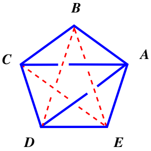
$$u_1' = G w_1, \qquad w_1' = w_2 w_1$$
$$u_2' = -u_2 w_1 u_1^{-1}, \qquad w_2' = G w_1 u_1^{-1}$$

G – free gauge parameter

$$\phi_{BD} = \left(\phi_{BC} u_1 w_1^{-1} + \phi_{CD} u_2\right) G^{-1} = \left(\phi_{AB} w_1^{-1} - \phi_{AD} w_2\right) G^{-1}$$

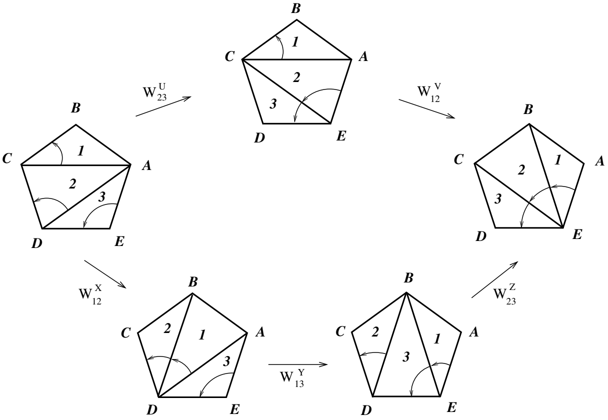
The Desargues configuration





The Desargues configuration and its 4-simplex combinatorics

The pentagon property of the Veblen flip



The pentagon relation for Veblen flips in the 4-simplex representation

The pentagon property of the Veblen map

Proposition

The Veblen flip map $W^G : \mathbb{D}^2 \times \mathbb{D}^2 \dashrightarrow \mathbb{D}^2 \times \mathbb{D}^2$

$$u_1' = G w_1, \qquad w_1' = w_2 w_1$$
$$u_2' = -u_2 w_1 u_1^{-1}, \qquad w_2' = G w_1 u_1^{-1}$$

satisfies the functional dynamical pentagon relation

$$W_{12}^{Z} \circ W_{13}^{Y} \circ W_{23}^{X} = W_{23}^{V} \circ W_{12}^{U}$$

provided the parameters/functions U, V on the right hand side are related to the left hand functions X, Y, Z by the conditions

$$V = Y w_2, \qquad ZX = -U w_2$$

Ultra-local/quantum reduction of the Veblen flip map

"Fix" the gauge parameter G of the Veblen flip map

$$G(u_1, w_1, u_2, w_2) = (\alpha u_2 + \beta w_2 u_1) w_1^{-1}, \qquad \alpha, \beta \in \Bbbk,$$

Under the ultra-locality condition the map $W^{(\alpha,\beta)}$

$$\begin{aligned} u_1' &= \alpha u_2 + \beta u_1 w_2, & \qquad w_1' &= w_1 w_2 \\ u_2' &= -w_1 u_1^{-1} u_2, & \qquad w_2' &= \alpha u_1^{-1} u_2 + \beta u_1 \end{aligned}$$

- selects the Weyl (q-plane) commutation relations
- satisfies the functional pentagon equation

$$W_{12}^{(\alpha_V,\beta_V)} \circ W_{23}^{(\alpha_U,\beta_U)} = W_{23}^{(\alpha_Z,\beta_Z)} \circ W_{13}^{(\alpha_Y,\beta_Y)} \circ W_{12}^{(\alpha_X,\beta_X)},$$

provided

$$\alpha_X = \alpha_V \beta_Z, \quad \alpha_Y = \alpha_U \alpha_V, \quad \alpha_Z = \alpha_U \beta_X, \quad \beta_U = \beta_Y \beta_Z, \quad \beta_V = \beta_X \beta_Y$$

Girard Desargues (1591–1661)

The relevance of a geometric theorem is determined by what the theorem tells us about space, and not by the eventual difficulty of the proof. The Desargues' theorem of projective geometry comes as close as a proof can to the Zen ideal. It can be summarized in two words: "I see!" Nevertheless, Desargues' theorem, far from trivial despite the simplicity of its proof, has many more applications both in geometry and beyond than any theorem in number theory, maybe even more then all the theorems in analytic number theory combined.

[Gian Carlo Rota, The Phenomenology of Mathematical Proof]

Conclusion

- the pentagonal property of the Veblen and normalization maps are equivalent to four dimensional consistency of the Desargues map equation
- from the ultra-locality assumption of the normalization map (and the Veblen map as well) we obtain for free the quantum plane (Weyl) commutation relations and Poisson property of the corresponding classical map
- integrability (multidimensional consistency) of the ultra-local normalization map provides with the bi-algebra structure maps of the quantum plane
- the pentagonal property of the q-commutative maps can be then reformulated in the operator form
- relation to quantum integrable systems?
- relation to Topological Quantum Field Theory (via Pachner moves)?