The Hirota equation and its reductions from the point of view of root lattices

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Integrable Systems in Newcastle

Newcastle upon Tyne, 26–27 September, 2014

Outline

1. The affine Weyl group symmetry of Desargues maps
   - The A_N root lattice and its affine $W(A_N)$ Weyl group
   - Desargues maps of the $Q(A_N)$ root lattice
   - The non-commutative Hirota system
2. Planar quadrilaterals lattices and their reductions
   - The quadrilateral lattice
   - B and C reductions of the Hirota system
3. Periodic reduction of Desargues maps
   - Gel'fand–Dikii systems
   - Yang–Baxter maps
   - Self-similarity $(2,2)$ reduction to $q - P_{VI}$

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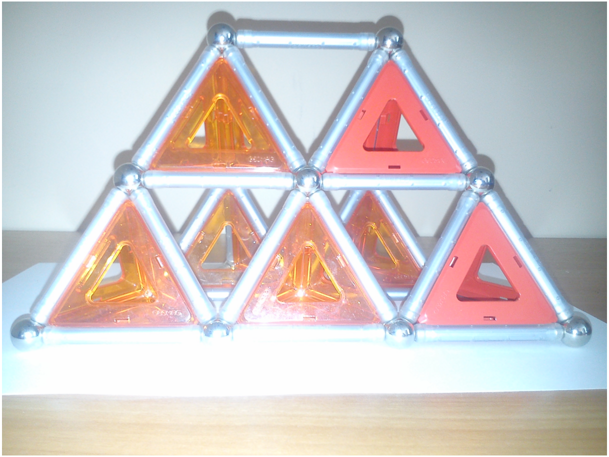
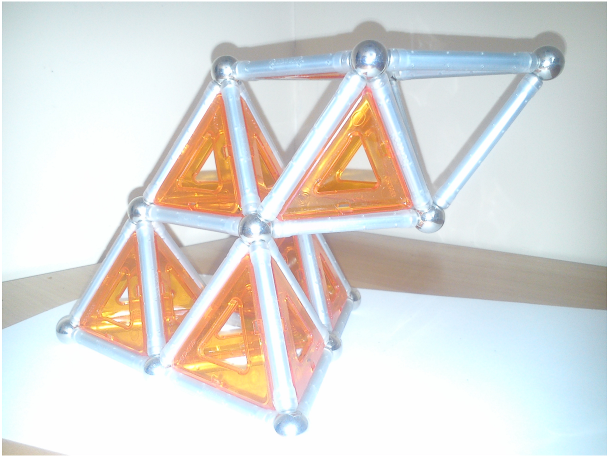




The A_N root lattice

$Q(A_N)$ is the lattice generated by vectors along the edges of regular N-simplex. If we take the vertices of the simplex to be the vectors of the canonical basis in $\mathbb{R}^{N+1}$

$$\boldsymbol{e}_i = (0, \dots, \overset{i}{1}, \dots, 0), \qquad 1 \leq i \leq N+1$$

then the generators are

$$\varepsilon^i_j = \boldsymbol{e}_i - \boldsymbol{e}_j, \qquad 1 \leq i \neq j \leq N+1$$

$$Q(A_N) = \{(n_1, \dots, n_{N+1}) \in \mathbb{Z}^{N+1} | n_1 + \dots + n_{N+1} = 0\}$$

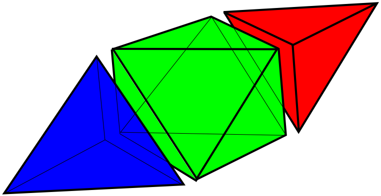
$\boldsymbol{\alpha}_i = \boldsymbol{e}_i - \boldsymbol{e}_{i+1}$ - simple root vectors

$$(\boldsymbol{\omega}_i | \boldsymbol{\alpha}_j) = \delta_{ij}, \qquad i, j = 1, \dots, N$$

$\boldsymbol{\omega}_i$ - fundamental weights

Tiles (Delunay polytopes) of the A_N root lattice

Holes - points locally maximally distant from the lattice
Delaunay polytope - convex hull of the lattice points closest to the hole



The Delaunay polytopes of $Q(A_N)$: $P(k,N)$, $k = 1, \ldots N$
truncations of order $k-1$ of the regular N-simplex

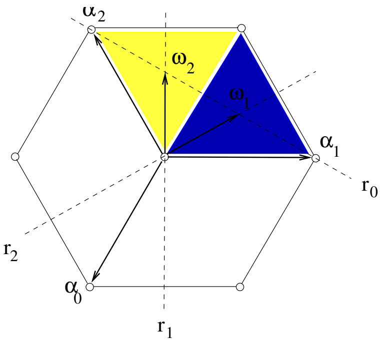
$\omega_k + Q(A_N)$ - centers of tiles of type $P(k,N)$

The A_N Weyl group

The *Weyl group* $W_0(A_N)$ is generated by the reflections r_i

$$r_i : \boldsymbol{v} \mapsto \boldsymbol{v} - \frac{2(\boldsymbol{v}|\boldsymbol{\alpha}_i)}{(\boldsymbol{\alpha}_i|\boldsymbol{\alpha}_i)}\boldsymbol{\alpha}_i, \qquad i = 1, \ldots, N$$

$W_0(A_N) \equiv S_{N+1}$, where r_i is identified with transposition $\sigma_i = (i, i+1)$



The A_N affine Weyl group

The *affine Weyl group* $W(A_N)$ is generated by the reflections r_i, $1 \leq i \leq N$, and by the affine reflection r_0

$$r_0 : \boldsymbol{v} \mapsto \boldsymbol{v} - \left(1 - \frac{2(\boldsymbol{v}|\tilde{\alpha})}{(\tilde{\alpha}|\tilde{\alpha})}\right)\tilde{\alpha}$$

$\tilde{\alpha} = -\boldsymbol{\alpha}_0 = \boldsymbol{\alpha}_1 + \cdots + \boldsymbol{\alpha}_N = \boldsymbol{e}_1 - \boldsymbol{e}_{N+1}$ - the highest root vector

$$W(A_N) = Q(A_N) \rtimes W_0(A_N)$$

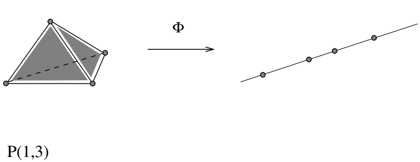
Theorem (Coxeter)

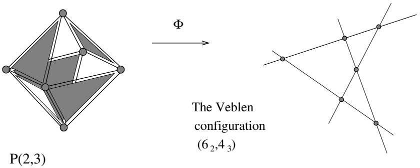
The affine Weyl group acts on the Delaunay tiling by permuting tiles within each class $P(k, N)$.

Desargues maps and their affine Weyl group symmetry

Definition of Desargues maps

Maps $\phi : Q(A_N) \to \mathbb{P}^M$ such that for any translate of the N-simplex its vertices are mapped into collinear points *[AD 2011]*





By the Coxeter theorem we have

Theorem

If $\phi : Q(A_N) \to \mathbb{P}^M$ is a Desargues map then for an arbitrary $w \in W(A_N)$ acting on $Q(A_N)$ the map $\phi \circ w$ is a Desargues map.

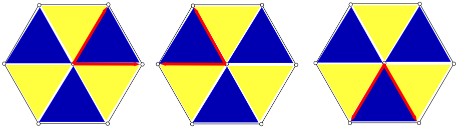
$\mathbb{Z}^N$ - Desargues maps

Identify $\mathbb{Z}^N = \sum_{i=1}^{N} \mathbb{Z}\varepsilon_i^{N+1} = Q(A_N)$

$\mathbb{Z}^N$ – Desargues maps

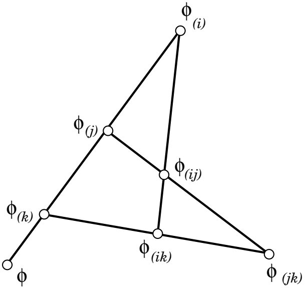
Maps $\phi : \mathbb{Z}^N \to \mathbb{P}^M$, such that the points $\phi(n)$, $\phi_{(i)}(n)$ and $\phi_{(j)}(n)$ are collinear, for all $n \in \mathbb{Z}^N$, $i \neq j$ *[AD 2010]*

Notation: $\phi_{(i)}(n_1, \ldots, n_i, \ldots, n_N) = \phi(n_1, \ldots, n_i + 1, \ldots, n_N)$



Observation: There are $N+1$ equivalent choices of $\mathbb{Z}^N$ coordinates in $Q(A_N)$ (with fixed origin) respecting geometrically the Desargues map condition!

Linear problem for Desargues maps

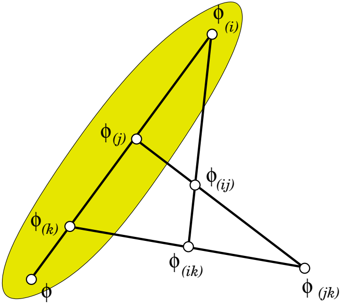


Algebraic description in homogeneous coordinates $\mathbf{\Phi} : \mathbb{Z}^N \to \mathbb{D}^{M+1}$

$$\mathbf{\Phi} + \mathbf{\Phi}_{(i)} A_{ij} + \mathbf{\Phi}_{(j)} A_{ji} = 0, \qquad i \neq j, \qquad A_{ij} : \mathbb{Z}^N \to \mathbb{D}^\times$$

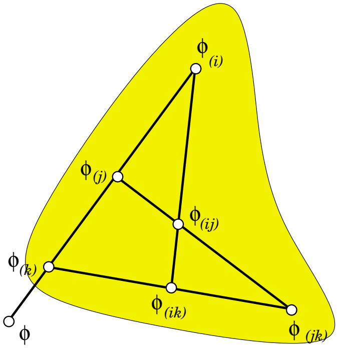
$\mathbb{D}$ – arbitrary division ring (skew field)

The first part of the Desargues map equations



$$A_{ij}^{-1}A_{ik} + A_{kj}^{-1}A_{ki} = 1, \qquad i,j,k \quad \text{distinct}$$

The Veblen configuration and the second part of the Desargues map equations



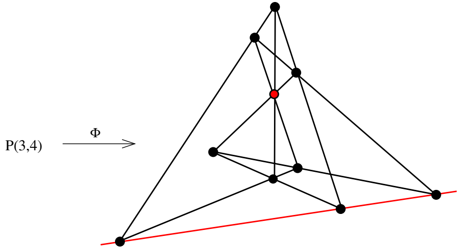
$$A_{ik(j)}A_{jk} = A_{jk(i)}A_{ik}, \qquad i, j, k \quad \text{distinct}$$

Four dimensional consistency of Desargues maps

Desargues theorem

In projective space two triangles are in perspective from a point if and only if they are in perspective from a line

Desargues configuration is the image of $P(3,4)$ cell



Remark: The four dimensional consistency of the Hirota–Miwa equation or the discrete Schwarzian KP equation has combinatorics of the Desargues configuration *[Schief 2009]*

The non-commutative Hirota system

Gauge transformations: $\boldsymbol{\Phi} = \tilde{\boldsymbol{\Phi}} F$, where $F : \mathbb{Z}^N \to \mathbb{D}^\times$ - gauge function results in $\tilde{A}_{ij} = F_{(i)} A_{ij} F^{-1}$

One can find homogeneous coordinates such that $A_{ji} = -A_{ij} = U_{ij}^{-1}$

$$\boldsymbol{\Phi}_{(i)} - \boldsymbol{\Phi}_{(j)} = \boldsymbol{\Phi} U_{ij}, \qquad 1 \leq i \neq j \leq N,$$

Fact to remember

The gauge functions which do not change the structure of the above linear problem are characterized by the condition $F_{(i)} = F_{(j)}$ for all pairs of indices, i.e. F is a function of $n_\sigma = n_1 + n_2 + \cdots + n_N$.

$$U_{ij} + U_{ji} = 0, \qquad U_{ij} + U_{jl} + U_{li} = 0,$$

$$U_{li} U_{lj(i)} = U_{lj} U_{li(j)} \qquad \Longrightarrow \qquad U_{ij} = \rho_i^{-1} \rho_{i(j)}$$

[Nimmo 2006]

The Hirota equation

When $\mathbb{D} = \mathbb{F}$ is commutative then the functions U_{ij} can be parametrized in terms of a single potential $\tau : \mathbb{Z}^N \to \mathbb{F}$

$$U_{ij} = \frac{\tau \tau_{(ij)}}{\tau_{(i)} \tau_{(j)}}, \qquad 1 \leq i < j \leq N$$

The nonlinear system reads *[Hirota 1981], [Miwa 1982]*

$$\tau_{(i)}\tau_{(jl)} - \tau_{(j)}\tau_{(il)} + \tau_{(l)}\tau_{(ij)} = 0, \qquad 1 \leq i < j < l \leq N$$

Remark: Other gauges lead to

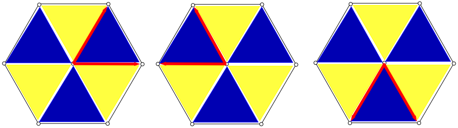
- the non-commutative discrete mKP system *[Nijhoff-Capel 1990]*
- the generalized lattice spin system *[Nijhoff-Capel 1990]*
- non-commutative Schwarzian discrete KP system *[Konopelchenko-Schief 2005]*

Back to the root lattice

In the $N+1$th sector $\mathbb{Z}^N = \sum_{j=1}^{N} \mathbb{Z}\varepsilon_j^{N+1} = Q(A_N)$

$$\phi^{N+1}(n+\varepsilon_i^{N+1}) - \phi^{N+1}(n+\varepsilon_j^{N+1}) = \phi^{N+1}(n) U_{ij}^{N+1}(n), \quad 1 \leq i \neq j \leq N$$

$$U_{ij}^{N+1}(n) = \left[\rho_i^{N+1}(n)\right]^{-1} \rho_i^{N+1}(n+\varepsilon_j^{N+1})$$



In the ith sector $\mathbb{Z}^N = \sum_{j=1, j\neq i}^{N+1} \mathbb{Z}\varepsilon_j^{i} = Q(A_N)$

The "rotated" linear problems

Theorem

The functions $\phi^i : \mathbb{Z}^N = \sum_{j=1, j\neq i}^{N+1} \mathbb{Z}\varepsilon_j^i \to \mathbb{D}_*^{M+1}$ *given by*

$$\phi^i(n) = (-1)^{(n|\varepsilon_i^{N+1})} \phi^{N+1}(n) \left[\rho_i^{N+1}(n)\right]^{-1}$$

satisfy the linear system

$$\phi^i(n+\varepsilon_j^i) - \phi^i(n+\varepsilon_k^i) = \phi^i(n) U_{jk}^i(n), \qquad i, j, k \quad \text{distinct,}$$

where

$$U_{jk}^i(n) = \left[\rho_j^i(n)\right]^{-1} \rho_j^i(n+\varepsilon_k^i),$$

$$\rho_j^i(n) = \begin{cases} \rho_j^{N+1}(n) \left[\rho_i^{N+1}(n)\right]^{-1}, & j \neq N+1, \\ \left[\rho_i^{N+1}(n)\right]^{-1}, & j = N+1. \end{cases}$$

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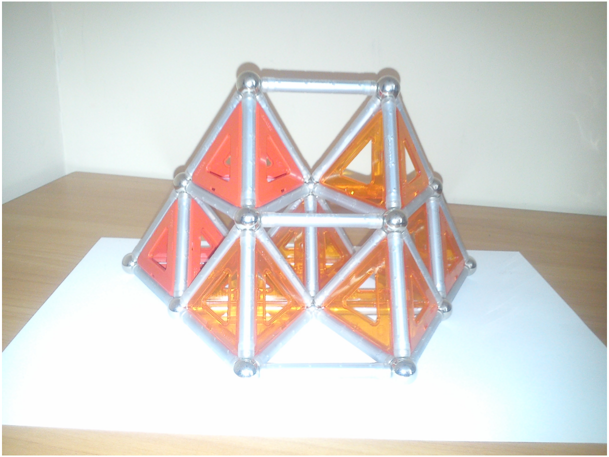


Embedding of B_K into A_{2K-1}

Algebraic description of $Q(A_N)$

$(\mathbf{e}_i)_{i=1}^{N+1}$ – the standard orthonormal basis of $\mathbb{E}^{N+1}$

$\mathbb{E}^{N+1} \supset Q(A_N) \ni \sum_{i=1}^{N+1} x_i \mathbf{e}_i$, $x_i \in \mathbb{N}$, such that $x_1 + x_2 + \cdots + x_{N+1} = 0$

$\varepsilon_i = \mathbf{e}_{N+1} - \mathbf{e}_i$, $i = 1, \ldots, N$ a parallelogram basis of $Q(A_N)$

Fix $N = 2K - 1$ then the vectors $\mathbf{E}_i = \mathbf{e}_{2i-1} - \mathbf{e}_{2i}$, $i = 1, \ldots, K$ satisfy $(\mathbf{E}_i | \mathbf{E}_j) = 2\delta_{ij}$ and generate the $\mathbb{Z}^K = Q(B_K)$ sub-lattice in $Q(A_{2K-1})$

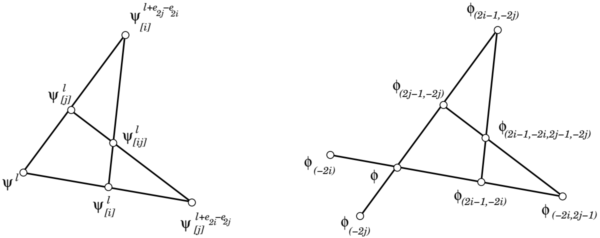
$$\sum_{i=1}^{2K-1} n_i \varepsilon_i = -\sum_{j=1}^{K} m_j \mathbf{E}_j + \sum_{j=1}^{K} \ell_j \mathbf{e}_{2j}, \qquad \ell = \sum_{j=1}^{K} \ell_j \mathbf{e}_{2j} \in Q(A_{K-1})$$

m — quadrilateral lattice variables, $[i]$ — shift by $\mathbf{E}_i$

ℓ — Laplace transformation variables

Discrete Darboux equations

Fix $\ell \in Q(A_{K-1})$ define $\psi^\ell : \mathbb{Z}^K \to \mathbb{P}^M$ by $\psi^\ell(m) = \phi(n)$



- the points ψ^ℓ, $\psi^\ell_{[i]}$, $\psi^\ell_{[j]}$, and $\psi^\ell_{[ij]}$ are coplanar
- the functions $\beta^\ell_{ij} = \operatorname{sgn}(j-i)\left(\frac{\tau^{\ell+\mathbf{e}_{2i}-\mathbf{e}_{2j}}}{\tau^\ell}\right)_{[j]}$, $i \neq j$, satisfy the discrete

Darboux equations *[Bogdanov, Konopelchenko 1995]*

$$\beta^\ell_{ij[k]} = \beta^\ell_{ij} + \beta^\ell_{ik[j]}\beta^\ell_{kj}, \qquad i,j,k \quad \text{distinct}$$

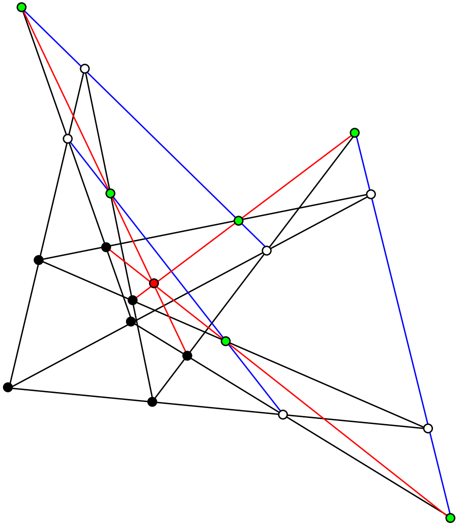
$$\tau^\ell \tau^{\ell+\mathbf{e}_{2i}-\mathbf{e}_{2j}}_{[k]} = \tau^\ell_{[k]}\tau^{\ell+\mathbf{e}_{2i}-\mathbf{e}_{2j}} + \operatorname{sgn}(j-i)\operatorname{sgn}(k-j)\operatorname{sgn}(i-k)\tau^{\ell+\mathbf{e}_{2i}-\mathbf{e}_{2k}}_{[k]}\tau^{\ell+\mathbf{e}_{2k}-\mathbf{e}_{2j}}$$

Quadrilteral lattices and their Laplace transformations

Quadrilateral lattice is a map $\psi : \mathbb{Z}^K \to \mathbb{P}^M(\mathbb{D})$, $2 \leq K \leq M$, whose all elementary quadrilaterals are planar.

- 2D lattices of planar quadrilaterals — discrete conjugate nets *[Sauer 1937]*
- Laplace sequence of 2D discrete conjugate nets — geometric interpretation of the Hirota–Miwa equation in the 2D discrete Toda system form *[AD 1997]*
- multidimensional quadrilateral lattices — geometric interpretation of the discrete Darboux equations *[AD, Santini 1997]*
- Laplace transformations of generic K-dimensional quadrilateral lattices are parametrized by points of the root lattice $Q(A_{K-1})$ *[AD, Mañas, Martínez Alonso, Medina, Santini 1999]*
- FCC $= Q(A_3)$ description of 2D quadrilatral lattice and its Laplace sequence *[Schief 2007]*

$(20_3, 15_4)$ configuration as the image of $P(3,5)$ cell, and the quadrilateral lattice construction



The discrete *C*-KP system

Problem

Find constraints on τ which result in a single equation involving fixed ℓ

$$\tau_{[j]}^{\ell_C+\mathbf{e}_{2i}-\mathbf{e}_{2j}} + \tau_{[i]}^{\ell_C+\mathbf{e}_{2j}-\mathbf{e}_{2i}} = 0, \qquad i \neq j,$$

[AD, Santini 2000]

$$\left(\tau_{[i]}\tau_{[jk]} - \tau_{[j]}\tau_{[ik]} + \tau_{[k]}\tau_{[ij]} - \tau\tau_{[ijk]}\right)^2 - 4\left(\tau_{[i]}\tau_{[jk]}\tau_{[k]}\tau_{[ij]} + \tau_{[j]}\tau_{[ik]}\tau\tau_{[ijk]}\right) + \\ +4\tau_{[i]}\tau_{[j]}\tau_{[k]}\tau_{[ijk]} + 4\tau\tau_{[ij]}\tau_{[jk]}\tau_{[ik]} = 0 \qquad \tau = \tau^{\ell_C}$$

[Kashaev 1996] , [Schief 2003]

Remark

"Half" of the discrete KP variables is fixed

The discrete *B*-KP system

$$\left(\tau_{[j]}^{\ell_B+\mathbf{e}_{2i}-\mathbf{e}_{2j}} - \tau_{[i]}^{\ell_B+\mathbf{e}_{2j}-\mathbf{e}_{2i}}\right)^2 = 4\tau_{[i]}^{\ell_B}\tau_{[j]}^{\ell_B}, \qquad i \neq j \qquad (*)$$

$$\left[\left(\tau_{[i]}\tau_{[jk]} - \tau_{[j]}\tau_{[ik]} + \tau_{[k]}\tau_{[ij]} - \tau\tau_{[ijk]}\right)^2 - 4\left(\tau_{[i]}\tau_{[jk]}\tau_{[k]}\tau_{[ij]} + \tau_{[j]}\tau_{[ik]}\tau\tau_{[ijk]}\right)\right]^2 =
= 64\tau\tau_{[i]}\tau_{[j]}\tau_{[k]}\tau_{[ij]}\tau_{[jk]}\tau_{[ik]}\tau_{[ijk]}, \qquad \tau = \tau^{\ell_B} \qquad (**)$$

Proposition

One can consistently parametrize $(*)$ by $\mu\colon \mathbb{Z}^K \to \mathbb{F}$ such that $\mu^2 = \tau^{\ell_B}$

$$\tau_{[j]}^{\ell_B+\mathbf{e}_{2i}-\mathbf{e}_{2j}} = -(-1)^{\sum_{i\leq k<j} m_k}\left(\mu\mu_{[ij]} + \mu_{[i]}\mu_{[j]}\right) \qquad i<j$$

$$\tau_{[i]}^{\ell_B+\mathbf{e}_{2j}-\mathbf{e}_{2i}} = -(-1)^{\sum_{i\leq k<j} m_k}\left(\mu\mu_{[ij]} - \mu_{[i]}\mu_{[j]}\right)$$

Then equation $(**)$ gives *[Miwa 1982]*

$$\mu\mu_{[ijk]} = \mu_{[i]}\mu_{[jk]} - \mu_{[j]}\mu_{[ik]} + \mu_{[k]}\mu_{[ij]}, \qquad i<j<k$$

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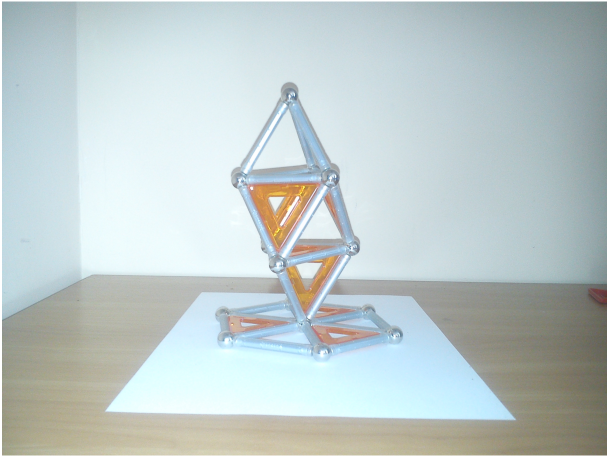


The orthogonal projection of $Q(A_{N+1}) \subset \mathbb{E}^{N+1}$ onto the hyperplane of $Q(A_N)$ gives the weight lattice $P(A_N)$

The non-commutative KP hierarchy

Replace $N \to N+1$, and distinguish the last variable $k = n_{N+1}$, denote also

$$n = (n_1, \dots, n_N), \quad \mathbf{\Phi}(n,k) = \mathbf{\Psi}_k(n), \quad U_{N+1,i}(n,k) = u_{i,k}(n)$$

which allows the rewrite a part (that with the distinguished variable) of the linear problem in the form

$$\mathbf{\Psi}_{k+1} - \mathbf{\Psi}_{k(i)} = \mathbf{\Psi}_k u_{i,k}, \qquad i = 1, \dots, N.$$

[Kajiwara, Noumi, Yamada 2002]

The compatibility of the above linear system reads

$$u_{j,k} u_{i,k(j)} = u_{i,k} u_{j,k(i)}, \qquad i \neq j,$$
$$u_{i,k(j)} + u_{j,k+1} = u_{j,k(i)} + u_{i,k+1}.$$

The first part allows to define potentials $r_k(n) = \rho_{N+1}(n,k)$ such that $u_{i,k} = r_k^{-1} r_{k(i)}$, while the other equations give the system

$$(r_{k(j)}^{-1} - r_{k(i)}^{-1}) r_{k(ij)} = r_{k+1}^{-1}(r_{k+1(i)} - r_{k+1(j)}), \qquad i \neq j$$

Periodic Desargues maps: $\phi_{k+P}(n) = \phi_k(n)$

$$\boldsymbol{\Psi}_{k+P}(n) = \boldsymbol{\Psi}_k(n)\mu_k(n), \quad \mu_{k+1}(n) = \mu_{k(i)}(n), \quad r_{k+P} = r_k\mu_k$$

Matrix linear problem

$$(\boldsymbol{\Psi}_1, \dots, \boldsymbol{\Psi}_P)_{(i)} = (\boldsymbol{\Psi}_1, \dots, \boldsymbol{\Psi}_P) \begin{pmatrix} -u_{i,1} & 0 & \cdots & 0 & \mu_1 \\ 1 & -u_{i,2} & 0 & \dots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & & & -u_{i,P-1} & 0 \\ 0 & 0 & \dots & 1 & -u_{i,P} \end{pmatrix}$$

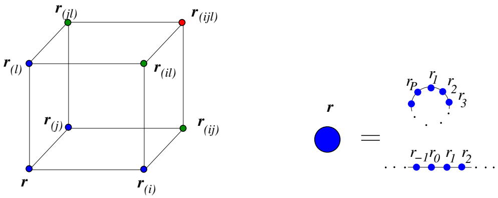
where μ_1 is a function of the variable $n_\sigma = n_1 + \cdots + n_N$.
The corresponding (lattice non-isospectral non-commutative modified Gel'fand–Dikii) system of non-linear equations

$$\begin{aligned} (r^{-1}_{k(j)} - r^{-1}_{k(i)}) r_{k(ij)} &= r^{-1}_{k+1}(r_{k+1(i)} - r_{k+1(j)}), \quad k = 1, \dots, P-1, \\ (r^{-1}_{P(j)} - r^{-1}_{P(i)}) r_{P(ij)} &= \mu_1^{-1} r_1^{-1}(r_{1(i)} - r_{1(j)})\mu_{1(\sigma)} \qquad i \neq j. \end{aligned}$$

Comutative and iso-spectral case *[Nijhoff, Papageorgiou, Capel, Quispel 1992]*

Three dimensional consistency of the GD systems

$\boldsymbol{r} = (r_k)$ where $k \in \mathbb{Z}/(P\mathbb{Z})$ – periodic case, or $k \in \mathbb{Z}$ in the full KP case



Multidimensional consistency of a discrete system — possibility of extending the number of independent variables of the system by adding its copies in different directions

Fact

The lattice non-isospectral non-commutative modified Gel'fand–Dikii system is three-dimensionally consistent.

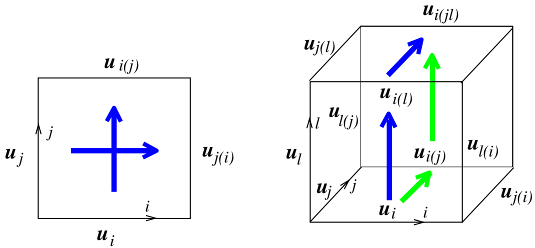
Multidimensional consistency of the KP map

Theorem

The non-commutative KP map (edge system $u_{i,k} = r_k^{-1} r_{k(i)}$)

$$u_{i,k(j)} = (u_{i,k} - u_{j,k})^{-1} u_{i,k}(u_{i,k+1} - u_{j,k+1}), \qquad 1 \leq i \neq j \leq N,$$

is multidimensionaly consistent



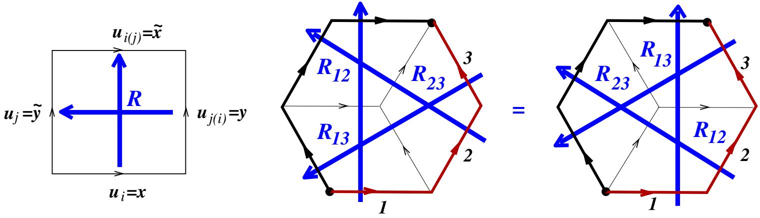
$\boldsymbol{u}_i = (u_{i,k})$, $k \in \mathbb{Z}$ or $k \in \mathbb{Z}/(P\mathbb{Z})$, $u_{i,k+P} = \mu_k^{-1} u_{i,k} \mu_{k(i)}$

From KP map to Yang-Baxter map

A map $R\colon \mathcal{X}\times\mathcal{X}$ is called Yang–Baxter map if

$$R_{12}\circ R_{13}\circ R_{23} = R_{23}\circ R_{13}\circ R_{12}, \qquad \text{in} \quad \mathcal{X}\times\mathcal{X}\times\mathcal{X}$$

If moreover $\pi\circ R\circ\pi\circ R = \mathrm{Id}_{\mathcal{X}\times\mathcal{X}}$, where π is the transposition, then R is called reversible YB map



Non-commutative rational Yang–Baxter maps

Theorem

Given two assemblies of non-commuting variables $\boldsymbol{x} = (x_1, \dots, x_P)$, $\boldsymbol{y} = (y_1, \dots, y_P)$ define polynomials

$$\mathcal{P}_k = \sum_{a=0}^{P-1} \left(\prod_{i=0}^{a-1} y_{k+i} \prod_{i=a+1}^{P-1} x_{k+i} \right), \qquad k = 1, \dots, P,$$

where subscripts in the formula are taken modulo P. If the products $\alpha = x_1 x_2 \dots x_P$ and $\beta = y_1 y_2 \dots y_P$ are central then the map

$$R(\boldsymbol{x}, \boldsymbol{y}) = (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}), \qquad \tilde{x}_k = \mathcal{P}_k x_k \mathcal{P}_{k+1}^{-1}, \qquad \tilde{y}_k = \mathcal{P}_k^{-1} y_k \mathcal{P}_{k+1},$$

is reversible Yang–Baxter map

commutative case *[Kajiwara, Noumi, Yamada 2001], [Etingov 2003]*

Non-commutative F_{III} map

Fact

The products α and β are conserved (for arbitrary P)

The simplest case: $P = 2$ we put $x = x_1$, $y = y_1$ to get a parameter dependent reversible Yang–Baxter map $R(\alpha, \beta) : (x, y) \mapsto (\tilde{x}, \tilde{y})$

$$\tilde{x} = \left(\alpha x^{-1} + y\right) x \left(x + \beta y^{-1}\right)^{-1},$$
$$\tilde{y} = \left(\alpha x^{-1} + y\right)^{-1} y \left(x + \beta y^{-1}\right),$$

which in the commutative case is equivalent to the F_{III} map in the list of *[Adler, Bobenko, Suris 2004]*

Non-commutative Gel'fand–Dikii systems with centrality assumptions

Proposition

In the P-periodic reduction $u_{i,k+P} = \mu_k^{-1} u_{i,k} \mu_{k(i)}$ of the non-commutative KP system assume centrality of the monodromy factors μ_k and of the products $\mathcal{U}_i = u_{i,1} u_{i,2} \dots u_{i,P} \mu_1^{-1}$. Then $\mathcal{U}_i$ is a function of n_i only.

In particular, for $P = 2$ we obtain the non-autonomous, non-isospectral lattice modified KdV equation for non-commutative variable $r = r_1$

$$\left(r_{(j)}^{-1} - r_{(i)}^{-1} \right) r_{(ij)} = \left(r_{(i)}^{-1} \mathcal{U}_i - r_{(j)}^{-1} \mathcal{U}_j \right) r \mu_1 \qquad \text{(nc-ni-na-l-mKdV)}$$

iso-spectral case *[Bobenko, Suris 2002]*

Self-similarity $(2,2)$ reduction to $q-P_{VI}$

In nc-ni-na-l-mKdV take $N=2$, $x_{(1122)}=x$

$$\frac{\mathcal{U}_{i(ii)}}{\mathcal{U}_i}=\frac{\mu}{\mu_{(\sigma\sigma\sigma\sigma)}}, \qquad i=1,2$$

By separation of variables there exists a non-zero central constant q

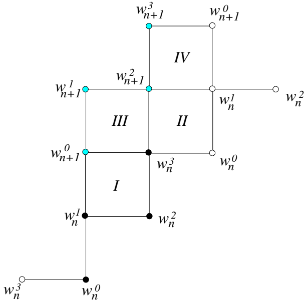
$$\begin{aligned} \mu(n_\sigma) &= \alpha_k q^{n_\sigma}, & k &= n_\sigma \mod 4, & & \\ \mathcal{U}_i(n_i) &= \beta_{i,k} q^{-2n_i}, & k &= n_i \mod 2, & i &= 1,2, \end{aligned}$$

for certain non-zero parameters α_k, $\beta_{i,k}$

Remark: We will need only $\alpha_{k+2}=\alpha_k$

The repeating pattern for q-P_{VI}

$w_n^0 = x(n_1, n_2 - 1)$, $w_n^1 = x(n_1, n_2)$, $w_n^2 = x(n_1 + 1, n_2)$, $w_n^3 = x(n_1 + 1, n_2 + 1)$



shift in $n \equiv$ double shift in n_2

$$f_n = \frac{1}{q\alpha_0\sqrt{\beta_{1,0}\beta_{2,0}}} w_n^0 (w_n^2)^{-1} \mathcal{U}_1(n_1)\mu(n_\sigma),$$

$$g_n = \frac{1}{\alpha_0\sqrt{\beta_{1,0}\beta_{2,1}}} w_n^1 (w_n^3)^{-1} \mathcal{U}_2(n_2)\mu(n_\sigma)$$

A non-commutative q-P_{VI} system

$$t_n = t_0 \lambda^n, \qquad \lambda = q^4, \qquad t_0 = \sqrt{\frac{\beta_{1,0}\beta_{1,1}}{\beta_{2,0}\beta_{2,1}}},$$

$$c_1 = \alpha_0 \sqrt{\beta_{1,1}\beta_{2,0}}, \quad c_2 = \alpha_0 \sqrt{\beta_{1,0}\beta_{2,1}}, \quad c_3 = \alpha_1 \sqrt{\beta_{1,1}\beta_{2,1}}, \quad c_4 = \alpha_1 \sqrt{\beta_{1,0}\beta_{2,0}}$$

nc q-P_{VI}

$$f_{n+1} = \frac{g_n + t_n c_1^{-1}}{g_n + c_2^{-1}} f_n^{-1} \frac{g_n + t_n c_1}{g_n + c_2}, \qquad t_{n+1} = \lambda t_n,$$

$$g_{n+1} = \frac{f_{n+1} + t_n \sqrt{\lambda} c_3^{-1}}{f_{n+1} + c_4^{-1}} g_n^{-1} \frac{f_{n+1} + t_n \sqrt{\lambda} c_3}{f_{n+1} + c_4}$$

[Ramani, Grammaticos 1992], [Jimbo, Sakai 1996]

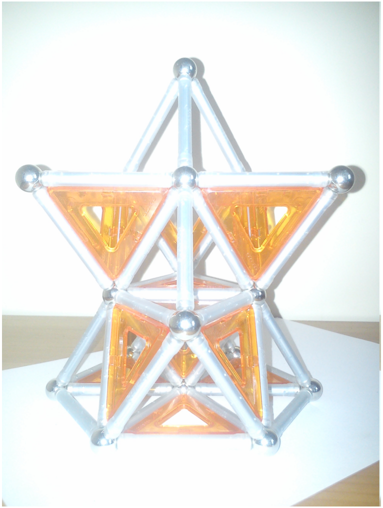
reduction in commutative and iso-spectral case *[Ormerod 2012]*

Conclusion

- we recalled (SIDE IX, Varna 2010) the *A*-type root lattice description of Desargues maps and of the Hirota equation
- *K* dimensional lattices of planar quadrilaterals can be described from the corresponding $Q(B_k) \subset Q(A_{2K-1})$ perspective
- the discrete *C*-KP and *B*-KP equations were given as reductions of the discrete (*A*-)KP equation
- periodicity in one direction of the lattice gives nc-ni-na-l-mGD systems and corresponding YB maps
- self-similarity $(2,2)$ reduction of nc-ni-na-l-mKdV equaion gives nc q-P_{VI} system

References

- A. Doliwa, *Desargues maps and the Hirota–Miwa equation*, Proc. R. Soc. A **466** (2010) 1177–1200.
- A. Doliwa, *The affine Weyl group symmetry of Desargues maps and of the non-commutative Hirota-Miwa system*, Phys. Lett. A **375** (2011) 1219–1224.
- A. Doliwa, *Non-commutative lattice modified Gel'fand-Dikii systems*, J. Phys. A: Math. Theor. **46** (2013) 205202 (14pp).
- A. Doliwa, *Desargues maps and their reductions*, [in:] Nonlinear and Modern Mathematical Physics, W.X. Ma, D. Kaup (eds.), AIP Conference Proceedings, Vol. 1562, AIP Publishing 2013, pp. 30-42.
- A. Doliwa, *Non-commutative q-Painlevé VI equation*, J. Phys. A: Math. Theor. **47** (2014) 035203 (8pp).
- A. Doliwa, *Non-commutative rational Yang-Baxter maps*, Lett. Math. Phys. **104** (2014) 299–309.



THANK YOU!