Hopf algebras of trees and Dyck languages

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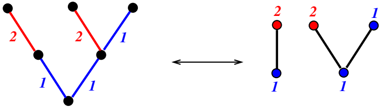
Outline

1. ROC-tree algebra of Foissy and Dyck languages
2. Hopf algebra diagrams
3. The second Hopf algebra structure on Dyck words
4. Distinguished subalgebras

Dyck coding of ROC-trees

Rooted (distinguished vertex) Ordered (set of children of any vertex ordered) Colored (n-colors of edges) tree can be coded by a Dyck word on n pairs of letters $\{a_1, b_1, \dots, a_n, b_n\}$



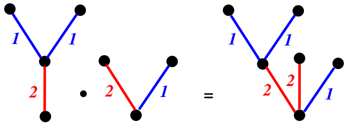
ROC–tree coded by the Dyck word $w = a_1 a_2 b_2 b_1 a_1 a_2 b_2 a_1 b_1 b_1 \in D_2$ and its L. Foissy [2002] version in terms of rooted ordered (vertex-) colored forest

The number of ROC-trees with k edges equals

$$n^k C_k = \frac{n^k}{k+1} \binom{2k}{k}$$

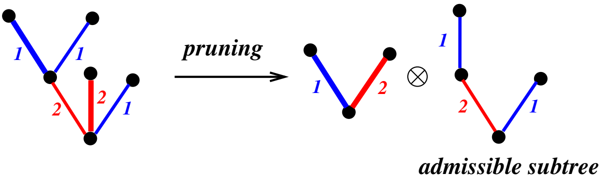
The Foissy-type Hopf algebra structure of Dyck words

$(\mathbb{k}D_n, \bullet, \eta)$ – algebra with concatenation product and $\emptyset$ as unit



$$a_2 a_1 b_1 a_1 b_1 b_2 \bullet a_2 b_2 a_1 b_1 = a_2 a_1 b_1 a_1 b_1 b_2 a_2 b_2 a_1 b_1$$

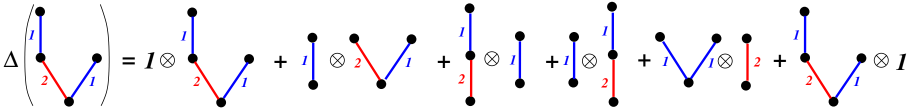
Admissible subtrees and subwords



$$a_2 \boldsymbol{a_1} \boldsymbol{b_1} a_1 b_1 b_2 \boldsymbol{a_2} \boldsymbol{b_2} a_1 b_1 \longrightarrow \boldsymbol{a_1} \boldsymbol{b_1} \boldsymbol{a_2} \boldsymbol{b_2} \otimes a_2 a_1 b_1 b_2 a_1 b_1$$

The Foissy-type Hopf algebra structure of Dyck words

$$\Delta(w) = \sum_{w_s \in A(w)} (w \setminus w_s) \otimes w_s$$



$$\Delta(a_2 a_1 b_1 b_2 a_1 b_1) = 1 \otimes a_2 a_1 b_1 b_2 a_1 b_1 + a_1 b_1 \otimes a_2 b_2 a_1 b_1 + a_2 a_1 b_1 b_2 \otimes a_1 b_1 + \\ + a_1 b_1 \otimes a_2 a_1 b_1 b_2 + a_1 b_1 a_1 b_1 \otimes a_2 b_2 + a_2 a_1 b_1 b_2 a_1 b_1 \otimes 1$$

Proposition

$(\mathbb{k}D_n, \bullet, \Delta, \eta, \epsilon)$ – is graded, locally finite, connected bialgebra

$$\epsilon(w) = \begin{cases} 1_{\mathbb{k}} & \text{if} \quad w = \emptyset \\ 0_{\mathbb{k}} & \text{otherwise} \end{cases}$$

Algebra diagrams

Associativity of multiplication

$$\begin{array}{ccc} \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} & \xrightarrow{\mathrm{id} \otimes \bullet} & \mathcal{H} \otimes \mathcal{H} \\ {\scriptstyle \bullet \otimes \mathrm{id}} \big\downarrow & & \big\downarrow {\scriptstyle \bullet} \\ \mathcal{H} \otimes \mathcal{H} & \xrightarrow{\bullet} & \mathcal{H} \end{array}$$

Unity ($\mathbb{k}$-linear) map in $\mathbb{k}$-algebra $\mathcal{H}$

$$\eta \colon \mathbb{k} \to \mathcal{H}, \qquad \eta(1_{\mathbb{k}}) = 1_{\mathcal{H}}$$

$$\begin{array}{ccccc} \mathcal{H} \otimes \mathbb{k} & = & \mathcal{H} & = & \mathbb{k} \otimes \mathcal{A} \\ {\scriptstyle \mathrm{id} \otimes \eta} \big\downarrow & & {\scriptstyle \mathrm{id}} \big\downarrow & & \big\downarrow {\scriptstyle \eta \otimes \mathrm{id}} \\ \mathcal{H} \otimes \mathcal{H} & \xrightarrow{\bullet} & \mathcal{H} & \xleftarrow{\bullet} & \mathcal{H} \otimes \mathcal{H} \end{array}$$

Coalgebra diagrams

Coassociativity of comultiplication

$$
\begin{tikzcd}
\mathcal{H} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathcal{H} \otimes \mathcal{H} \arrow[d, "\Delta \otimes \mathrm{id}"] \\
\mathcal{H} \otimes \mathcal{H} \arrow[r, "\mathrm{id} \otimes \Delta"] & \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}
\end{tikzcd}
$$

Counit ($\mathbb{k}$-linear) map $\epsilon \colon \mathcal{H} \to \mathbb{k}$

$$
\begin{tikzcd}
\mathcal{H} \otimes \mathcal{H} \arrow[d, "\mathrm{id} \otimes \epsilon"'] & \mathcal{H} \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}"'] & \mathcal{H} \otimes \mathcal{H} \arrow[d, "\epsilon \otimes \mathrm{id}"] \\
\mathcal{H} \otimes \mathbb{k} \arrow[r, equal] & \mathcal{H} \arrow[r, equal] & \mathbb{k} \otimes \mathcal{H}
\end{tikzcd}
$$

Two $(\mathcal{H}, \bullet, \eta)$ – algebra, and $(\mathcal{H}, \Delta, \epsilon)$ – coalgebra structures on $\mathcal{H}$ give bialgebra structure on $\mathcal{H}$ when Δ and ϵ are unital algebra morphisms

Hopf algebra

is a bialgebra $(\mathcal{H}, \bullet, \Delta, \eta, \epsilon)$ with a linear map $T\colon \mathcal{H} \to \mathcal{H}$ satisfying

$$\bullet(T \otimes \mathrm{id}) \circ \Delta = \bullet(\mathrm{id} \otimes T) \circ \Delta = \eta \circ \epsilon$$

In all cases considered here all bialgebras are graded $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}^{(n)}$

$$\mathcal{H}^{(n)} \otimes \mathcal{H}^{(m)} \xrightarrow{\bullet} \mathcal{H}^{(n+m)}, \quad \mathcal{H}^{(n)} \xrightarrow{\Delta} \bigoplus_{i+j=n} \mathcal{H}^{(i)} \otimes \mathcal{H}^{(j)}, \quad \epsilon(\mathcal{H}^{(n)}) = 0, \quad n > 0$$

locally finite and connected

$$\dim \mathcal{H}^{(n)} < \infty, \qquad \dim \mathcal{H}^{(0)} = 1$$

thus (by Takeuchi theorem) they are Hopf algebras

Applications of Hopf algebras

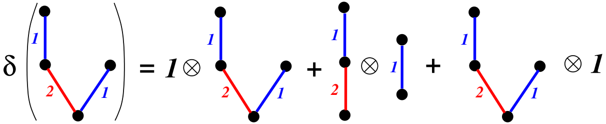
topology of manifolds (H. Hopf), combinatorics (G.-C. Rota), quantum groups and noncommutative geometry, renormalization in quantum field theory, ...

The dual coproduct δ to the concatenation product $\bullet$

The prime Dyck words are of the form $a_i w b_i$, $w \in D_n$, $i = 1, \ldots, n$

For Dyck words $w = u_1 \bullet u_2 \bullet \cdots \bullet u_k$ decomposed into prime factors define

$$\delta(u_1 \bullet u_2 \bullet \cdots \bullet u_k) = \sum_{i=0}^{k} u_1 \bullet \cdots \bullet u_i \otimes u_{i+1} \bullet \cdots \bullet u_k$$



$$\delta(a_2 a_1 b_1 b_2 a_1 b_1) = 1 \otimes a_2 a_1 b_1 b_2 a_1 b_1 + a_2 a_1 b_1 b_2 \otimes a_1 b_1 + a_2 a_1 b_1 b_2 a_1 b_1 \otimes 1$$

Proposition

The second coproduct of the characteristic series S of the n-th Dyck language D_n decomposes as follows

$$\delta(S) = \left(\sum_{i=1}^{n} a_i S b_i\right)^{*} \otimes S$$

Proof: Equations

$$S = 1 + a_1 S\, b_1 S + \cdots + a_n S\, b_n S$$

$$\delta(a_i u b_i v) = 1 \otimes a_i u b_i v + (a_i u b_i \otimes 1) \bullet \delta(v)$$

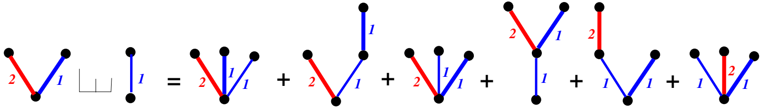
give

$$\delta(S) = 1 \otimes S + \left(\sum_{i=1}^{n} a_i S b_i \otimes 1\right) \bullet \delta(S)$$

Graded dual bialgebra to $(\Bbbk D_n, \bullet, \Delta, \eta, \epsilon)$

Define the asymmetric shuffle product $\sqcup\!\sqcup : \Bbbk D_n \otimes \Bbbk D_n \to \Bbbk D_n$ on two Dyck words w and v as shuffling of prime factors of w into letters of v

$$\boldsymbol{a_2 b_2\, a_1 b_1} \sqcup\!\sqcup\, a_1 b_1 = \boldsymbol{a_2 b_2\, a_1 b_1} a_1 b_1 + \boldsymbol{a_2 b_2} a_1 \boldsymbol{a_1 b_1} b_1 + \boldsymbol{a_2 b_2} a_1 b_1 \boldsymbol{a_1 b_1} + a_1 \boldsymbol{a_2 b_2 a_1 b_1} b_1 + a_1 \boldsymbol{a_2 b_2} b_1 \boldsymbol{a_1 b_1} + a_1 b_1 \boldsymbol{a_2 b_2\, a_1 b_1}$$



Facts (Foissy)

1. $\sqcup\!\sqcup$ is associative
2. $(\Bbbk D_n, \sqcup\!\sqcup, \delta, \eta, \epsilon)$ is a graded dual bialgebra to $(\Bbbk D_n, \bullet, \Delta, \eta, \epsilon)$
3. the above duality is self-duality (!)

Connection to the theorem of Reutenauer

The Hopf subalgebra of $(\Bbbk D_n, \bullet, \Delta, \eta, \epsilon)$ generated by $\boldsymbol{\alpha}_1 = a_1 b_1$, $\boldsymbol{\alpha}_2 = a_2 b_2, \ldots \boldsymbol{\alpha}_n = a_n b_n$ is the free Hopf algebra $\Bbbk\langle A\rangle$ on the set $A = \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n\}$ with its natural concatenation product
The dual (deconcatenation) coproduct

$$\delta(\boldsymbol{\alpha}_{i_1} \cdots \boldsymbol{\alpha}_{i_m}) = \sum_{k=0}^{m} \boldsymbol{\alpha}_{i_1} \cdots \boldsymbol{\alpha}_{i_k} \otimes \boldsymbol{\alpha}_{i_{k+1}} \cdots \boldsymbol{\alpha}_{i_m}$$

Theorem (Reutenauer)

Let S_L be characteristic series of a language $L \subset A^*$. Then L is recognizable if and only if there exists finite decomposition

$$\delta(S_L) = \sum_{i=1}^{k} S_i \otimes T_i, \qquad S_i, T_i \in \mathbb{Q}\langle\langle A\rangle\rangle$$

i.e. S_L belongs to the Sweedler dual of $\mathbb{Q}\langle A\rangle$

Hopf algebra of noncommutative symmetric functions

$$\mathrm{NSym} = \mathbb{Q}\langle \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \ldots \rangle, \qquad \Delta(\boldsymbol{e}_n) = \sum_{k=0}^{n} \boldsymbol{e}_k \otimes \boldsymbol{e}_{n-k}$$

Gelfand, Krob, Lascoux, Leclerc, Retakh, Thibon [1995]

$\mathrm{Sym} = \mathbb{Q}[e_1, e_2, e_3, \ldots]$ where

$$e_1 = \sum_{1 \leq i} x_i, \quad e_2 = \sum_{1 \leq i < j} x_i x_j, \quad e_3 = \sum_{1 \leq i < j < k} x_i x_j x_k, \ldots$$

are elementary symmetric functions (in infinitely many variables)

Proposition

1. NSym is a Hopf subalgebra of $(\mathbb{Q}D_1, \bullet, \Delta, \eta, \epsilon)$ generated by $\boldsymbol{e}_1 = ab$, $\boldsymbol{e}_2 = a^2b^2$, $\boldsymbol{e}_3 = a^3b^3$, ... (sticks)
2. graded dual QSym (quasisymmetric functions by Gessel) of NSym inherits its Hopf algebra structure from $(\mathbb{Q}D_1, \sqcup\!\sqcup, \delta, \eta, \epsilon)$

THANK YOU

Related works

- Darli Grinberg, Victor Reiner, *Hopf algebras in combinatorics*, `arXiv:1409.8356`.
- Loïc Foissy, *Les algébres de Hopf des arbres enracinés décorés. I, II*, Bull. Sci. Math. **126** (2002) no. 3, 193–239, no. 4, 249–298.
- Marcelo Aguiar, N. Bergeron, Frank Sottile, *Combinatorial Hopf algebras and generalized Dehn-Sommerville relations*, Compositio Mathematica **142** (2006) 1–30.