Reconstruction in the Cone-Beam Vector Tomography With Two Sources

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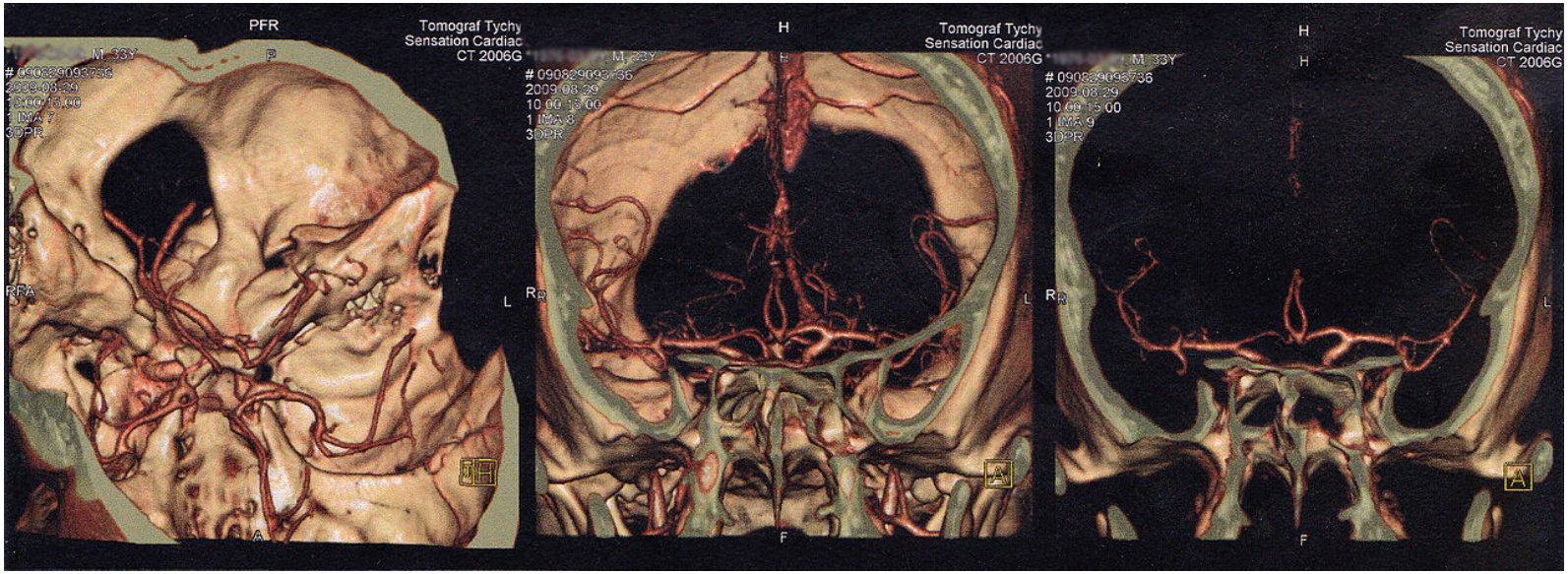
Introduction

Integral Geometry

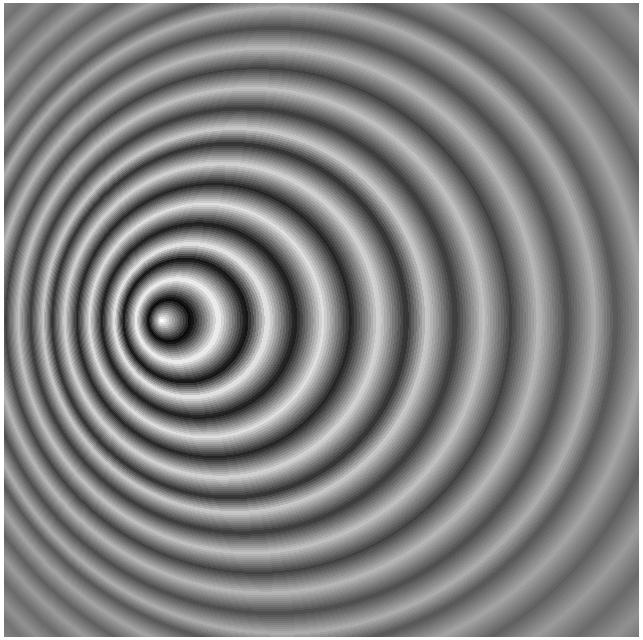
- Integral transform: $f(x) \mapsto F(L) = \int_L f(x)\,\sigma$
- Johann Radon: Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten, Ber. Verh. Säche. Akad. Wiss. (1917)
- Paul Funk: Über Flächen mit lauter geschlossenen geodätischen Linien, Mathematische Annalen (1913)

X-ray tomography



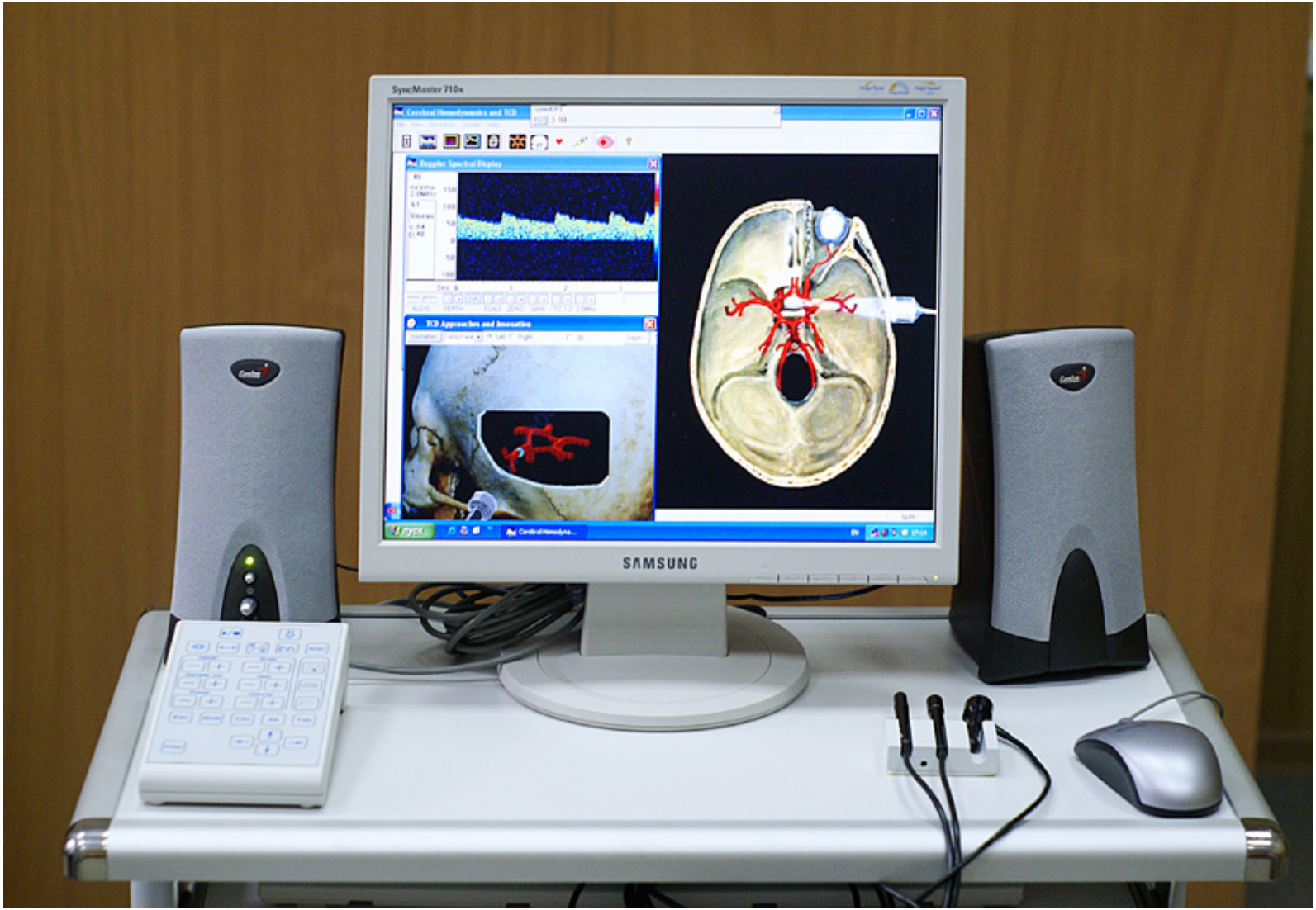
- $I = I_0 \exp(-\int_L \mu(x)\, dl)$
- X-ray transform: $\mu(x) \mapsto \int_L \mu(x)\, dl$

Doppler effect



- $\Delta\omega = \frac{2cu\omega}{c^2-u^2} \approx \frac{2u\omega}{c}$
- Vector x-ray tranform: $u(x) \mapsto \int_L u'_L(x)\, dl$
  - u'_L — projection of the field u to the direction L

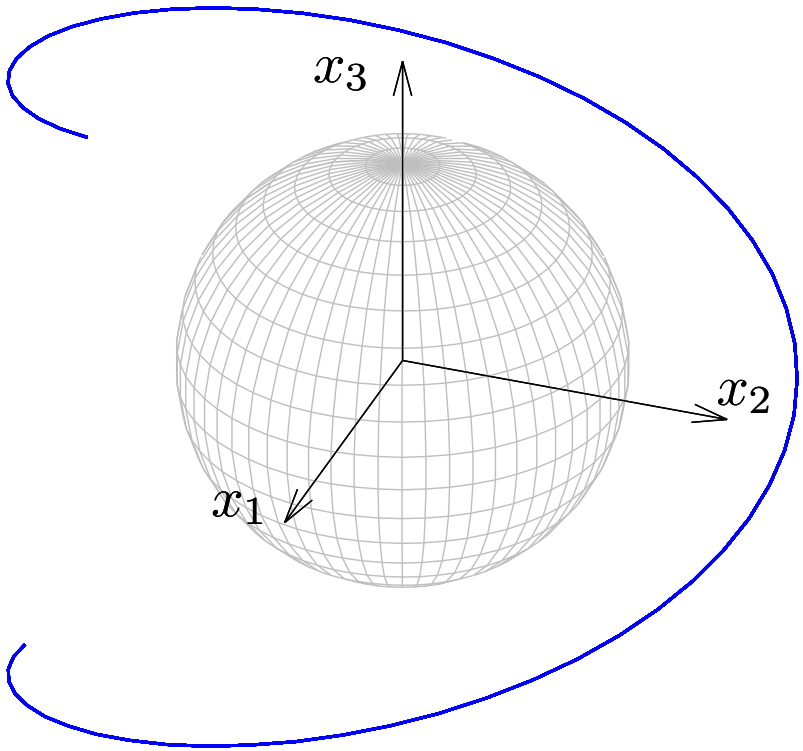
Doppler ultrasonography



Cone-beam 3D tomography

- X-ray $\mu(x) \mapsto \int_0^\infty \mu(x_0 + \xi t)\, dt$
- Doppler (vector) $u(x) \mapsto \int_0^\infty u(x_0 + \xi t) \cdot \xi\, dt$
- Overdetermined problem
  - ✦ Sources on a curve $\mathbb{R}^3$

Completeness condition (X-ray)



- Every plane intersecting the object should intersect the curve

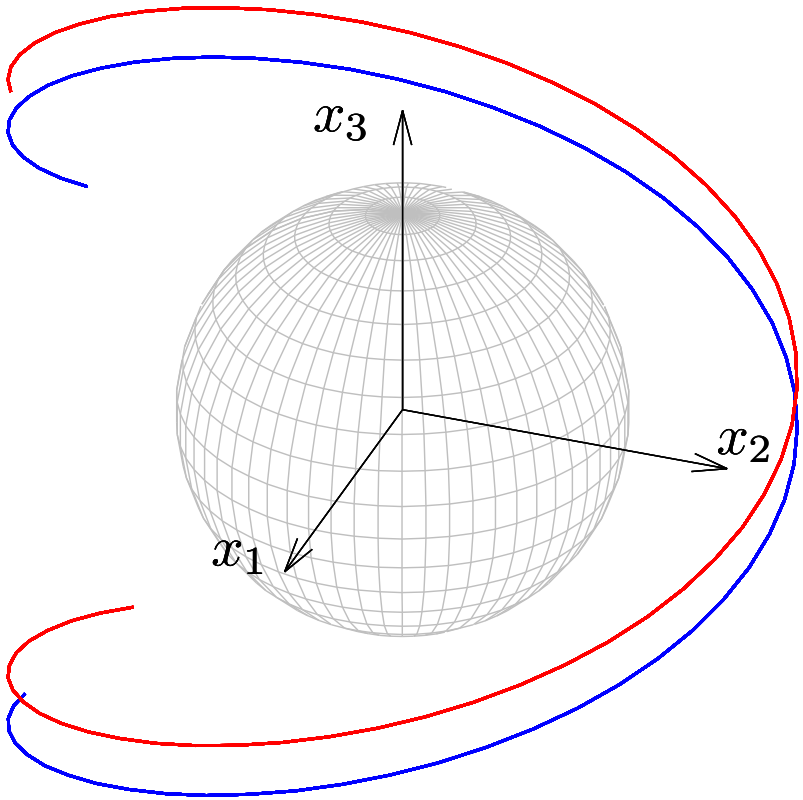
Previous works

- Kirillov A A: On a problem of I. M. Gelfand, Sov. Math. Dokl. (1961)
- Tuy H K: An Inversion Formula for Cone-Beam Reconstruction, SIAM J. Appl. Math (1983)
- Vertgeim L: Integral geometry problems for symmetric tensor fields with incomplete data, J. Inv. Ill-posed Problems (2000)
- Palamodov V P: Reconstructive Integral Geometry (2004)
- A D: Inversion of the x-ray transform for 3D symmetric tensor fields with sources on a curve, Inverse Problems (2006)

Recent works

- Schuster T, Theis D and Louis A K: A Reconstruction Approach for Imaging in 3D Cone Beam Vector Field Tomography, International Journal of Biomedical Imaging (2008)
- Katsevich A and Schuster T: An exact inversion formula for cone beam vector tomography, Inverse Problems (2013)
- Katsevich A, Rothermel D and Schuster T: An improved exact inversion formula for solenoidal fields in cone beam vector tomography, Inverse Problems (2017)

Completeness condition



- Every plane intersecting the object should intersect the curve twice

Mathematical model

The X-Ray transform

- Vector field $(f^1(x), f^2(x), f^3(x))$, $x \in \mathbb{R}^3$
- Ray in $\mathbb{R}^3$: $x_0 + \boldsymbol{\xi} t, \quad t > 0,\ x_0, \boldsymbol{\xi} \in \mathbb{R}^3,\ \|\boldsymbol{\xi}\| = 1$
- The X-ray transform

  $$f(x) \mapsto \boldsymbol{X}\, f(x_0, \boldsymbol{\xi}) = \int_0^\infty \langle f(x + \boldsymbol{\xi} t), \boldsymbol{\xi} \rangle \, dt,$$

  - $\langle f, \boldsymbol{\xi} \rangle = \sum f_i \xi^i.$
- Differential form $f(x) = f_1(x)dx^1 + f_2(x)dx^2 + f_3(x)dx^3$
- To recover f from $\boldsymbol{X}\, f$
- Assumption: $\operatorname{supp} f \subset \mathbb{B} = \{\, \|x\| < 1 \,\}$

Two sources on a curve

- Two curves are $\gamma_1(\lambda_1), \gamma_2(\lambda_2) \in \mathbb{R}^3$
- Data:
  - $\boldsymbol{X}_1 f(\lambda_1, \boldsymbol{\xi}) = \int \langle f(\gamma_1(\lambda_1) + \boldsymbol{\xi} t), \boldsymbol{\xi} \rangle \, dt,$
  - $\boldsymbol{X}_2 f(\lambda_2, \boldsymbol{\eta}) = \int \langle f(\gamma_2(\lambda_2) + \boldsymbol{\eta} \tau), \boldsymbol{\eta} \rangle \, d\tau,$
- Assumption: $\gamma_1 \cap \operatorname{supp} f = \gamma_2 \cap \operatorname{supp} f = \emptyset$

Decomposition and kernel

- Every differential form f, $\operatorname{supp} f \subset \mathbb{B}$ can be uniquely decomposed into sum of solenoidal and potential parts:

$$f = {}^s f + dh,$$

where $h|_{\partial\mathbb{B}} = 0$ and $\delta\, {}^s f = 0$

- $\ker \boldsymbol{X} = \{\, dh : h|_{\partial\mathbb{B}} = 0 \,\}$

Problem statement

- Given two x-ray transforms

$$\boldsymbol{X}_1 f(\lambda_1, \boldsymbol{\xi}), \quad \boldsymbol{X}_2 f(\lambda_2, \boldsymbol{\eta}).$$

- To reconstruct ${}^s f$, the solenoidal part of f.

The Kirillov-Tuy Condition

- Every plane $H_{\omega,p} \cap \mathbb{B} \neq \emptyset$ intersects curves γ_1 and γ_2 at $\Gamma_1 = \gamma_1(\lambda_1(\omega, p))$ and, correspondingly, at $\Gamma_2 = \gamma_2(\lambda_2(\omega, p))$.

Analytical Inversion

Strategy

1. $X_1 f(\lambda_1, \boldsymbol{\xi}), X_2 f(\lambda_2, \boldsymbol{\eta}) \longmapsto \frac{\partial}{\partial p} \mathbf{R}[df](\omega, p)$
2. $\frac{\partial}{\partial p} \mathbf{R}[df](\omega, p) \longmapsto df$
3. $df \longmapsto {}^{s}f$

The Radon transform

- Plane $H_{\omega,p} = \{ \langle \omega, x \rangle = p \}$
- $\mathbf{R}[g](\omega, p) = \int_{H_{\omega,p}} g(x)\, ds$
- $g(x) = -\frac{1}{8\pi^2} \int_{\mathbb{S}^2} \frac{\partial^2 \mathbf{R}[g](\omega, \langle \omega, x \rangle)}{\partial p^2}\, d\omega$
- For any vector a:

  $\mathbf{R}[\sum a_i \frac{\partial}{\partial x_i} g](\omega, p) = \langle \omega, a \rangle\, \partial/\partial p\, \mathbf{R}[g](\omega, p)$

The Radon transform of df

- For any two vectors $\boldsymbol{u}, \boldsymbol{v}$:

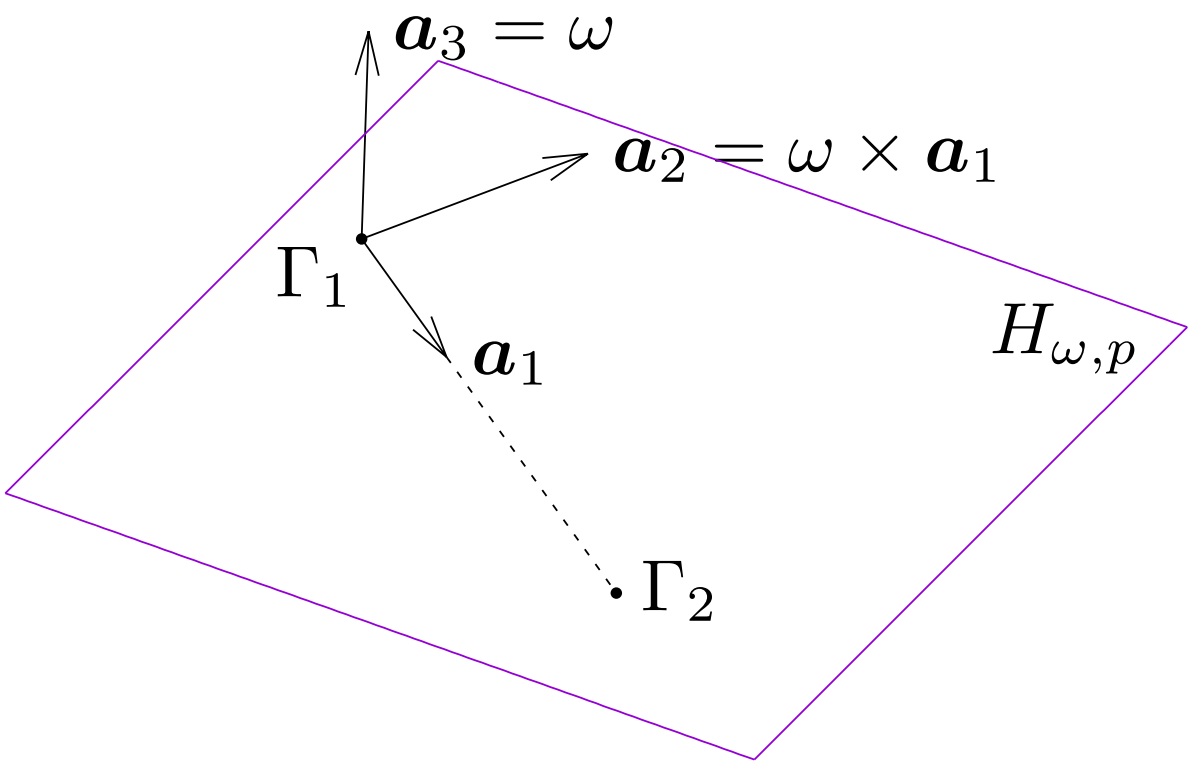
  $\langle df, (\boldsymbol{u}, \boldsymbol{v})\rangle = \sum u^i \frac{\partial}{\partial x^i} \langle f, \boldsymbol{v}\rangle - \sum v^i \frac{\partial}{\partial x^i} \langle f, \boldsymbol{u}\rangle$
- Hence: $\mathbf{R}[\langle df, (\boldsymbol{u}, \boldsymbol{v})\rangle](\omega, p) =$

  $\langle \omega, \boldsymbol{u}\rangle \, \partial/\partial p \, \mathbf{R}[\langle f, \boldsymbol{v}\rangle](\omega, p) - \langle \omega, \boldsymbol{v}\rangle \, \partial/\partial p \, \mathbf{R}[\langle f, \boldsymbol{u}\rangle](\omega, p)$
- Specifically: $\mathbf{R}[df_{ij}](\omega, p) = \mathbf{R}[\langle df, (\boldsymbol{e}_i, \boldsymbol{e}_j)\rangle](\omega, p) =$

  $\langle \omega, \boldsymbol{e}_i\rangle \, \partial/\partial p \, \mathbf{R}[\langle f, \boldsymbol{e}_j\rangle](\omega, p) - \langle \omega, \boldsymbol{e}_j\rangle \, \partial/\partial p \, \mathbf{R}[\langle f, \boldsymbol{e}_i\rangle](\omega, p),$

  where $i, j = 1, 2, 3$ and $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is the standard basis in $\mathbb{R}^3$

Change of basis



$$\boldsymbol{a}_1 = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \boldsymbol{a}_2 = \begin{pmatrix} \omega_2 v_3 - \omega_3 v_2 \\ -\omega_1 v_3 + \omega_3 v_1 \\ \omega_1 v_2 - \omega_2 v_1 \end{pmatrix}, \quad \boldsymbol{a}_3 = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

The Radon transform of df

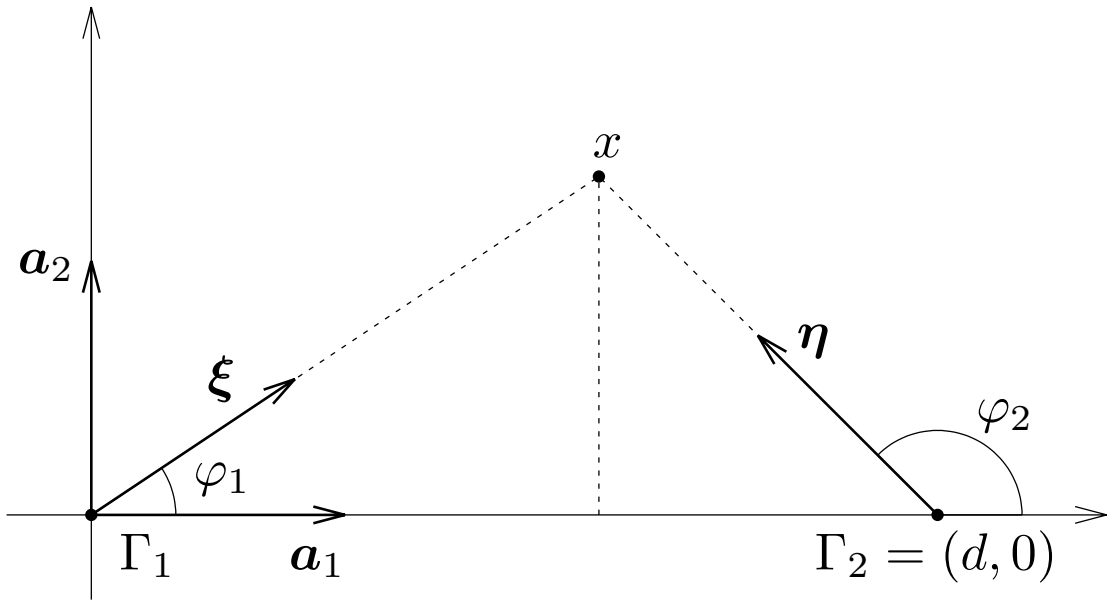
- $\mathbf{R}[df_{12}](\omega, p) =$

  $(\omega_1 v_2 - \omega_2 v_1)\frac{\partial}{\partial p}\mathbf{R}[\langle f, \boldsymbol{a}_1\rangle](\omega, p) - v_3\frac{\partial}{\partial p}\mathbf{R}[\langle f, \boldsymbol{a}_2\rangle](\omega, p)$
- $\mathbf{R}[df_{13}](\omega, p) =$

  $(\omega_1 v_3 - \omega_3 v_1)\frac{\partial}{\partial p}\mathbf{R}[\langle f, \boldsymbol{a}_1\rangle](\omega, p) + v_2\frac{\partial}{\partial p}\mathbf{R}[\langle f, \boldsymbol{a}_2\rangle](\omega, p)$
- $\mathbf{R}[df_{23}](\omega, p) =$

  $(\omega_2 v_3 - \omega_3 v_2)\frac{\partial}{\partial p}\mathbf{R}[\langle f, \boldsymbol{a}_1\rangle](\omega, p) - v_1\frac{\partial}{\partial p}\mathbf{R}[\langle f, \boldsymbol{a}_2\rangle](\omega, p)$

On the plane $H_{\omega,p}$



- $$\begin{cases} \boldsymbol{\xi} = \boldsymbol{a}_1 \cos\varphi_1 + \boldsymbol{a}_2 \sin\varphi_1 \\ \boldsymbol{\eta} = \boldsymbol{a}_1 \cos\varphi_2 + \boldsymbol{a}_2 \sin\varphi_2 \end{cases}$$

- $x = \boldsymbol{\xi} t = (d, 0) + \boldsymbol{\eta}\tau$
- $$\begin{cases} t\sin\varphi_1 = \tau\sin\varphi_2 \\ t\cos\varphi_1 - \tau\cos\varphi_2 = d \end{cases}$$

The Radon transform of df

- $\boldsymbol{a}_1 = \frac{t}{d}\boldsymbol{\xi} - \frac{\tau}{d}\boldsymbol{\eta}$
- $\boldsymbol{a}_2 = \left(\frac{1}{\sin\varphi_1} - \frac{t}{d}\operatorname{ctg}\varphi_1\right)\boldsymbol{\xi} + \left(\frac{1}{\sin\varphi_2} + \frac{\tau}{d}\operatorname{ctg}\varphi_2\right)\boldsymbol{\eta}$
- $\langle f, \boldsymbol{a}_1\rangle = \frac{t}{d}\langle f, \boldsymbol{\xi}\rangle - \frac{\tau}{d}\langle f, \boldsymbol{\eta}\rangle$
- $\langle f, \boldsymbol{a}_2\rangle = \frac{1}{\sin\varphi_1}\langle f, \boldsymbol{\xi}\rangle - \frac{t}{d}\operatorname{ctg}\varphi_1\langle f, \boldsymbol{\xi}\rangle + \frac{1}{\sin\varphi_2}\langle f, \boldsymbol{\eta}\rangle + \frac{\tau}{d}\operatorname{ctg}\varphi_2\langle f, \boldsymbol{\eta}\rangle$
- $\frac{\partial}{\partial p}\mathbf{R}[\langle f, \boldsymbol{a}_1\rangle] = \frac{\partial}{\partial p}\mathbf{R}[\frac{t}{d}\langle f, \boldsymbol{\xi}\rangle] - \frac{\partial}{\partial p}\mathbf{R}[\frac{\tau}{d}\langle f, \boldsymbol{\eta}\rangle]$
- $\frac{\partial}{\partial p}\mathbf{R}[\langle f, \boldsymbol{a}_2\rangle] = \frac{\partial}{\partial p}\mathbf{R}[\frac{1}{\sin\varphi_1}\langle f, \boldsymbol{\xi}\rangle] - \frac{\partial}{\partial p}\mathbf{R}[\frac{t}{d}\operatorname{ctg}\varphi_1\langle f, \boldsymbol{\xi}\rangle] + \frac{\partial}{\partial p}\mathbf{R}[\frac{1}{\sin\varphi_2}\langle f, \boldsymbol{\eta}\rangle] + \frac{\partial}{\partial p}\mathbf{R}[\frac{\tau}{d}\operatorname{ctg}\varphi_2\langle f, \boldsymbol{\eta}\rangle]$

Rays with origin at $\gamma_1\big(\lambda_1(\omega, p)\big)$

- $\boldsymbol{\xi} = \boldsymbol{a}_1 \cos\varphi_1 \cos\theta_1 + \boldsymbol{a}_2 \sin\varphi_1 \cos\theta_1 + \boldsymbol{a}_3 \sin\theta_1$
- $x = \gamma_1\left(\lambda_1(\omega, p)\right) + t \begin{pmatrix} \cos\varphi_1 \cos\theta_1 \\ \sin\varphi_1 \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}$
- t, φ_1, θ_1 are the spherical coordinates in $\mathbb{R}^3$
- For any function $g(\varphi_1)$:

$$\mathbf{R}\left[g(\varphi_1)\left\langle f, \boldsymbol{\xi}\right\rangle\right](\omega, p) = \int\limits_{H_{\omega,p}} g(\varphi_1)\left\langle f, \boldsymbol{\xi}\right\rangle \, ds$$

$$= \int\limits_0^{2\pi} \left(\int\limits_0^{\infty} t\left\langle f, \boldsymbol{\xi}\right\rangle \, dt \right) g(\varphi_1)\, d\varphi_1$$

Derivative of the Radon transform

- For a function $h(x)$, $x \in \mathbb{R}^3$,

  $$\frac{\partial}{\partial p}\,\mathbf{R}[h](\omega, p) = \mathbf{R}\left[\sum \omega_i \frac{\partial}{\partial x^i} h\right](\omega, p)$$

- In coordinates (y^1, y^2, y^3), associated with the basis $(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$: $\sum \omega_i \frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^3}$

- In the spherical coordinates (t, φ_1, θ_1):

  $$\frac{\partial}{\partial y^3} = \sin\theta_1 \frac{\partial}{\partial t} + \frac{1}{t}\cos\theta_1 \frac{\partial}{\partial \theta_1}$$

- Specifically, on the plane $H_{\omega,p}$: $\frac{\partial}{\partial y^3} = \frac{1}{t}\frac{\partial}{\partial \theta_1}$

- Hence $\frac{\partial}{\partial p}\,\mathbf{R}[h](\omega, p) = \int_0^{2\pi} \frac{\partial}{\partial \theta_1}\left(\int_0^\infty h(\gamma_1 + t\boldsymbol{\xi})\,dt\right)_{\theta_1=0} d\varphi_1$,

  $$\frac{\partial^2}{\partial p^2}\,\mathbf{R}[th](\omega, p) = \int_0^{2\pi} \frac{\partial^2}{\partial \theta_1^2}\left(\int_0^\infty h(\gamma_1 + t\boldsymbol{\xi})\,dt\right)_{\theta_1=0} d\varphi_1$$

The Radon tansform of df

$$\frac{\partial}{\partial p}\,\mathbf{R}[df_{12}](\omega,p) = \frac{\omega_1 v_2 - \omega_2 v_1}{d}\left(\boldsymbol{IX}_1 f(\omega,p) - \boldsymbol{IX}_2 f(\omega,p)\right)$$
$$+ \frac{v_3}{d}\left(\boldsymbol{I_cX}_1 f(\omega,p) - \boldsymbol{I_cX}_2 f(\omega,p)\right)$$
$$- v_3\frac{\partial}{\partial p}\left(\boldsymbol{I_sX}_1 f(\omega,p) + \boldsymbol{I_sX}_2 f(\omega,p)\right),$$
$$= F_{12}(\omega,p) + \frac{\partial}{\partial p}G_{12}(\omega,p),$$

where for $i = 1, 2$:

- $\boldsymbol{IX}_i = \int_0^{2\pi} \frac{\partial^2}{\partial\theta_i^2}\,\boldsymbol{X}_i f(0,\varphi_i)\,d\varphi_i,$
- $\boldsymbol{I_cX}_i = \int_0^{2\pi} \operatorname{ctg}\varphi_i \frac{\partial^2}{\partial\theta_i^2}\,\boldsymbol{X}_i f(0,\varphi_i)\,d\varphi_i,$
- $\boldsymbol{I_sX}_i = \int_0^{2\pi} \frac{1}{\sin\varphi_i}\frac{\partial}{\partial\theta_i}\,\boldsymbol{X}_i f(0,\varphi_i)\,d\varphi_i,$

Reconstruction of df

$$df_{\alpha\beta}(x) = -\frac{1}{8\pi^2} \int_{\mathbb{S}^2} \frac{\partial}{\partial p} \left(\frac{\partial}{\partial p} \mathbf{R}[df_{\alpha\beta}] \right) (\omega, \langle \omega, x \rangle) \, d\omega$$
$$= R^* \left(\frac{\partial}{\partial p} F_{\alpha\beta}(\omega, \langle \omega, x \rangle) + \frac{\partial^2}{\partial p^2} G_{\alpha\beta}(\omega, \langle \omega, x \rangle) \right).$$

- $1 \leqslant \alpha < \beta \leqslant 3$

Reconstruction of ${}^s f$

$$\begin{cases} d\,{}^s f(x) = u(x), & \text{for } x \in \mathbb{B}, \\ \delta\,{}^s f(x) = 0, & \text{for } x \in \mathbb{B}, \\ \boldsymbol{X}\ {}^s f(x, \boldsymbol{\xi}) = g(x, \boldsymbol{\xi}) & \text{for } x \in \gamma = \gamma_1 \cup \gamma_2, \quad \boldsymbol{\xi} \in \mathbb{S}^2, \end{cases}$$

- $u(x)$ is the reconstructed df
- $g(x, \boldsymbol{\xi})$ are the measured data
- There is a unique solution
  - ✦ new kind of boundary conditions

Discretization

Integration over sphere

$$\int_{\mathbb{S}^2} g(\omega)\, d\omega \approx \frac{2\pi^2}{(m+1)^2} \sum_{j=0}^{m} \sin\psi_j \sum_{i=0}^{m} g(\omega_{ij}),$$

- $\omega_{ij} = (\sin\upsilon_i \sin\psi_j, \cos\upsilon_i \sin\psi_j, \cos\psi_i)$
- $\psi_j = \pi\frac{2j+1}{2(m+1)}$
- $\upsilon_i = \pi\frac{i}{m+1}$
- $i, j = 0, \ldots, m.$

The data

- Let $h = 1/q$ be a discretization parameter, $s_l = hl$, $l = -q, \dots, q$
- For ω_{ij}, $s_l = hl$, where $i, j = 0, \dots, m$, $l = -q, \dots, q$

  1. Calculate Γ_1, Γ_2, and the basis $(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$
  2. Collect data: $\boldsymbol{X}_\alpha f(0, \varphi_k)$, $\boldsymbol{X}_\alpha f(h_\theta, \varphi_k)$, $\boldsymbol{X}_\alpha f(-h_\theta, \varphi_k)$ for rays intersecting the ball $\mathbb{B}$

     - $\varphi_k = kh_\varphi$, $k = 0, \dots, K-1$, $h_\varphi = 2\pi/K$
     - $i = 1, 2$
     - $h_\theta = h/\pi$
     - K is chosen so that $h_\varphi = 2\pi/K \approx h/\pi$

$\frac{\partial}{\partial\theta}\,\mathbf{X}\,f$

- $D^1_{i,k} = \frac{1}{2h_\theta}\left(\boldsymbol{X}_i\, f(h_\theta, \varphi_k) - \boldsymbol{X}_i\, f(-h_\theta, \varphi_k)\right)$
- $D^2_{i,k} = \frac{1}{h^2_\theta}\left(\boldsymbol{X}_i\, f(h_\theta, \varphi_k) + \boldsymbol{X}_i\, f(-h_\theta, \varphi_k) - \boldsymbol{X}_i\, f(0, \varphi_k)\right)$

Operators I

- The trapezoids rule:
  - $I_{jl} = h_\varphi \sum_{k=0}^{K-1} \left(D^2_{1,k} - D^2_{2,k} \right)$
  - $Ic_{jl} = h_\varphi \sum_{k=0}^{K-1} \left(D^2_{1,k} - D^2_{2,k} \right) \operatorname{ctg} \varphi_k$
  - $Is_{jl} = h_\varphi \sum_{k=0}^{K-1} \left(D^1_{1,k} - D^1_{2,k} \right) \frac{1}{\sin \varphi_k}$

Parts F and G

- $F_{12,jl} = \frac{\omega_1 v_2 - \omega_2 v_1}{d} I_{\alpha\beta} + \frac{v_3}{d} Ic_{jl}$
- $G_{12,jl} = -v_3 \, Is_{jl}$
- $F_{13,jl} = \frac{\omega_1 v_3 - \omega_3 v_1}{d} I_{\alpha\beta} - \frac{v_2}{d} Ic_{jl}$
- $G_{13,jl} = v_2 \, Is_{jl}$
- $F_{23,jl} = \frac{\omega_2 v_3 - \omega_3 v_2}{d} I_{\alpha\beta} + \frac{v_1}{d} Ic_{jl}$
- $G_{23,jl} = -v_1 \, Is_{jl}$

Derivatives

$$\mathcal{V}_{\alpha\beta,ijl} = \frac{1}{8\pi^2}\left(\frac{F_{\alpha\beta,ij(l-1)} - F_{\alpha\beta,ij(l+1)}}{2h} + \frac{2G_{\alpha\beta,ijl} - G_{\alpha\beta,ij(l-1)} - G_{\alpha\beta,ij(l+1)}}{h^2}\right),$$

- where $1 \leqslant \alpha < \beta \leqslant 3$, $i, j = 0, \ldots, m$, $l = -q, \ldots, q$

Backprojection

$$df_{\alpha\beta}(x) = \frac{2\pi^2}{(m+1)^2} \sum_{j=0}^{m} \sin\psi_j \sum_{i=0}^{m} \Big((1-u)\mathcal{V}_{\alpha\beta,ij(k-1)} + u\mathcal{V}_{\alpha\beta,ijk} \Big),$$

- $s_{k-1} \leqslant \langle \omega_j, x \rangle < s_k,$
- $u = \frac{\langle \omega_j, x \rangle - s_{k-1}}{h}.$

Filtering

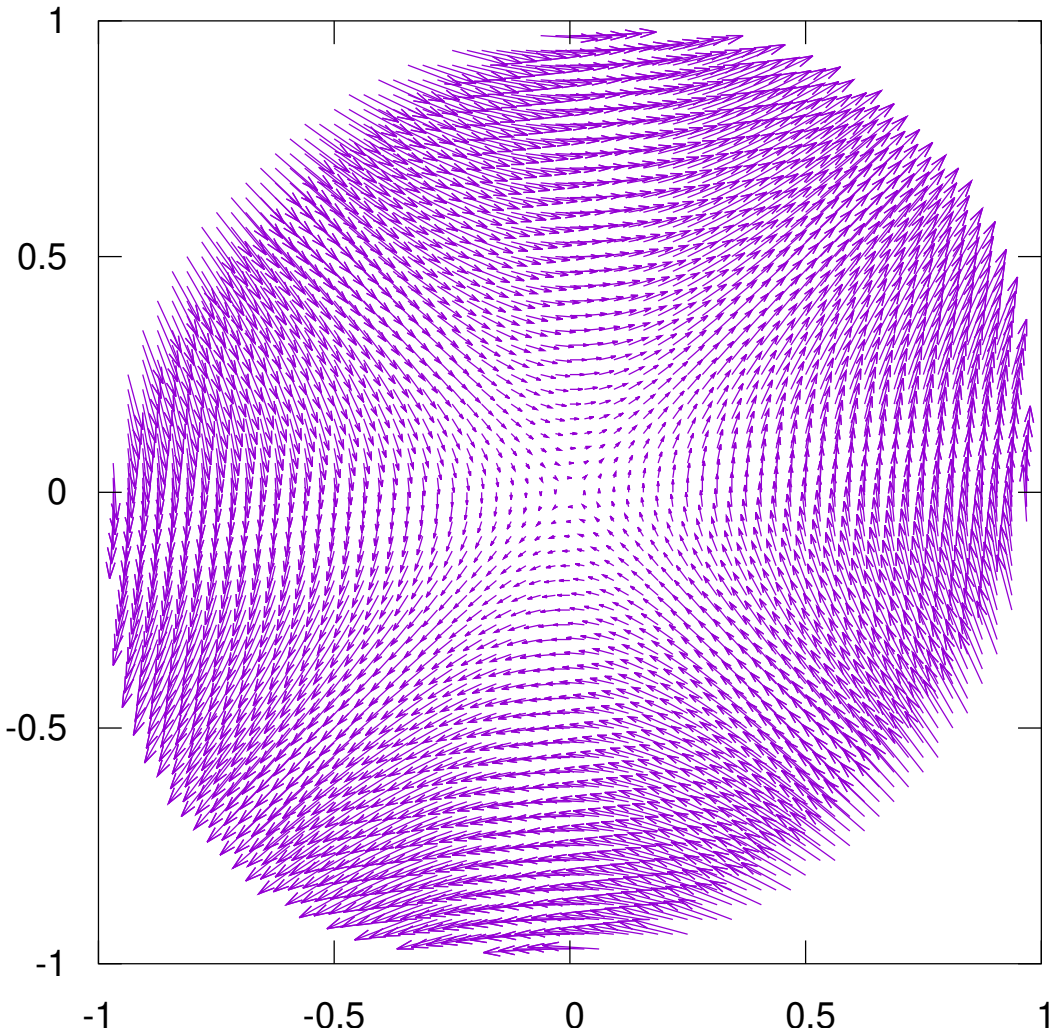
$$\frac{1}{(\sqrt{2\pi}\sigma)^3} e^{-\frac{x_1^2+x_2^x+x_3^2}{2\sigma^2}}$$

$$\begin{cases} f_{2,(i+1)jk} - f_{2,(i-1)jk} - f_{1,i(j+1)k} + f_{1,i(j-1)k} = 2hu_{12,ijk}, \\ f_{3,i(j+1)k} - f_{3,i(j-1)k} - f_{2,ij(k+1)} + f_{2,ij(k-1)} = 2hu_{23,ijk}, \\ f_{3,(i+1)jk} - f_{3,(i-1)jk} - f_{1,ij(k+1)} + f_{1,ij(k-1)} = 2hu_{13,ijk}, \\ f_{1,(i+1)jk} - f_{1,(i-1)jk} + f_{2,i(j+1)k} - f_{2,i(j-1)k} \\ \qquad + f_{3,ij(k+1)} - f_{3,ij(k-1)} = 0, \\ \sum\limits_{\alpha,i,j,k} a_{\mu,\alpha,ijk} f_{\alpha,ijk} = Xf(\Gamma_\mu, \boldsymbol{\xi}_\mu) \end{cases} \tag{1}$$

- The Kaczmarz method

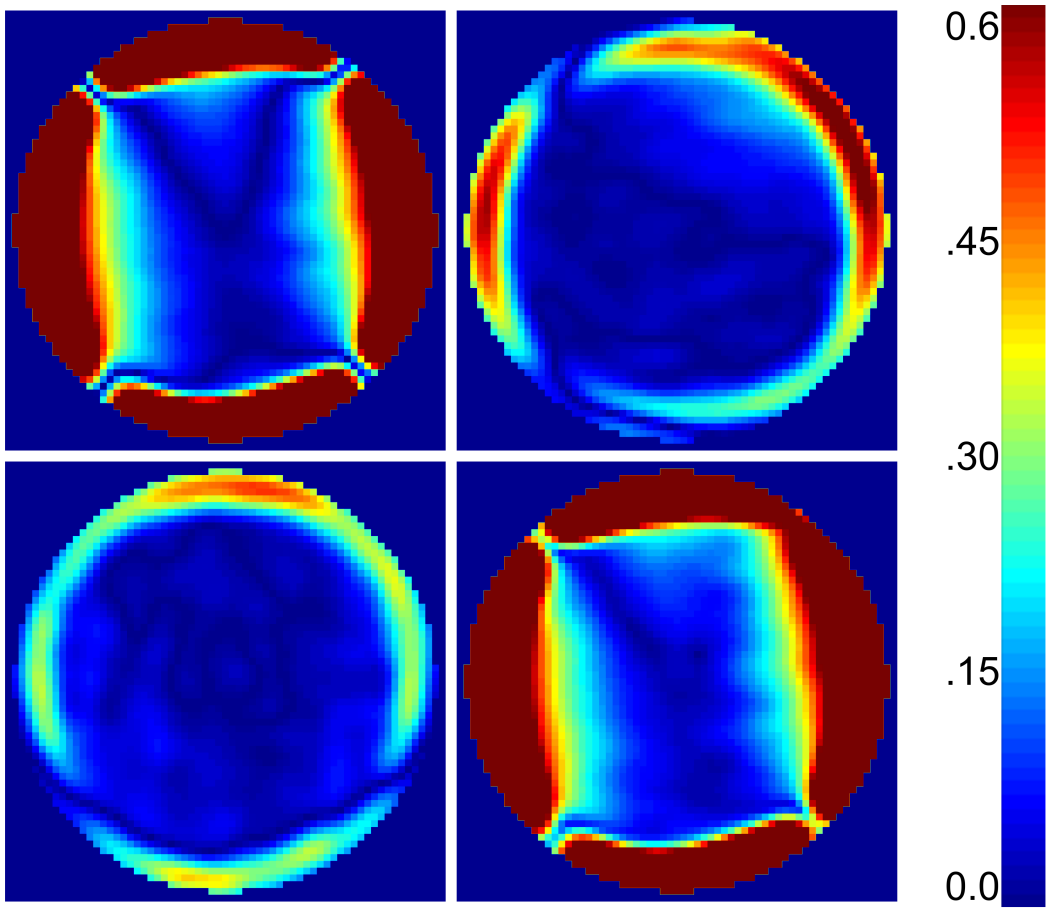
Numerical Experiments

Numerical Experiments



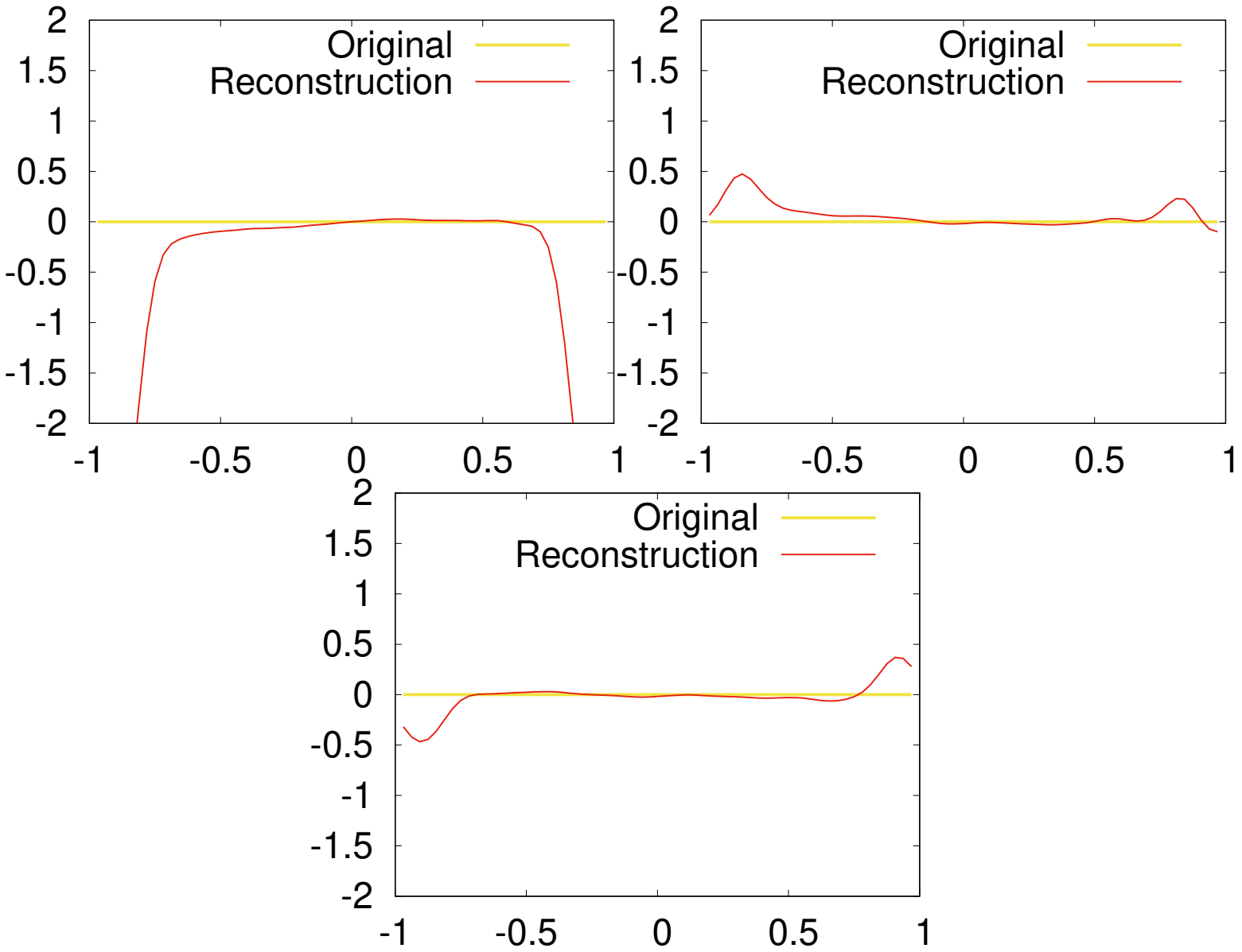
- $f_a = x^2 dx^1 + x^1 dx^2 \ (z = 0.125)$

Error of reconstruction of df at $z = 0.125$



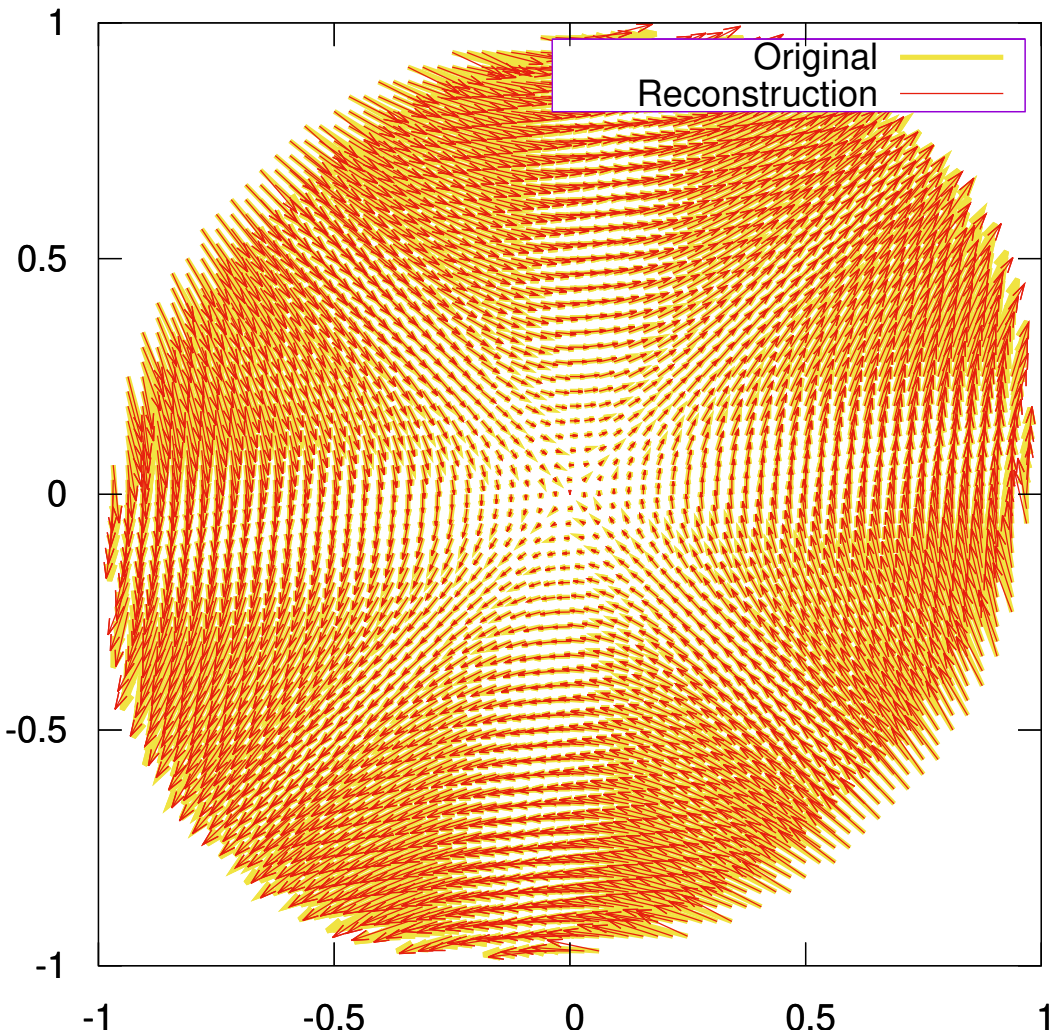
- df_{12}, df_{13}, df_{23}, $\|df - df_{\text{original}}\|$

Reconstruction of df at $z = 0.125$, $y = 0$

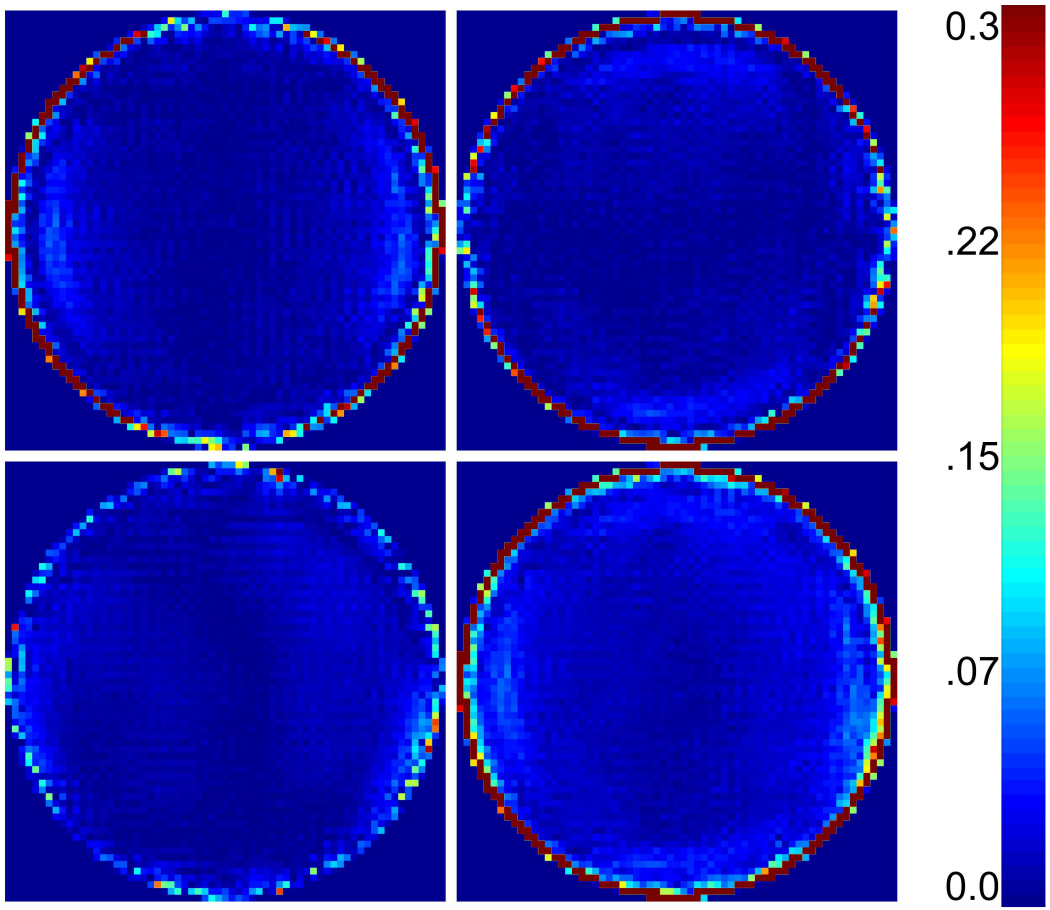


- df_{12}, df_{13}, df_{23}

Reconstruction of f at $z = 0.125$

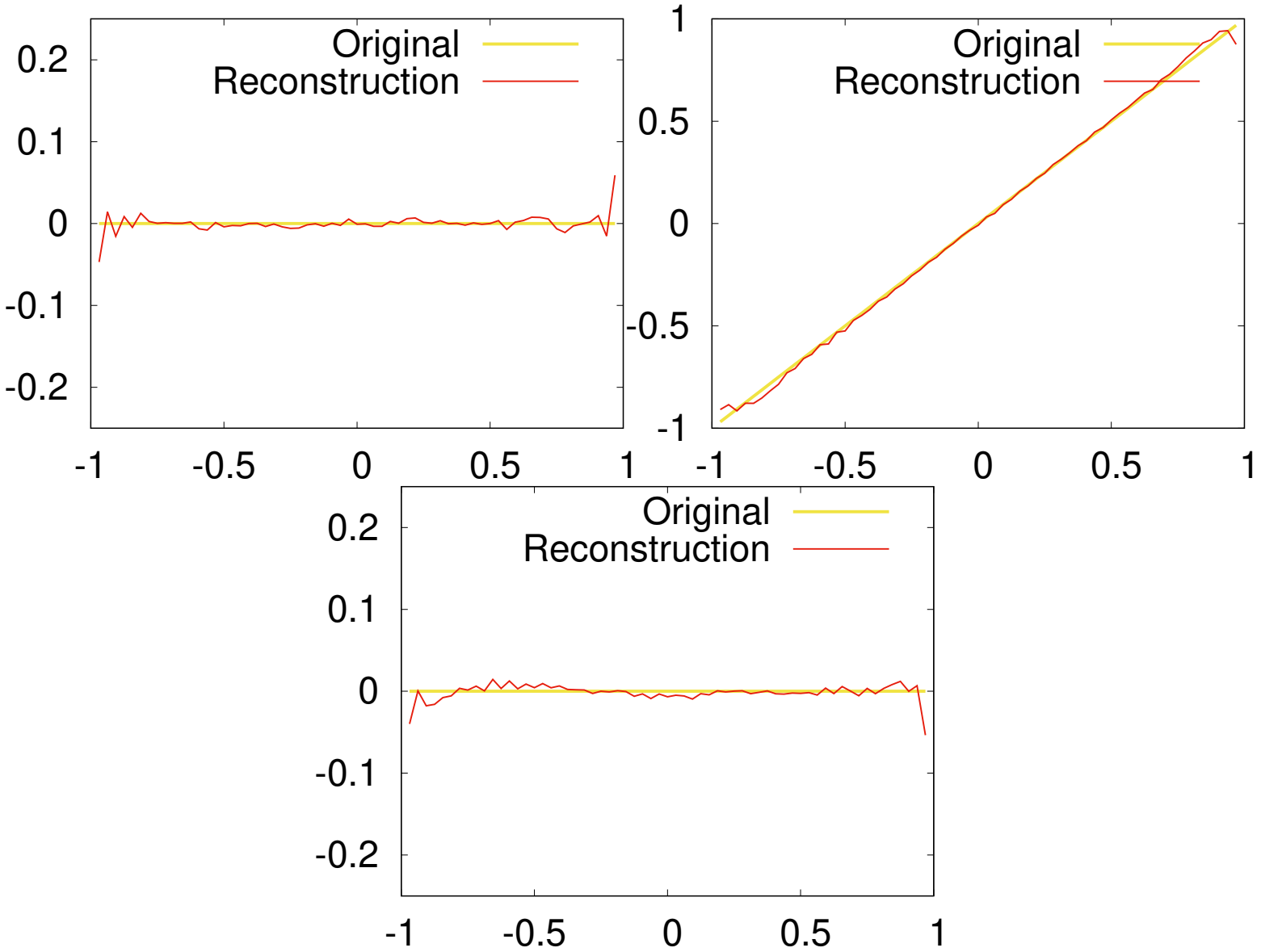


Error of reconstruction of f at $z = 0.125$



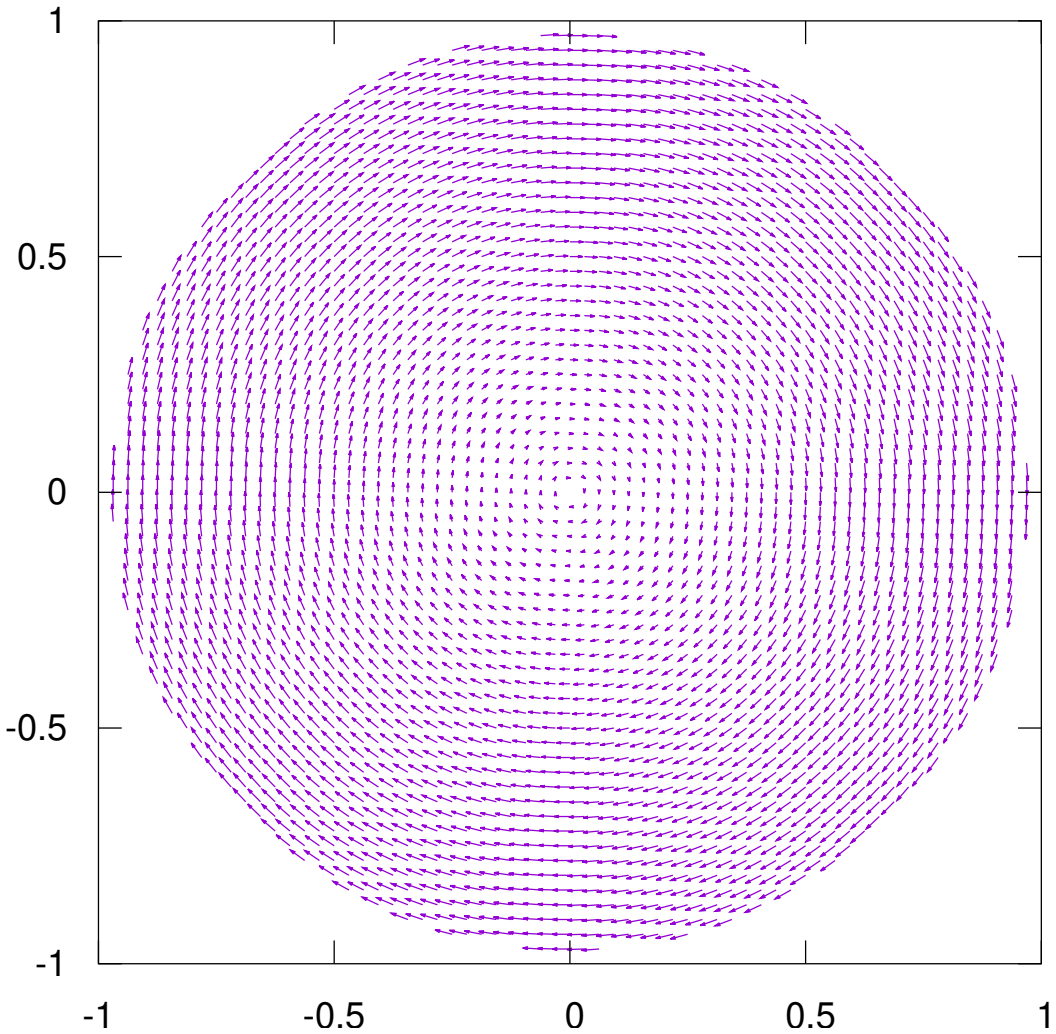
- $f_1,\ f_2,\ f_3,\ \|f - f_{\text{original}}\|$

Reconstruction of f at $z = 0.125$, $y = 0$



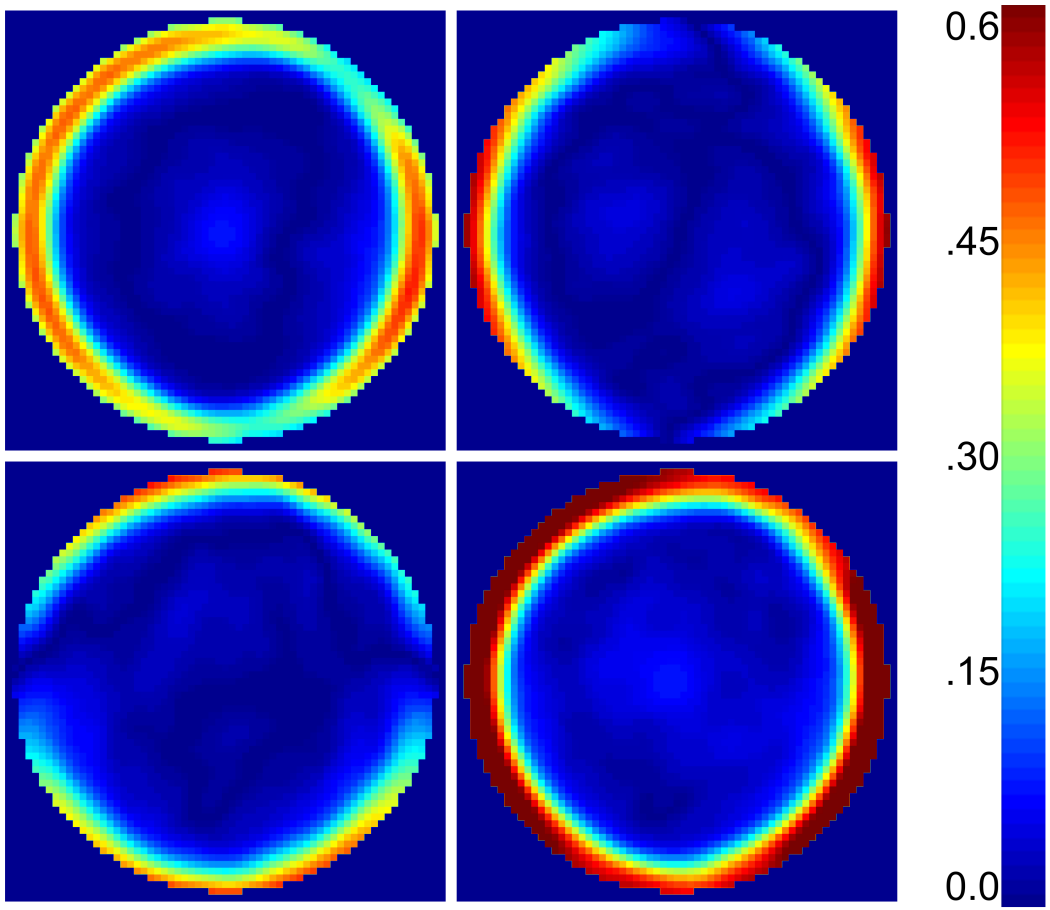
- f_1, f_2, f_3

Numerical Experiments



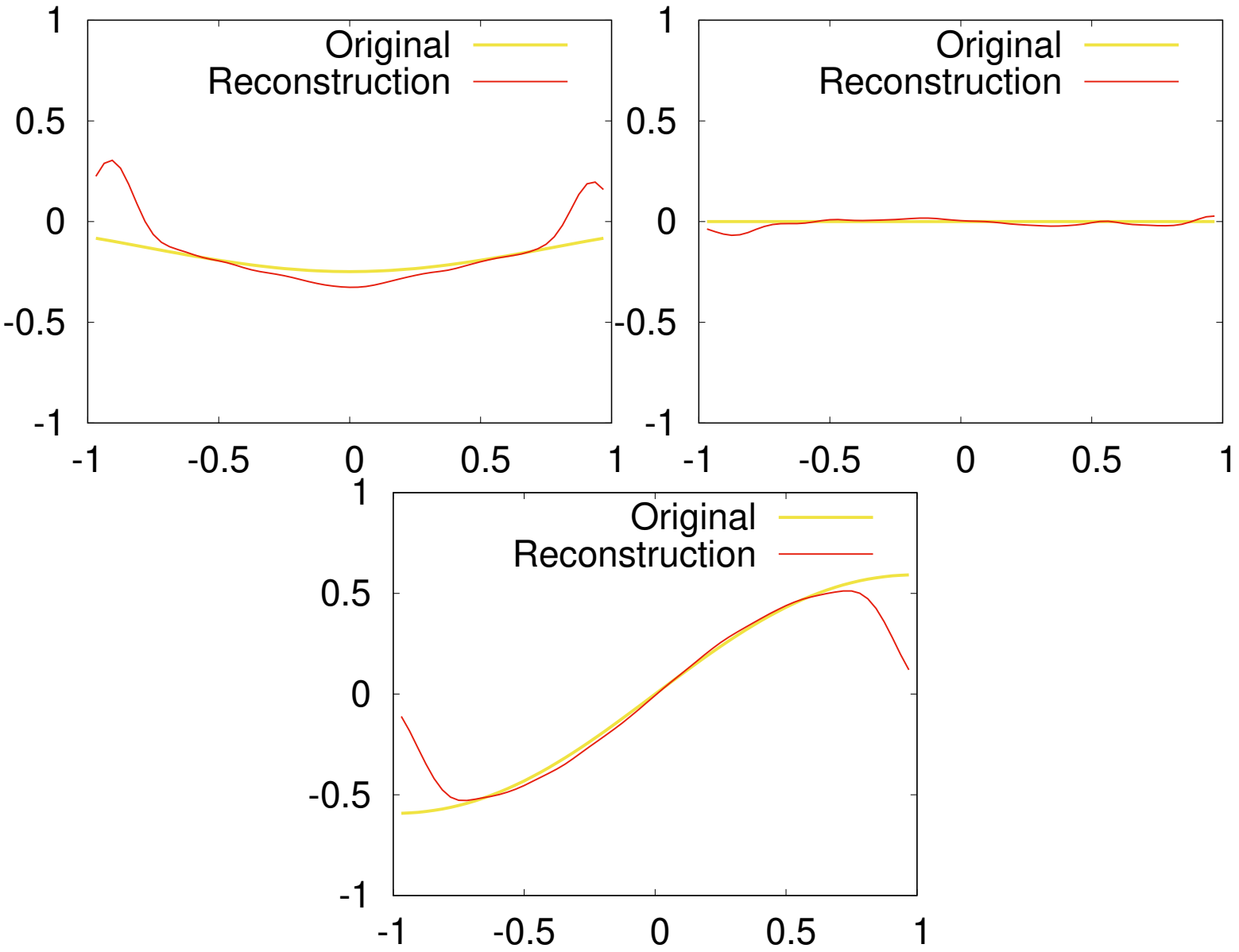
- $f_b = *d(\frac{1}{2}e^{-\frac{|x|^2}{2}}(x^1dx^1 + x^2dx^2 - x^3dx^3))\ (z = 0.125)$

Error of reconstruction of df at $z = 0.125$



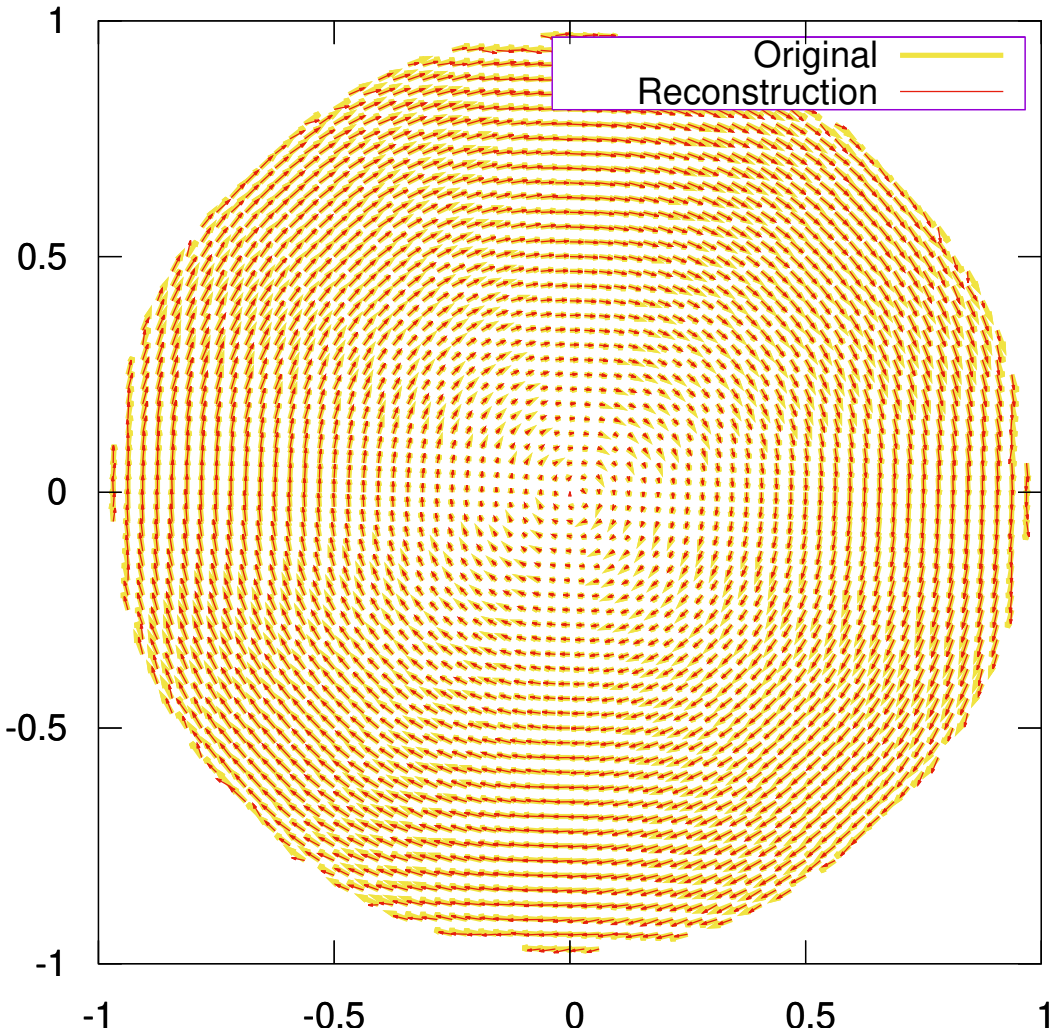
- df_{12}, df_{13}, df_{23}, $\|df - df_{\text{original}}\|$

Reconstruction of df at $z = 0.125$, $y = 0$

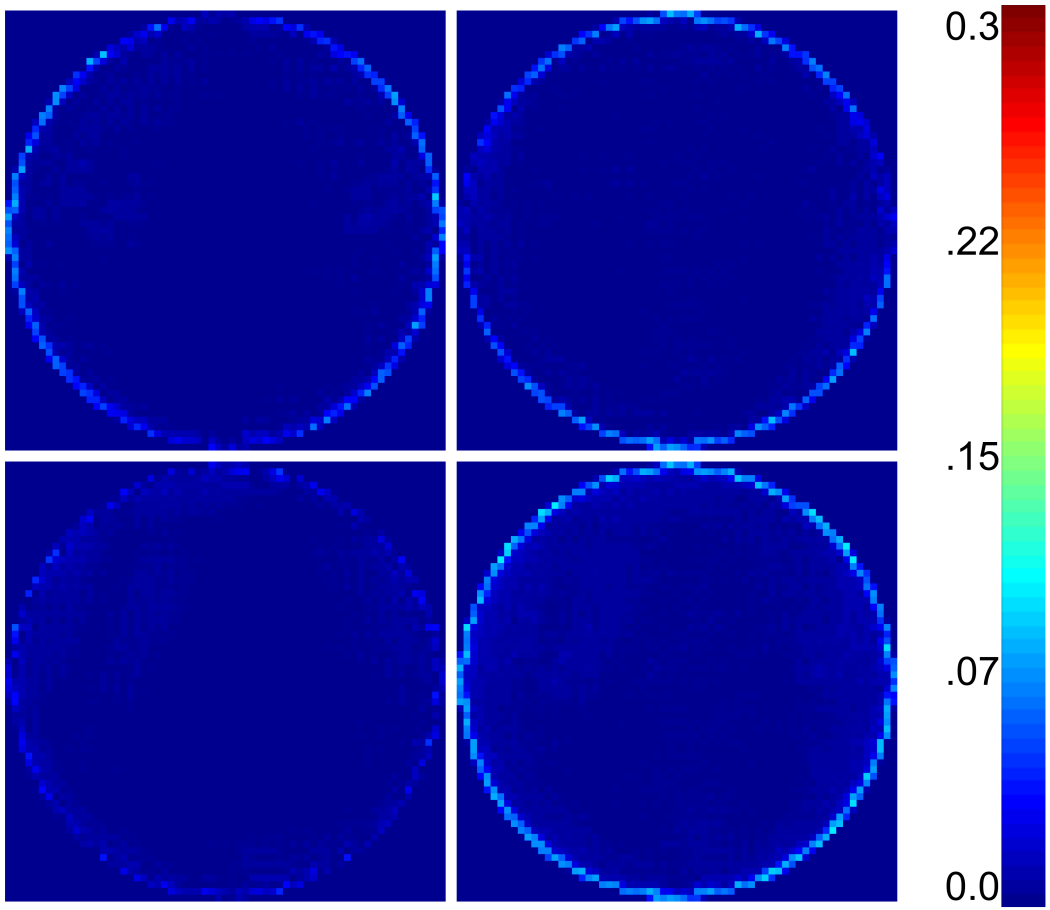


- df_{12}, df_{13}, df_{23}

Reconstruction of f at $z = 0.125$

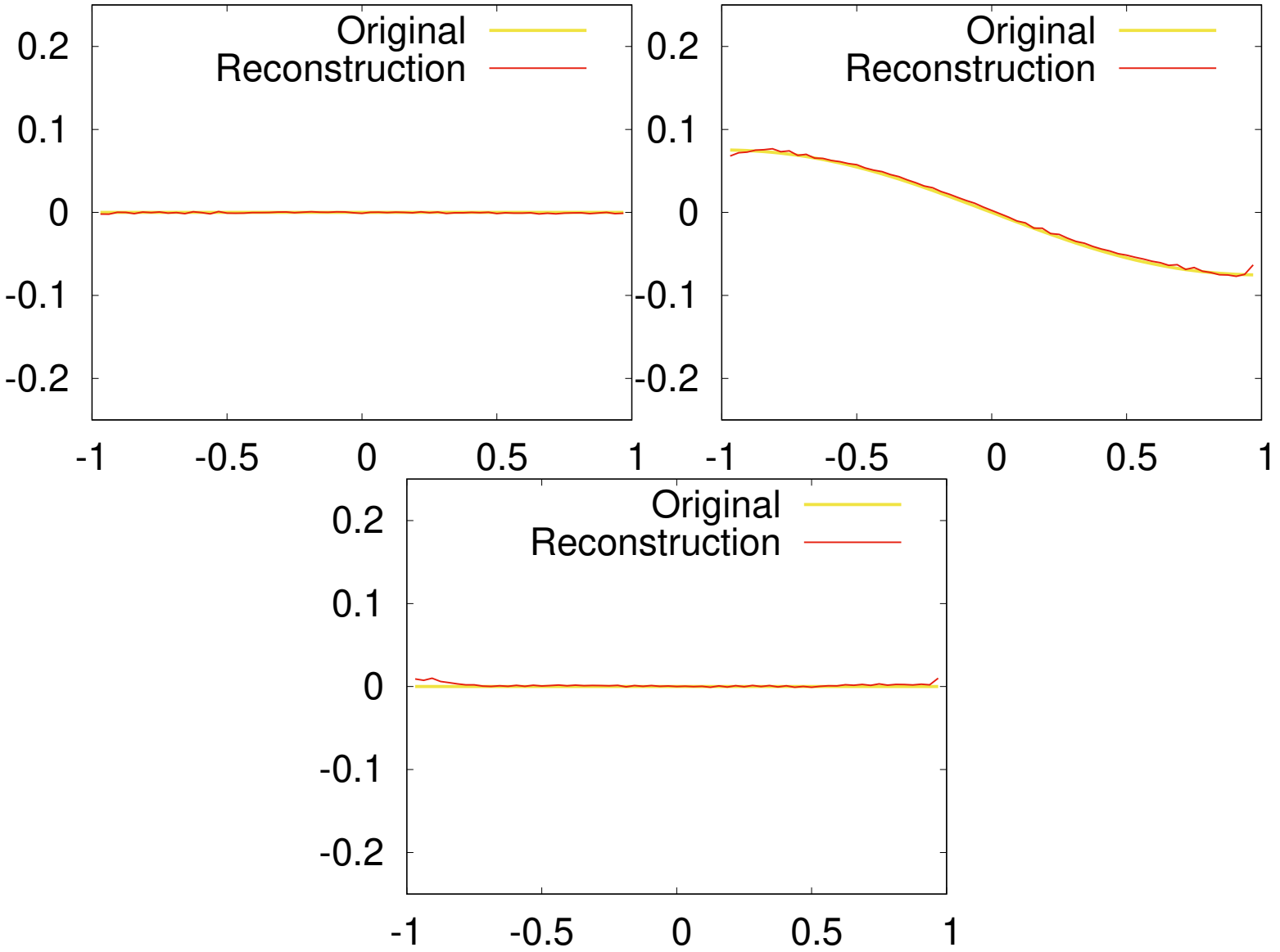


Error of reconstruction of f at $z = 0.125$

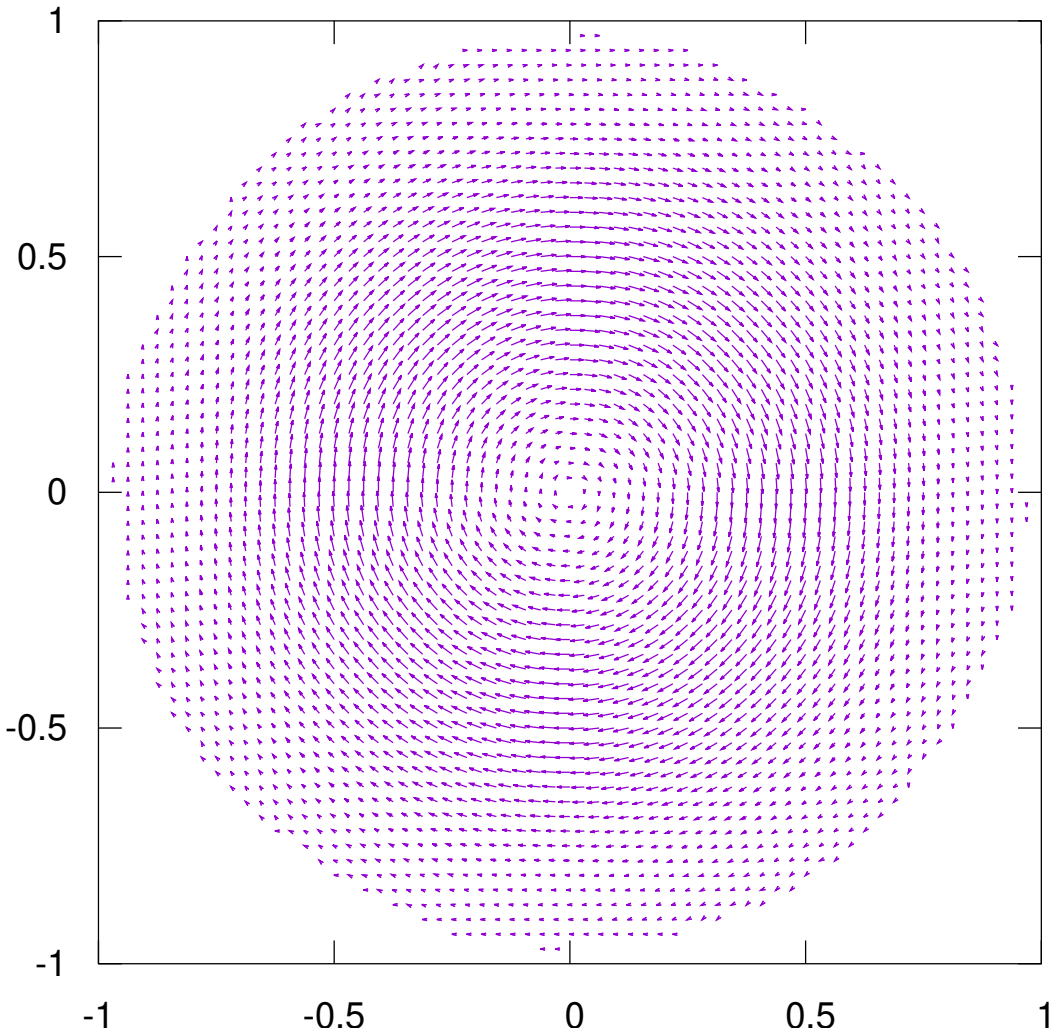


- $f_1,\ f_2,\ f_3,\ \|f - f_{\text{original}}\|$

Reconstruction of f at $z = 0.125$, $y = 0$

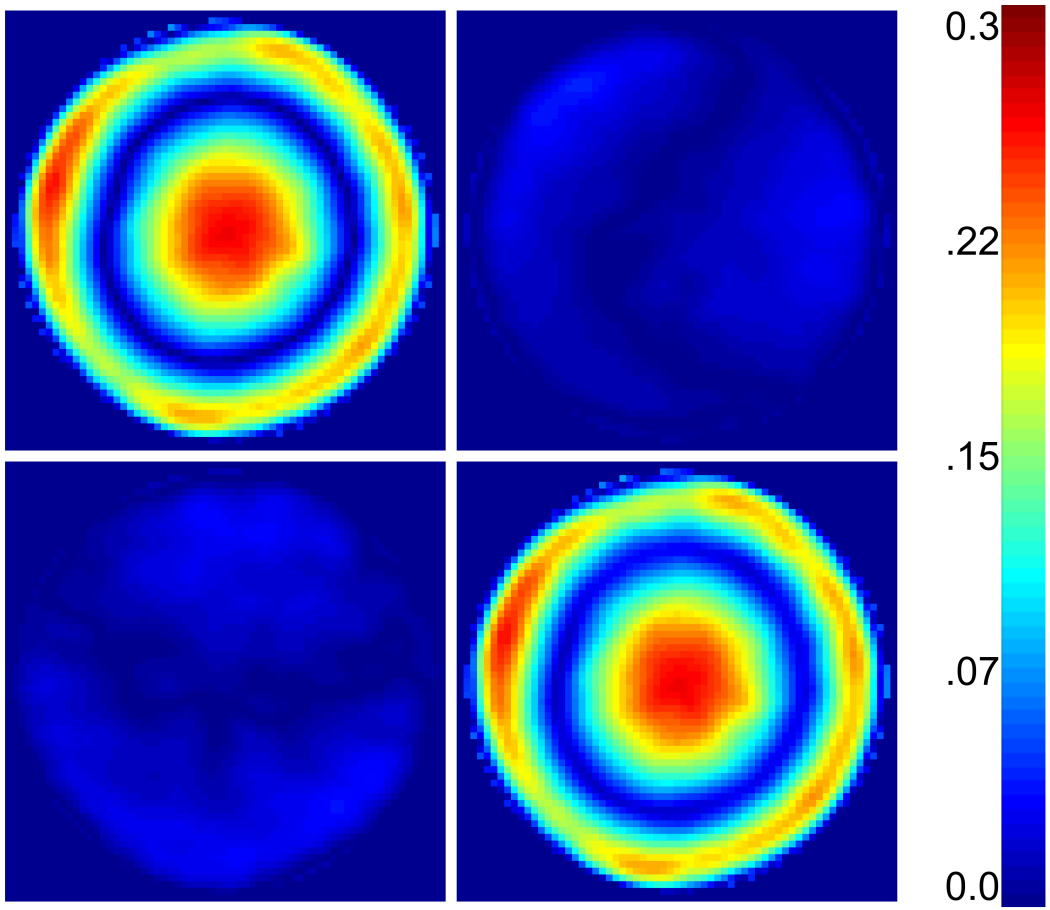


Numerical Experiments



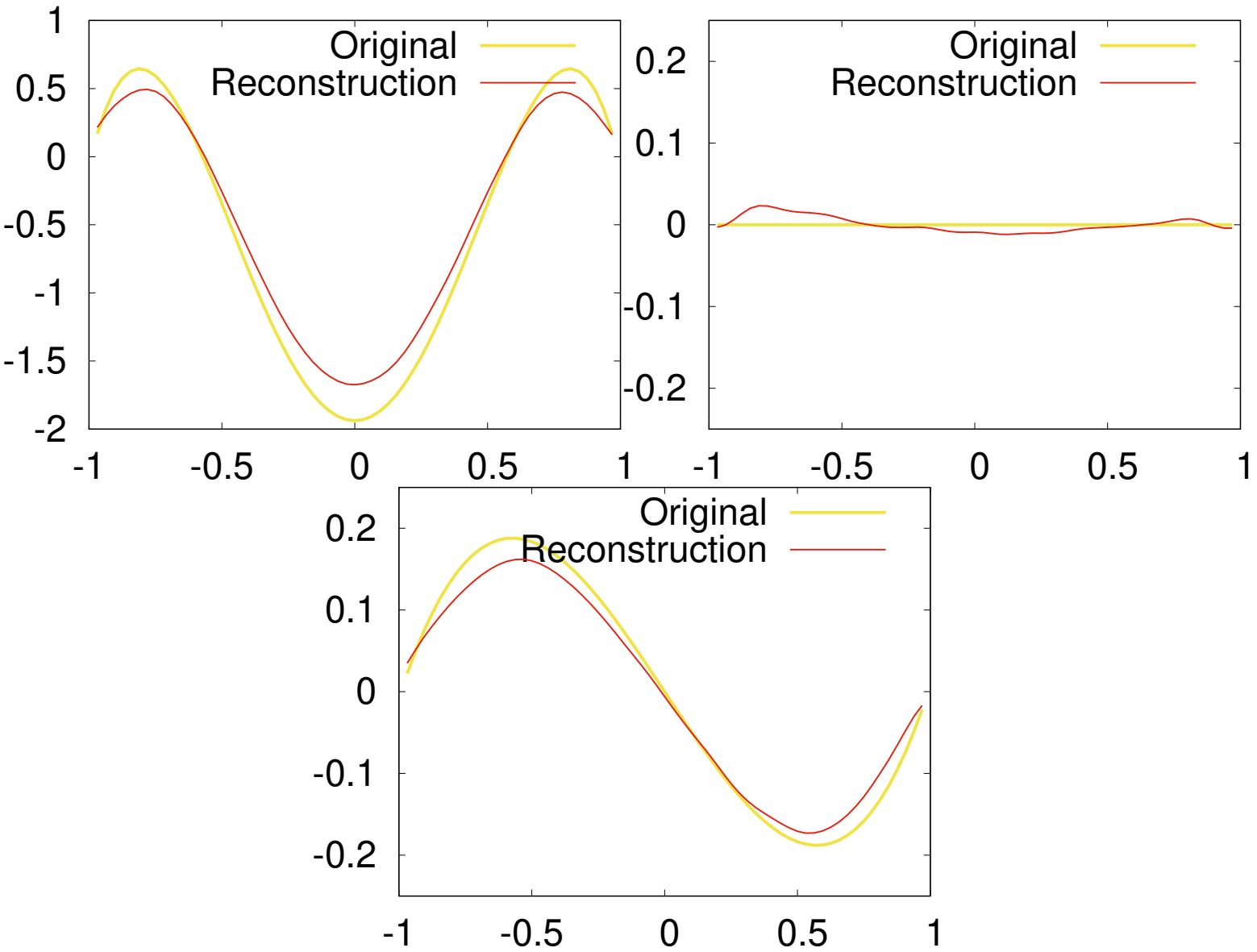
- $f_c = x^2(1-|x|^2)^2 dx^1 - x^1(1-|x|^2)^2 dx^2 \ (z = 0.125)$

Error of reconstruction of df at $z = 0.125$



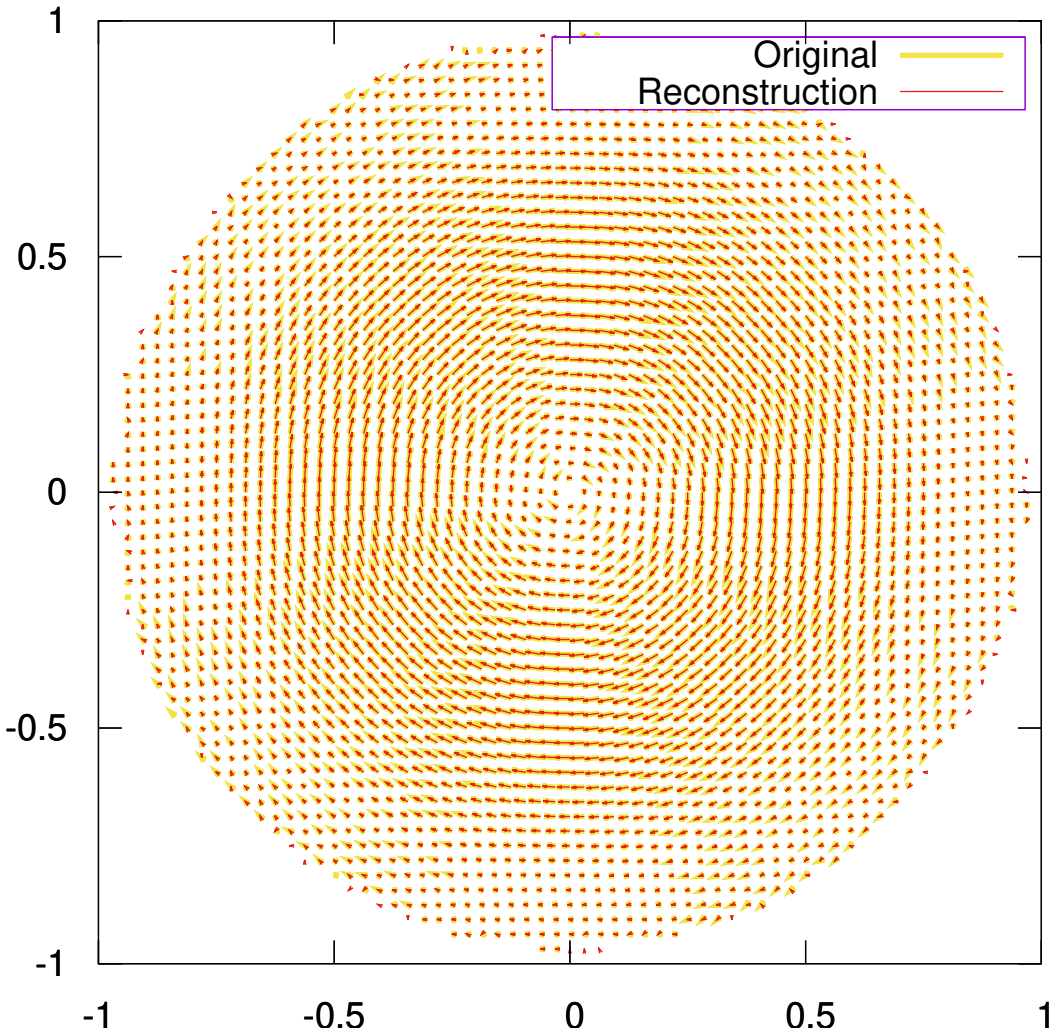
- df_{12}, df_{13}, df_{23}, $\|df - df_{\text{original}}\|$

Reconstruction of df at $z = 0.125$, $y = 0$

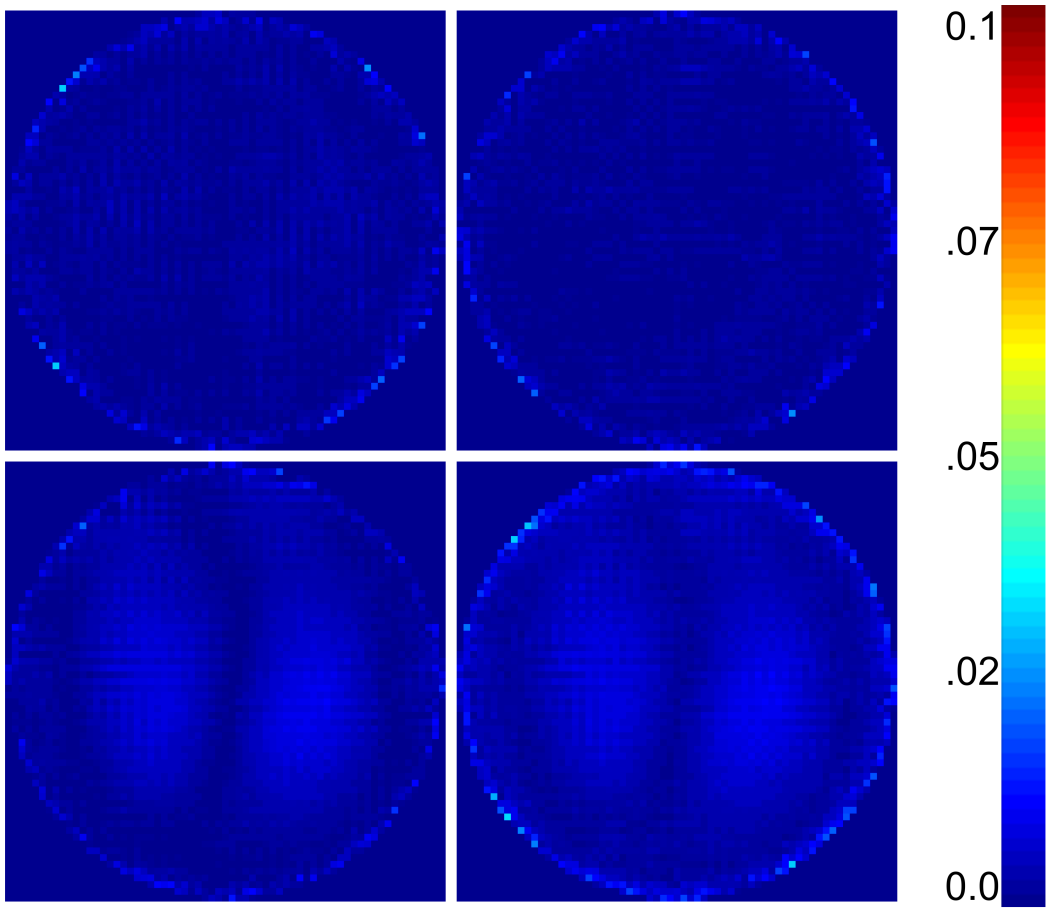


- df_{12}, df_{13}, df_{23}

Reconstruction of f at $z = 0.125$

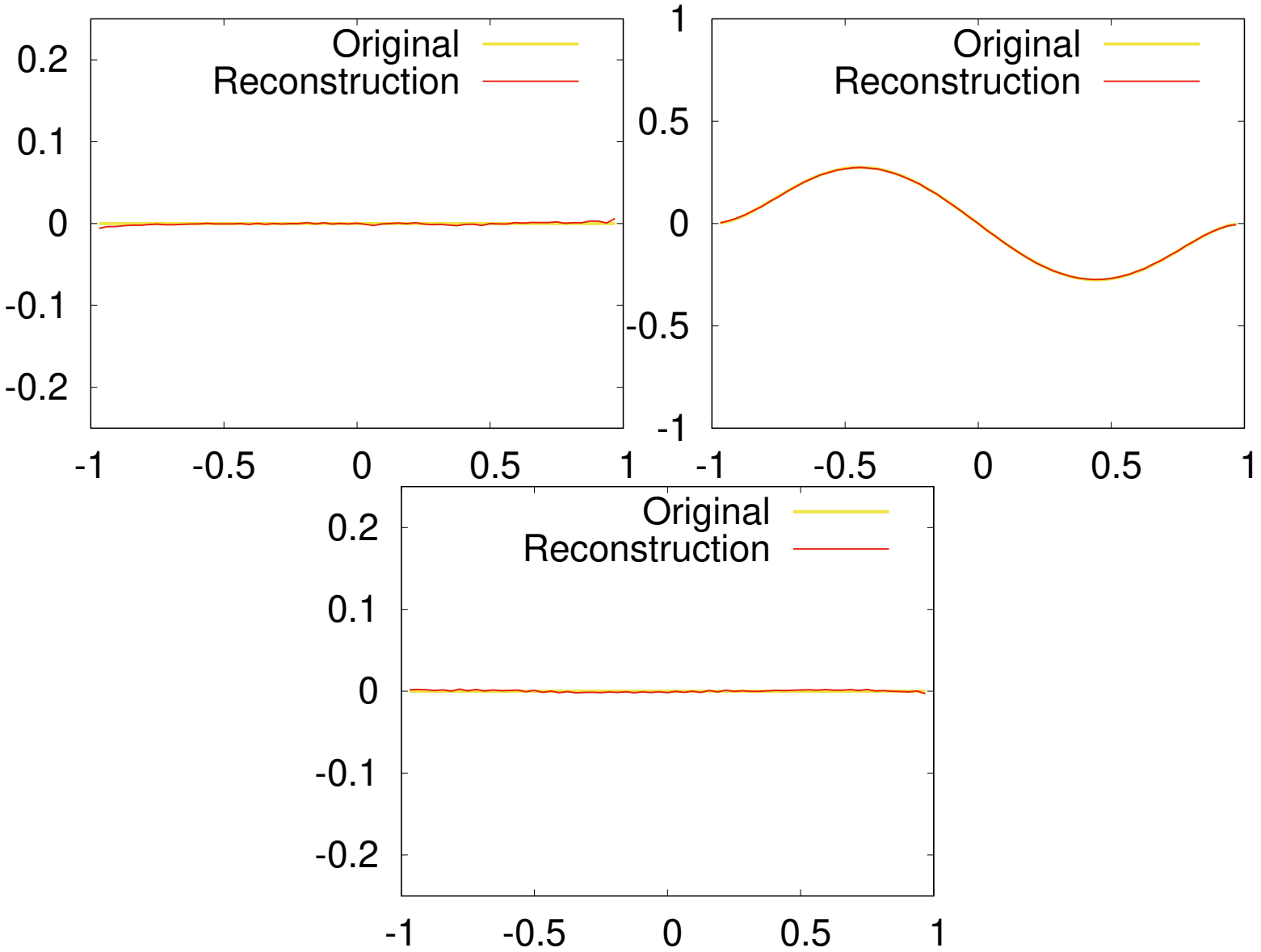


Error of reconstruction of f at $z = 0.125$



- $f_1,\ f_2,\ f_3,\ \|f - f_{\text{original}}\|$

Reconstruction of f at $z = 0.125$, $y = 0$



- f_1, f_2, f_3

Generalization

Tensor x-ray transform

- Tensor field $f = \{ f_{i_1 \dots i_m} \} \in \mathcal{S}_m(\mathbb{R}^n)$
- X-ray transform: $f \mapsto \int_{-\infty}^{\infty} f_{i_1 \dots i_m}(x + \xi t)\xi^{i_1} \dots \xi^{i_m}\, dt$
- $(df)_{i_1 \dots i_{m+1}} = \sigma(i_1 \dots i_{m+1}) \dfrac{\partial}{\partial x_{i_{m+1}}} f_{i_1 \dots i_m}$
- $(\delta f)_{i_1 \dots i_{m-1}} = \dfrac{\partial}{\partial x_i} f_{i_1 \dots i_{m-1} i}$
- $f = {}^s f + dv$, where $\delta\, {}^s f = 0$
- $(Wf)_{i_1 j_1 \dots i_m j_m} = \alpha(i_1 j_1) \dots \alpha(i_m j_m) \dfrac{\partial f_{i_1 \dots i_m}}{\partial x_{j_1} \dots \partial x_{j_m}}$
- $\boldsymbol{X}(f) = 0 \iff f = dv \iff Wf = 0$

Reconstruction

- $\mathbf{R}\left[\langle Wf, (\xi_1, \eta_1, \ldots, \xi_m, \eta_m)\rangle\right](\omega, p) =$

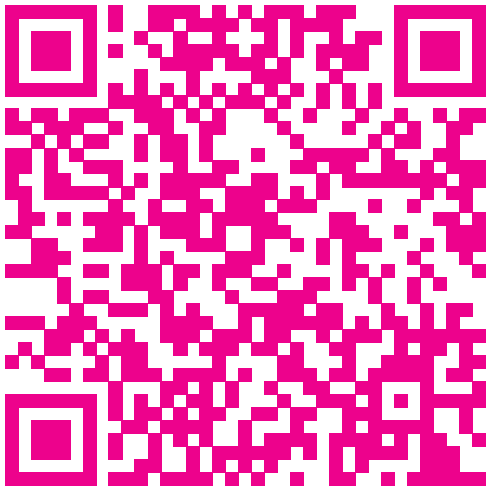
  $\alpha(\xi_1\eta_1)\ldots\alpha(\xi_m\eta_m)\,\langle\omega, \eta_1\rangle\ldots\langle\omega, \eta_m\rangle \times$

  $\frac{\partial^m}{\partial p^m}\,\langle f, (\xi_1, \ldots, \xi_m)\rangle\,(\omega, p)$
- Strategy $\boldsymbol{X}\, f \to \mathbf{R}[Wf] \to {}^s f$
- $C^m_{n+m-2} = \frac{(n+m-2)!}{m!(n-2)!}$ vs $C^m_{n+m-1} = \frac{(n+m-1)!}{m!(n-1)!}$ sources
  - For $n = 3$: $m + 1$ vs $\frac{(m+1)(m+2)}{2}$

Further Information

Next from Here

- This presentation:



- The article:
  - ✦ Aleksander Denisiuk: Reconstruction in the cone-beam vector tomography with two sources, Inverse Problems, Vol. 34, No. 12, 124008, (2018) DOI: 10.1088/1361-6420/aae9ac