On support theorems for X-Ray transform with incomplete data

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Introduction

Weighted X-ray Transform

- $X \subset \mathbb{R}^n$ is open
- $Y \subset \mathbb{G}_n$ is an immersed real-analytic n-dimensional submanifold of the set of lines—*line complex*
- $Z = \{\, (x, l) \in X \times Y \mid x \in l \,\}$—the incidence relation
- $\mu(x, l) \in C^\infty(Z)$ is a weight function
- $l(a, \xi) = \{\, x = a + \xi t \,\}$ is a line parameterization
- $R_\mu f(l) = R_\mu f(a, \xi) = \int_{l(a,\xi)} f(x)\mu(x, l(a, \xi))\, dt$

Boman-Quinto support theorems [BQ]

Admissible complexes in $\mathbb{R}^3$

Type I: Given a non-planar real analytic surface $W \subset \mathbb{R}^3$. Y is the set of all lines l, tangent to W, such that W has nonzero directional curvature along l at point of tangency.

Type II: Given a nonsingular real analytic curve $\gamma \in \mathbb{R}^3$. Y is the set of lines intersecting this curve non-tangentially

Type III: Given a closed simple nonsingular real analytic curve of directions $\theta \subset \mathbb{S}^2$. Y is the set of lines with directions on θ.

Type I

Theorem 1. *Let Y be an open connected subset of type I complex defined by W. Assume that Y is an embedded submanifold of the set of all lines. In case there is a plane $\mathcal{P}$ tangent to W at non-discrete set of points, assume that no line in Y is contained in $\mathcal{P}$.*

Let X be an open set in $\mathbb{R}^3$ disjoint from W and let $\mu(x,l)$ be real analytic function on Z that is never zero. Let $f \in \mathcal{E}'(X)$. If $R_\mu f|_Y = 0$ and some line in Y is disjoint from $\operatorname{supp} f$, then every line in Y is disjoint from $\operatorname{supp} f$.

Type II

Theorem 2. *Let Y be an open connected subset of type II complex defined by γ. Assume that Y is an embedded submanifold of the set of all lines. If γ is a plane curve, assume that no line in Y is contained in a plane containing γ. Let X be an open set in $\mathbb{R}^3$ disjoint from γ and let $\mu(x, l)$ be real analytic function on Z that is never zero. Let $f \in \mathcal{E}'(X)$. If $R_\mu f|_Y = 0$ and some line in Y is disjoint from $\operatorname{supp} f$, then every line in Y is disjoint from $\operatorname{supp} f$.*

Type III

Theorem 3. *Let Y be an open connected subset of type II complex defined by θ. Assume that θ is not a great circle of $\mathbb{S}^2$.*

Let X be an open set in $\mathbb{R}^3$ disjoint from γ and let $\mu(x,l)$ be real analytic function on Z that is never zero. Let $f \in \mathcal{E}'(X)$. If $R_\mu f|_Y = 0$ and some line in Y is disjoint from $\operatorname{supp} f$, then every line in Y is disjoint from $\operatorname{supp} f$.

Theorem of Hörmander

Theorem 4. *Let* X *be an open subset of* $\mathbb{R}^n$*,* $f \in \mathcal{D}'(x)$*, and* x_0 *a boundary point of the support of* f*, and assume that there is a* C^2 *function* F *such that* $F(x_0) = 0$*,* $dF(x_0) \neq 0$*, and* $F(x) \leq 0$ *on* $\operatorname{supp} f$*. Then* $(x_0, \pm dF(x_0)) \in WF_A(f)$*.*

Double fibration

$$\begin{array}{ccc} & Z & \\ {}^{p_X}\swarrow & & \searrow^{p_Y} \\ X & & Y \end{array} \qquad \begin{array}{ccc} & \Lambda = N^*(Z) & \\ {}^{\pi_X}\swarrow & & \searrow^{\pi_Y} \\ T^*X \setminus 0 & & T^*Y \setminus 0 \end{array}$$

- $N^*(Z) \subset T^*X \setminus 0 \times T^*Y \setminus 0$
- $p_X : Z \to X$ has surjective differential (Y is a *regular* line complex)

Admissible complexes

- cone $C_x = \cup p_Y\left(p_X^{-1}(x)\right) \subset X$
- for *non-critical* x C_x is two-dimensional
- $l \in Y$ is *non-critical*, if not all of its points are critical.
- complex of lines is *admissible,* if $\forall$ non-critical $x \in l$ C_x has the same tangent plane along l

Proposition

Proposition 5 (cf. [GU]). *Let Y be a regular real analytic admissible line complex. Let $l_0 \in Y$ and assume $f \in \mathcal{E}'(X)$ and $R_\mu f(l) = 0$ for all $l \in Y$ in a neighborhood of l_0. Let $x \in l_0 \cap X$ and let $\xi \in T_x^*(X)$ be conormal to l_0, but not conormal to the tangent plane to C_x along l_0. Then $(x, \xi) \notin WF_A(f)$.*

Proof of the proposition

- Let $\Lambda_0 \subset \Lambda$ be a set of (x, ξ, l, η) such that ξ is not conormal to C_x along l.
- R_μ as a Fourier integral operator with Lagrangian manifold Λ
- Λ_0 is a local canonical graph
- R_μ is analytic elliptic, when microlocally resticted to Λ_0
- $R_\mu f = 0$ near $l_0 \Rightarrow (x, \xi) \notin WF_A(f)$ for $(x, \xi, l_0, \eta) \in \Lambda_0$

Characteristic paths

- Let $x_0 \in \mathbb{RP}^n$. *Characteristic path* with pivot point x_0 is the smooth path in $p_Y\big(p_X^{-1}(x_0)\big)$.

Proposition 6. *Let the hypotheses of theorem Type I (Type II, Type III) hold. Let $f \in \mathcal{E}'(X)$ and assume $R_\mu f = 0$ on Y. Let $l(s) : [a, b] \to Y$ be a characteristic path and assume $l(a)$ does not meet $\operatorname{supp} f$ and the pivot point of the path is disjoint from $\operatorname{supp} f$. Then*

$$l(s) \cap \operatorname{supp} f = \emptyset \text{ for } a \leq s \leq b$$

Proof of proposition 6

- Reduce to the case of pivot point at infinity
- Construct a "wedge neighbourhood" of $l(s)$ in X:
  - $D(s,\tau)$, $D(s,0) = l(s)$, ($\tau = (\tau_1, \tau_2)$, $\|\tau\| \leq \varepsilon$)
  - $D(a,\tau) \cap \operatorname{supp} f = \emptyset$
  - no conormal $\bar{\xi}$ to $\partial D(\bar{s})$ at $\bar{x}$ is conormal to $C_{\bar{x}}$ along $l(\bar{s}) \ni \bar{x}$
- Let $\bar{s} = \sup \{ s_1 \in [a,b] \mid D(s) \cap \operatorname{supp} f = \emptyset$ for $a \leq s \leq s_1 \}$
- $D(\bar{s})$ meets $\operatorname{supp} f$ at some point $\bar{x} \in \partial D(\bar{s})$, $\bar{\xi} \perp \partial D(\bar{s})$
- Proposition 22 implies that $(\bar{x}, \bar{\xi}) \notin WF_A(f)$
- Hörmander's theorem implies that $f = 0$ near $\bar{x}$
- The only possibility is $\bar{s} = b$. So, $l(b) \cap \operatorname{supp} f = \emptyset$

Boman-Quinto support theorems—revised

Admissible complexes—revised [De]

Complexes of curves

- Let a smooth manifold X, $\dim X = n$ be given
- Let Y be a family of curves on X:
  - $\forall L \subset T_x X$, $\dim L = 1$ there is exactly one curve $y \in Y$, such that $x \in y$ and $T_x y = L$
  - then $\dim Y = 2n - 2$
- Assume that $\pi_Y : N^* Z \to T^* Y \setminus 0$ is bijective immersion
- $\Sigma = \operatorname{Im} \pi_Y \subset T^* Y \setminus 0$ is the *characteristic surface*
- Covector $\xi \in T^*_y Y$ is called *characteristic*, if $(y, \xi) \in \Sigma$
- Let $Y_x = \{ y \in Y \mid y \ni x \} \subset Y$
- $\Sigma = \bigcup_{x \in y} N^*_y Y_x$

Complexes of curves and critical points

- By a complex of curves we understand a submanifold $K \subset Y$, $\dim K \geq n-1$

Definition 7. Let $y \in K \subset Y$ and $\dim K = n-1+r$. A point $x \in y$ is a *critical* for the complex K if $R(x) = \dim(T_y K \cap T_y Y_x) > r$. A number $k(x) = R(x) - r$ is called the *multiplicity* of x

Critical points and characteristic covectors

Lemma 8 (cf. [Gu]). *There is a critical point $x \in y \in K$ of multiplicity $k(x) = k$ if and only if there is subspace $L_x \subset N^*_y K$ with $\dim L_x = k$ that consists of characteristic covectors.*

Proof.

- $\dim K = n - 1 + r, \quad \dim(T_y K \cap T_y Y_x) = r + k$
- $(T_y K \cap T_y Y_x)^\perp = (T_y K)^\perp \cup (T_y Y_x)^\perp$
- $2n - 2 - r - k = (n - 1 - r) + (n - 1) - \dim(N^*_y K \cap N^*_y Y_x)$

□

Characteristic and hyperbolic complexes

Definition 9. A complex of curves K is *hyperbolic,* if $\forall y \in K$ there exist critical points $x_1, \ldots, x_s \in y$ such that $N_y^* K = L_{x_1} \oplus \cdots \oplus L_{x_s}$, where L_{x_j} is the characteristic subspace corresponding to x_j

Definition 10. A complex of curves K is *characteristic,* $N^* K \subset \Sigma$

Lemma 11. *K is characteristic $\iff$ $\forall y \in K$ there exists critical point $x \in y$ of multiplicity $\operatorname{codim} K$*

Remark.
- The notion of hyperbolic complex of curves differs from the notion of *admissible* complex of curves
- For $\mathbb{C}$-complexes of lines notions coincide

Regular non-splitting critical points

Definition 12. A critical point $x \in y$ is *non-singular,* if there is a neighborhood $W \subset K$ of a curve y such that $\forall \xi \in L_x$ $T_\xi(N^*W \cap \Sigma) = T_\xi(N^*W) \cap T_\xi(\Sigma)$

Definition 13. Let K be a hyperbolic complex, $y_0 \in K$. We say that y_0 has *non-splitting* critical points if there is a neighborhood $W \subset K$ of a curve y_0 such that for $y \in W$ there are s non-singular critical points $x_1, \ldots, x_s \in y$ smoothly dependent in y with constant multiplicities $k_1, \ldots, k_s$ $(\sum k_i = \operatorname{codim} K)$ for which $N^*_y(W) = L_{x_1} \oplus \cdots \oplus L_{x_s}$

Local structure of hyperbolic complexes

Theorem 14. *Suppose that K is a hyperbolic complex, $y_0 \in K$ is a curve with non-splitting critical points. Then there exists s characteristic complexes W_j, such that $W = \cap W_j$ and $\forall y \in W$ the critical point $x_j \in y$ will be critical for exactly one complex W_j with the same multiplicity.*

Conversely, suppose that $W = \cap W_j$, where the W_j are hyperbolic complexes, and $\dim W \geq n - 1$. Then W is hyperbolic, and any $y \in W$ will have as critical points all the critical points of all the W_j with corresponding multiplicity.

Proof of the theorem 14

- $N^*W \cap \Sigma = \cap V_j$, where V_j is the bundle of characteristic covectors corresponding to the critical point $x_j = x_j(y)$
  - $\dim V_j = 2n - 2 + k_j - k$
  - V_j is isotropic submanifold in $T^*Y \setminus 0$
- Σ is an involutory submanifold of $T^*Y \setminus 0, \operatorname{codim} \Sigma = n - 2$
  - Ideal J of functions vanishing on Σ corresponds to the Lie algebra $\mathcal{V}$ of vector fields tangent to Σ:
    - $J \ni f \mapsto \operatorname{sgrad} f \in \mathcal{V}$ $(\operatorname{sgrad} f \vee \omega = -df)$
- Act on V_j by the Hamiltonian flow corresponding to $\mathcal{V}$
- We obtain a Lagrange manifold $\overline{W}_j = N^*W_j$

Local structure of characteristic complexes

Theorem 15. *Let K be a characteristic complex, $\operatorname{codim} K = k$. Then in a neighborhood of non-singular curve K consists*

1. *for $k > 1$ of curves intersecting given submanifold $M \subset X, \operatorname{codim} M = k + 1$*
2. *for $k = 1$ of either curves intersecting a given submanifold $M \subset X, \operatorname{codim} M = 2$, or curves tangent to a given submanifold $M \subset X, \operatorname{codim} M = 1$*

Proof. Compute a rank of $\varphi : y \mapsto x(y)$, where $x(y)$ is a critical point of y. $M = \operatorname{Im} \varphi$. □

Local structure of hyperbolic complexes—I

Theorem 16 (cf. [Ma]). *Let K be a hyperbolic complex, $\operatorname{codim} K = k$, and $y_0 \in K$ be a curve with non-splitting critical points. Then the critical points $x_j(y)$ circumscribe s manifolds $M_j \subset X$. Moreover, if $k_j > 1$ then $\operatorname{codim} M_j = k_j + 1$ and curves in W intersects M_j transversally; if $k_j = 1$, then either $\operatorname{codim} M_j = 2$ and curves in W intersect M_j transversally, or $\operatorname{codim} Y = 1$ and curves in W are tangent to M_j.*

Local structure of hyperbolic complexes—II

Theorem 17 (cf. [Ma]). *Conversely, let s submanifolds $M_j \subset X$ be given and let $k_j = \max\{1, \operatorname{codim} M_j - 1\}$; if $\sum k_j = k$ and the set of curves intersecting the submanifolds of codimension $k_j + 1$ and tangent to submanifolds of codimension 1 forms a submanifold in Y of codimension k, then this is a hyperbolic complex with critical points of multiplicities k_j lying on the M_j.*

Proposition—revised

Completeness condition

- Let K be a n-dimensional complex of lines
- $x(t) = \xi(u) + \beta(u)$ be a local parameterization
- $y_0 \in K$, $\omega \in \mathbb{R}^{n*}$, $\omega \neq 0$, $\omega \perp y_0$

Definition 18 (cf. [Pa]). Line y_0 satisfies a *weak completeness condition* for ω at $x_0 = x(t_0) \in y_0 = y(u_0)$, if a germ of the map $\Pi_\omega : K \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}^n$, $\Pi_\omega : (u, t) \mapsto (\langle \omega, \dot{x} \rangle, x(t))$ is a diffeomorphism at (u_0, t_0).

ω-critical points

- K is an n-dimensional line complex, $y \in K$, $\omega \perp y$

Definition 19.

- Point $x(t) \in y$ is ω-*critical,* if the weak completeness condition is not held at h
- Line y is ω-*critical,* if all its point are ω-critical
- The set of conormals $\omega \perp y$ for which y is ω-critical is called *the set of critical conormals,* and is denoted by Ω_y

ω-critical lines

Lemma 20. *Let* $y_0 = y(u_0) \in K$, $\omega \perp y_0$. *A point* $x = x(u_0, t_0)$ *is* ω*-critical* $\iff$ $P_\omega(t_0) = 0$, *where polynomial* $P_\omega(t) = \left\langle \omega, \sum_{k=1}^n \frac{\partial \xi}{\partial u_k} P_k(t) \right\rangle$.

Proof.

$$P_\omega(t) = \det \begin{vmatrix} \left\langle \omega, \frac{\partial \xi}{\partial u^1} \right\rangle & \left\langle \omega, \frac{\partial \xi}{\partial u^2} \right\rangle & \dots & \left\langle \omega, \frac{\partial \xi}{\partial u^n} \right\rangle & 0 \\ \frac{\partial \xi^1}{\partial u^1} t + \frac{\partial \beta^1}{\partial u^1} & \frac{\partial \xi^1}{\partial u^2} t + \frac{\partial \beta^1}{\partial u^2} & \dots & \frac{\partial \xi^1}{\partial u^n} t + \frac{\partial \beta^1}{\partial u^n} & \xi^1 \\ \frac{\partial \xi^2}{\partial u^1} t + \frac{\partial \beta^2}{\partial u^1} & \frac{\partial \xi^2}{\partial u^2} t + \frac{\partial \beta^2}{\partial u^2} & \dots & \frac{\partial \xi^2}{\partial u^n} t + \frac{\partial \beta^2}{\partial u^n} & \xi^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial \xi^n}{\partial u^1} t + \frac{\partial \beta^n}{\partial u^1} & \frac{\partial \xi^n}{\partial u^2} t + \frac{\partial \beta^n}{\partial u^2} & \dots & \frac{\partial \xi^n}{\partial u^n} t + \frac{\partial \beta^n}{\partial u^n} & \xi^n \end{vmatrix}$$

$\square$

Admissible complexes and critical normals

Theorem 21. *Let K be an n-dimensional line complex in $\mathbb{R}$. The following properties are equivalent:*

1. *K is admissible*
2. *For all non-critical line $y \in K$, for all $\omega \in \mathbb{R}^*$, $\omega \perp y$, either y is ω-critical, or all its ω-critical points are critical.*
3. *For all non-critical $y \in K$ $\dim \Omega_y = n - 2$.*

Proof of the theorem 21

- Tangent plane to C_{x_0} is spanned on vectors ξ and $\sum_{k=1}^{n}\left(\frac{\partial\xi}{\partial u^k}t+\frac{\partial\beta}{\partial u^k}\right)P_k(t_0)=\left(\sum_{k=1}^{n}\frac{\partial\xi}{\partial u^k}P_k(t)\right)(t-t_0)-P_0(t)\xi,$
- So, $\Omega_y=\left\{\omega\in\mathbb{R}^*\;\middle|\;\langle\omega,\xi\rangle=0,P_\omega(t)\equiv 0\right\}=\bigcap_{t\in\mathbb{R}}\left\{\omega\in\mathbb{R}^*\;\middle|\;\omega\perp\xi,\;\omega\perp\sum_{k=1}^{n}\frac{\partial\xi}{\partial u^k}P_k(t)\right\}=\bigcap_{t\in\mathbb{R}\setminus\mathrm{Crit}_y}\left\{\omega\in\mathbb{R}^*\;\middle|\;\omega\perp TC_{x(t)}\right\}=\bigcap_{t\in\mathbb{R}\setminus\mathrm{Crit}_y}\left(TC_{x(t)}\right)^{\perp}=\left(\bigcup_{t\in\mathbb{R}\setminus\mathrm{Crit}_y}TC_{x(t)}\right)^{\perp}$

Proposition (revised)

Proposition 22. *Let Y be a real analytic n-dimensional line complex in $\mathbb{R}^n$. Let $l_0 \in Y$ and assume $f \in \mathcal{E}'(X)$ and $R_\mu f(l) = 0$ for all $l \in Y$ in a neighborhood of l_0. Let $x_0 \in l_0 \cap X$ and let $\xi \in T^*_{x_0}(X)$ be conormal to l_0, and such that x_0 is not ξ-critical point for l_0. Then $(x_0, \xi) \notin WF_A(f)$.*

Proof of the revised proposition

- Let $\Lambda_0 \subset \Lambda$ be a set of (x, ξ, l, η) such that x is not ξ-critical for l.
- R_μ as a Fourier integral operator with Lagrangian manifold Λ
- Λ_0 is a local canonical graph
- R_μ is analytic elliptic, when microlocally resticted to Λ_0
- $R_\mu f = 0$ near $l_0 \Rightarrow (x, \xi) \notin WF_A(f)$ for $(x, \xi, l_0, \eta) \in \Lambda_0$

A common way to prove a support theorem

Principle 23. *Let Y be an n-dimensional complex of lines in $\mathbb{R}^n$. Let $f \in \mathcal{E}'(X)$ and assume $R_\mu f = 0$ on Y. Let $l(s) : [a, b] \to Y$ be a path and assume $l(a)$ does not meet $\operatorname{supp} f$. Suppose that there exists a "wedge neighbourhood" of $l(s)$ in X, such that*

1. $D(s, \tau)$, $D(s, 0) = l(s)$
2. $D(a, \tau) \cap \operatorname{supp} f = \emptyset$
3. *for no conormal $\bar{\xi}$ to $\partial D(\bar{s})$ at $\bar{x}$, line $l(\bar{s}) \ni \bar{x}$ is $\bar{\xi}$-critical*

Then

$$l(s) \cap \operatorname{supp} f = \emptyset \text{ for } a \leq s \leq b$$

Admissible complexes—characteristic paths

- $\Omega_y = \left(\bigcup_{t\in\mathbb{R}\setminus\mathrm{Crit}_y} TC_{x(t)}\right)^{\perp}$
- $\dim \Omega_y = n-2$

Non-admissible complex

- $$x = \begin{pmatrix} u_3 t + u_1 \\ u_1 t + u_2 \\ t \end{pmatrix}$$
- $P_\omega(t) = \omega_1 - t\omega_2$
- $l_0 = (0,0,0)$, $l_1 = (u_1, u_2, u_3)$, $l_s = (su_1, su_2, su_3)$
- $$D(s,\tau) = \begin{pmatrix} su_3 t + su_1 + \varepsilon(s)\cos(\tau) \\ (su_1 + \varepsilon(s)\cos(\tau))t + u_2 + \varepsilon(s)\sin(\tau) \\ t \end{pmatrix}$$
- For fixed s $P_\omega(t) = 0$ on $T_x(\partial D(s)) \iff \cos\tau = 0$:
  - ❖ you can chose $\varepsilon(s)$ such that it will be internal point of $D(s,\tau)$.

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