On X-ray transform of symmetric tensor fields

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Introduction

Symmetric tensor fields

- $S^m = S^m(T^*\mathbb{R}^n)$, $\mathbb{R}^n$ is the Euclidean space
- $\langle f, (\xi_1, \dots, \xi_m)\rangle\rangle = f_{i_1\dots i_m}\xi_1^{i_1}\dots\xi_m^{i_m}$
- Inner differentiation $d : C^\infty(S^m) \to C^\infty(S^{m+1})$,

  $(d\,f)_{i_1\dots i_{m+1}} = \sigma(i_1\dots i_{m+1})\frac{\partial}{\partial x^{i_{m+1}}} f_{i_1\dots i_m}$
- Divergence $\delta = (-\,d)^*$, $\delta : C^\infty(S^m) \to C^\infty(S^{m-1})$,

  $(\delta f)_{i_1\dots i_{m-1}} = \frac{\partial}{\partial x_i} f_{i_1\dots i_{m-1}i}$
- The Saint-Venant operator

  $R : C^\infty(S^m) \to C^\infty(S^m(\wedge^2 T^*\mathbb{R}^n))$,

  $(Rf)_{i_1 j_1\dots i_m j_m} = \alpha(i_1 j_1)\dots\alpha(i_m j_m)\frac{\partial f_{i_1\dots i_m}}{\partial x^{j_1}\dots\partial x^{j_m}}$

  - $Rf = d^{\text{ext}} f$ for $m = 1$

The X-ray transform

- $If(x,\xi) = \int \langle f(x+t\xi), \xi^m \rangle \, dt =$
  $\int f_{i_1 \dots i_m}(x+t\xi), \xi^{i_1} \dots \xi^{i_m} \, dt$
- $If(x,\xi) \in C^\infty \left(T(\mathbb{R}^n) \setminus O \right)$
  - $If(x,\lambda\xi) = \lambda^m |\lambda|^{-1} If(x,\xi), \quad 0 \neq \lambda \in \mathbb{R},$
  - $If(x+\lambda\xi,\xi) = If(x,\xi), \quad \lambda \in \mathbb{R}$
- $If(x,\xi) \in C^\infty(G_n)$, G_n is the manifold of lines in $\mathbb{R}^n$

Kernel of the X-ray transform

Theorem 1 (Sharafutdinov). *For the finite field $f \in C_0^\infty(S^m)$ the following statements are equivalent:*

1. $If(x,\xi) = 0$.
2. *There exists such smooth field $v \in C_0^\infty(S^{m-1})$, that $f = d\,v$.*
3. *Identically in $\mathbb{R}^n$ $Rf = 0$.*

Symmetric tensor field decomposition

Theorem 2 (Sharafutdinov)**.** *Let* $n \geqslant 2$*. For any field* $f \in \mathcal{S}(S^m)$ *there exist such uniquely defined fields* ${}^s\!f \in C^\infty(S^m)$ *and* $v \in C^\infty(S^{m-1})$*, that*

$$f = {}^s\!f + d\,v, \quad \delta\,{}^s\!f = 0, \tag{1}$$

$${}^s\!f(x) \to 0, \quad v(x) \to 0 \quad \textit{where} \quad x \to \infty,$$

$${}^s\!f, d\,v \in L^2(S^m).$$

The field ${}^s\!f$ is called *the solenoidal part* of f.

Inverse problem

- To recover the solenoidal part of f from the X-ray transform $If(x,\xi)$.
- Two-step approach:

1. to reconstruct the Saint-Venant operator Rf from If,
2. to reconstruct the solenoidal part ${}^{s}f$ from Rf.

Incomplete data problem

- $\dim G_n = 2n - 2 > n$.
- To reconstruct the solenoidal part ${}^s\!f$ from If known on n-dimensional *complex of lines*.

The case of complete data set

The John differential operator

$$D : C^{\infty}\left(S^p\left(\wedge^2(T\mathbb{R}^n \setminus O)\right)\right) \to C^{\infty}\left(S^{p+1}\left(\wedge^2(T\mathbb{R}^n \setminus O)\right)\right),$$

$$(D\varphi)_{i_1 j_1 \dots i_{p+1} j_{p+1}} = \frac{1}{2}\left(\frac{\partial^2 \varphi_{i_1 j_1 \dots i_p j_p}}{\partial x^{i_{p+1}} \partial \xi^{j_{p+1}}} - \frac{\partial^2 \varphi_{i_1 j_1 \dots i_p j_p}}{\partial x^{j_{p+1}} \partial \xi^{i_{p+1}}}\right).$$

Range condition for $I\,f$

Theorem 3 (Sharafutdinov). *A function $\varphi(x,\xi)$, defined on $T\mathbb{R}^n \setminus O$ ($n \geqslant 3$) which possesses the homogeneity properties (5) is the X-ray transform of symmetric tensor field of degree m if and only if $D^{m+1}\varphi(x,\xi) \equiv 0$.*

Theorem 4 (John, $n=3, m=0$).

$$\varphi(x_1,\xi_1,x_2,\xi_2) = \int f(x_1+\xi_1 t, x_2+\xi_2 t, t)\,dt \iff$$

$$\left(\frac{\partial^2}{\partial x_1 \partial \xi_2} - \frac{\partial^2}{\partial x_2 \partial \xi_1}\right)\varphi(x_1,\xi_1,x_2,\xi_2) \equiv 0$$

Reconstruction of Rf

Theorem 5. *The Saint-Venant operator Rf of a field $f \in \mathcal{S}(S^m)$ can be recovered after the X-ray transform $If(x,\xi)$ with the following formula*

$$Rf(x) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{4m!\pi^{\frac{n+1}{2}}}(-\Delta)^{\frac{1}{2}} \int_{\mathbb{S}^{n-1}} D^m If(x,\xi)\,\Omega_{n-1}(\xi). \qquad (2)$$

Even indices divergence

$$\delta_e : C^\infty(S^{k-1} \overset{s}{\otimes} S^{m-k+1}\big(\wedge^2(T^*\mathbb{R})\big) \to C^\infty(S^{k} \overset{s}{\otimes} S^{m-k}\big(\wedge^2(T^*\mathbb{R})\big)$$

$$(\delta_e f)_{i_1 \dots i_k i_{k+1} j_{k+1} \dots i_m j_m}(x) = \frac{\partial}{\partial x_{j_k}} f_{i_1 \dots i_{k-1} i_k j_k \dots i_m j_m}(x).$$

$$k = 1, \dots, m$$

- δ_e is the formal dual to $-\alpha(i_k, j_k)\partial/(\partial x^{j_k})$, $k = 1, \dots, m$
- Specifically, δ_e^m is the formal dual operator to $(-1)^m R = (-1)^m \alpha(i_1, j_1)\frac{\partial}{\partial x^{j_1}} \circ \dots \circ \alpha(i_m, j_m)\frac{\partial}{\partial x^{j_m}}$.
- For $m = 1$ one can consider Rf as a differential two-form, δ_e coincides with the standard divergence.

Reconstruction of the solenoidal part

Theorem 6. *For any field $f \in \mathcal{S}(S^m)$ the following relation is valid:*

$$\Delta^m\, {}^s\!f = 2^m \delta_e^m R f.$$

Theorem 7. *The solenoidal part ${}^s\!f$ of a field $f \in \mathcal{S}(S^m)$ can be recovered after the X-ray transform $If(x,\xi)$ with a formula*

$${}^s\!f(x) = \frac{(-2)^{m-2}\Gamma\left(\frac{n-1}{2}\right)}{m!\pi^{\frac{n+1}{2}}}(-\Delta)^{\frac{1}{2}-m}\delta_e^m \int_{\mathbb{S}^{n-1}} D^m If(x,\xi)\,\Omega_{n-1}(\xi). \tag{3}$$

Remark. In scalar case ($m = 0$) formulas (2) and (3) coincide with the inversion formula for the scalar X-ray transform.

The case of incomplete data set

Admissible comlexes of lines in $\mathbb{R}^3$

- The set of lines intersecting a curve in space
- The set of lines with direction vectors lying on a given curve on the sphere *(the case of curve belonging to infinity).*
- The set of lines tangent to a surface in space

The Kirillov-Tuy condition

- ($m = 0$) Every hyper-plane, intersecting the support of reconstructed function, intersects the curve or the surface, defining the line complex
- Every hyper-plane, intersecting the support of reconstructed function, intersects the curve or the surface, defining the line complex C^m_{n+m-2} times.

The cone-beam transform

- Consider the set K_γ of lines intersecting a given curve $\gamma \in \mathbb{R}^n$.
- Fix a parameterization $\gamma = \gamma(\lambda)$.
- $Xf(\lambda, \xi) = \int_0^\infty \langle f\big(\gamma(\lambda) + \xi t\big), \xi^m \rangle \, dt = \int_0^\infty f_{i_1 \dots i_m}\big(\gamma(\lambda) + \xi t\big)\xi^{i_1} \dots \xi^{i_m} \, dt.$

Generic vectors

Definition 8. Vectors $e_1, \dots, e_{\nu(m,n)} \in \mathbb{R}^{n-1}$, are *generic* ($\nu(m,n) = C^m_{n+m-2}$), if any symmetric (covariant) tensor f in $\mathbb{R}^{n-1}$ of order m is uniquely defined by the values $f(e_1), \dots, f(e_{\nu(m,n)})$.

Remark. In coordinates in $\mathbb{R}^{n-1}$:

$$f(e_k) = f_{i_1 \dots i_m} x_k^{i_1} \dots x_k^{i_m}, \quad k = 1, \dots, \nu(m,n), \tag{4}$$

where $e_k = (x_k^1, \dots, x_k^{n-1})$. The condition of unique solvability is equivalent to the condition that determinant of the system differs from zero.

Generic vectors for $n-1=2$

Theorem 9. *Let $m+1$ generic non-zero nodes $x_1,\ldots,x_{m+1}\in\mathbb{R}^2$ and $m+1$ values $y_1,\ldots,y_{m+1}\in\mathbb{R}$ be given. Then there exists unique homogeneous polynomial $F(x)$ of degree m such that $F(x_i)=y_i$ for $i=1,\ldots,m+1$. This polynomial is defined by the following (Lagrange) formula:*

$$F(x)=\sum_{i=1}^{m+1} y_i \prod_{j\neq i} \frac{x^1x_j^2-x^2x_j^1}{x_i^1x_j^2-x_i^2x_j^1}$$

- Condition 8 means that there is no pair of parallel vectors e_i and e_j for $i\neq j$.

Kirillov-Tuy condition of order m

Definition 10. Fix a domain $B \subset \mathbb{R}^n$. Say that the curve γ satisfies the *Kirillov-Tuy condition of order* m*,* if for almost any hyper-plane $H_{\omega,p}=\{\, x \in \mathbb{R}^n \mid \langle \omega, x \rangle = p \,\}$ intersecting the domain B, there is a set of points $\gamma_1, \ldots, \gamma_{\nu(m,n)} \in H_{\omega,p} \cap \gamma$, which locally smoothly depends on (ω, p), such that the for almost every $x \in H_{\omega,p} \cap B$ the vectors $x - \gamma_1, \ldots, x - \gamma_{\nu(m,n)}$ are generic.

Examples of curve γ

Example 11. For vector fields ($m = 1$) and the unit ball B. As a γ consider the union of two orthogonal big circles of the ball with sufficiently large radius (greater than $\sqrt{2}$) centered at the origin. Almost every plane intersecting the unit ball B has at least two different common points with γ.

Example 12. A set of $m + 1$ lines passing through the origin. Almost every plane in $\mathbb{R}^3$ have $m + 1$ intersection points with this union.

Explicit reconstruction

Theorem 13. *Let a domain* $B \subset \mathbb{R}^3$ *and a curve* $\gamma \subset \mathbb{R}^3$ *satisfying the Kirillov-Tuy condition of order* m *be given. Then, for any field* $f \in \mathcal{S}(S^m)$, $\operatorname{supp} f \subset B$, *the Saint-Venant operator* $Rf(x)$, $x \in B$, *can be reconstructed after the cone-beam transform known for lines from* K_γ *with the explicit formula.*

Corollary 14. *Under assumptions of the theorem 13 solenoidal part of the tensor field* ${}^s f(x)$, $x \in B$ *can be reconstructed after the cone-beam transform known for lines from* K_γ *with the explicit formula.*

The Radon transform

- For a function $f(x)$, $x \in \mathbb{R}^n$, define the Radon transform as follows:
$$f^\wedge(\omega, p) = \int_{H_{\omega,p}} f(x)\, d\, s,$$
where ds is the standard volume element on the hyper-plane $H_{\omega,p}$.
- A function $f \in L^2(\mathbb{R}^3)$ can be reconstructed after the Radon transform by explicit formula:
$$f(x) = -(8\pi^2)^{-1} \int_{\mathbb{S}^2} \frac{\partial^2}{\partial p^2} f^\wedge(\omega, \langle \omega, x \rangle)\, \Omega_2(\omega). \qquad (5)$$

The Radon transform of Rf—I

Lemma 15. *Let the set of vectors $\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_m \in \mathbb{R}^n$ be given. Then for any tensor field $f \in \mathcal{S}(S^m)$*

$$\langle Rf, (\xi_1, \eta_1, \dots, \xi_m, \eta_m)\rangle^\wedge (\omega, p)$$
$$= \alpha(\xi_1\eta_1) \dots \alpha(\xi_m\eta_m) \langle \omega, \eta_1\rangle \dots \langle \omega, \eta_m\rangle$$
$$\times \frac{\partial^m}{\partial p^m} \langle f, (\xi_1, \dots, \xi_m)\rangle^\wedge (\omega, p).$$

Proof.

$$\left(a_i \frac{\partial}{\partial x^i} f(x)\right)^\wedge (\omega, p) = \langle \omega, a\rangle \frac{\partial}{\partial p} f^\wedge(\omega, p).$$

□

The Radon transform of Rf—II

Corollary 16. *Let vectors $\xi_1, \dots, \xi_m \in \mathbb{R}^n$ be orthogonal to ω. Then for any tensor field $f \in \mathcal{S}(S^m)$*

$$\langle Rf, (\xi_1, \omega, \dots, \xi_m, \omega)\rangle^{\wedge}(\omega, p) = 2^{-m} \frac{\partial^m}{\partial p^m} \langle f, (\xi_1, \dots, \xi_m)\rangle^{\wedge}(\omega, p). \tag{6}$$

Reconstruction of the Radon transform

Lemma 17. *Let the curve γ intersect the plane $H_{\omega,p}$ at point γ_0. Consider the parameterization in $\mathbb{R}^3$: $x = \gamma_0 + \xi t$. For any tensor field $f \in \mathcal{S}(S^m)$ and weight function $w(\xi)$ the following derivative of the Radon transform:*

$$\partial^{m+1}/\partial p^{m+1} \left[\langle f(\gamma_0 + \xi t), \xi^m \rangle \, w(\xi) t^k \right]^{\wedge} (\omega, p), \quad 0 \le k \le m, \tag{7}$$

can be explicitly expressed from Xf.

Proof of the lemma 17

- $\int_{\mathbb{S}^1(\omega)} Xf\,\Omega(\xi) = \left[\langle f(\gamma_0 + \xi t), \xi^m\rangle \cdot t^{-1}\right]^{\wedge}(\omega, p)$
- $L = \sum \omega_i \partial/\partial\xi_i = t \sum \omega_i \partial/\partial x_i$
- $\int_{\mathbb{S}^1(\omega)} L\big(w(\xi)Xf\big)\,\Omega(\xi) = \frac{\partial}{\partial p}\left[\langle f(\gamma_0 + \xi t), \xi^m\rangle\, w(\xi)\right]^{\wedge}(\omega, p)$
- $\frac{\partial^{k+1}}{\partial p^{k+1}}\left[\langle f(\gamma_0 + \xi t), \xi^m\rangle \cdot w(\xi) \cdot t^k\right]^{\wedge}(\omega, p), \quad 0 \le k \le m,$

  can be explicitly expressed from Xf.
- $\frac{\partial}{\partial p}\left\{\frac{\partial^{k+1}}{\partial p^{k+1}}\left[\langle f(\gamma(\lambda) + \xi t), \xi^m\rangle \cdot w(\xi) \cdot t^k\right]^{\wedge}(\omega, p)\right\} =$
  $\frac{\partial^{k+2}}{\partial p^{k+2}}\left[\langle f(\gamma(\lambda) + \xi t), \xi^m\rangle \cdot w(\xi) \cdot t^k\right]^{\wedge}(\omega, p) +$
  $\frac{\partial^{k+1}}{\partial p^{k+1}}\left[\frac{\partial\lambda}{\partial p}\frac{\partial}{\partial\lambda}\langle f(\gamma(\lambda) + \xi t), \xi^m\rangle \cdot w(\xi) \cdot t^k\right]^{\wedge}(\omega, p)$

Reconstruction of the Radon transform—II

Lemma 18. *Let $H_{\omega,p}$ be a plane from the condition 10. $\forall\, f \in \mathcal{S}(S^m)$ and any vector $e \parallel H_{\omega,p}$*

$$\partial^{m+1}/\partial p^{m+1} \left\langle f, e^m \right\rangle^{\wedge} (\omega, p), \tag{8}$$

can be explicitly expressed from Xf.

Corollary 19. *Let $H_{\omega,p}$ be a plane from the condition 10. $\forall\, f \in \mathcal{S}(S^m)$ and any vectors $e_1, \ldots, e_m \parallel H_{\omega,p}$*

$$\partial^{m+1}/\partial p^{m+1} \left\langle f, (e_1, \ldots, e_m) \right\rangle^{\wedge} (\omega, p), \tag{9}$$

can be explicitly expressed from Xf.

Proof of the lemma 18

- Let $\gamma_1, \ldots, \gamma_{m+1}$ be intersection points from the condition 10. Then the following integrals are known: $X_i(\xi) = \int_0^\infty \langle f(\gamma_i + \xi_i t_i), \xi^m \rangle \, d\, t_i$ for $i = 1, \ldots, m+1$.
- For arbitrary vector $e = (e^1, e^2) \parallel H_{\omega,p}$ the following relation in fixed Cartesian coordinates in $H_{\omega,p}$ is valid:
  $\langle f(x), e \rangle = \sum_{i=1}^{m+1} \langle f(x), \xi_i^m \rangle \, t_i^m \prod_{j \neq i} \frac{e^1 \xi_j^2 t_j - e^2 \xi_j^1 t_j}{\xi_i^1 t_i \xi_j^2 t_j - \xi_i^2 t_i \xi_j^1 t_j}, =$
  $\langle f(x), e \rangle = \sum_{i=1}^{m+1} \langle f(x), \xi_i^m \rangle \prod_{j \neq i} \frac{e^1 \gamma_{ij}^2 - e^2 \gamma_{ij}^1 + (e^1 \xi_i^2 - e^2 \xi_i^1) t_i}{\xi_i^1 \gamma_{ij}^2 - \xi_i^2 \gamma_{ij}^1}.$
- $\frac{\partial^{m+1}}{\partial p^{m+1}} \langle f(x), e \rangle^{\wedge} (\omega, p) =$
  $\sum_{i=1}^{m+1} \frac{\partial^{m+1}}{\partial p^{m+1}} \int_{H_{\omega,p}} \langle f(x), \xi_i^m \rangle \prod_{j \neq i} \frac{e^1 \gamma_{ij}^2 - e^2 \gamma_{ij}^1 + (e^1 \xi_i^2 - e^2 \xi_i^1) t_i}{\xi_i^1 \gamma_{ij}^2 - \xi_i^2 \gamma_{ij}^1} \, d\, S.$

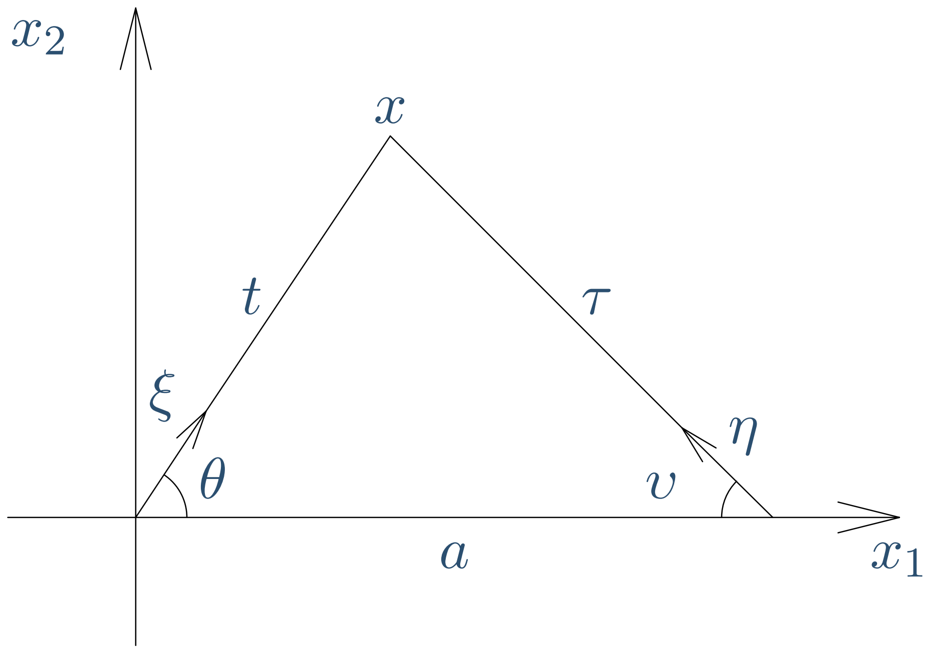
Proof of the theorem 13

- Consider arbitrary vector set $\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_m$ and hyper-plane $H_{\omega,p}$, which satisfies the condition 10.
- Decompose each vector into the sum $\xi_i = \xi_i' + \xi_i'', \quad \eta_i = \eta_i' + \eta_i'', \quad i = 1, \dots, m$, where $\xi_i', \eta_i' \parallel \omega, \quad \xi_i'', \eta_i'' \perp \omega$.
- $\frac{\partial}{\partial p} \langle Rf, (\xi_1, \eta_1, \dots, \xi_m, \eta_m) \rangle^\wedge (\omega, p)$ is a linear combination of terms $\frac{\partial^{m+1}}{\partial p^{m+1}} \langle f, (\chi_1, \dots, \chi_m) \rangle^\wedge (\omega, p)$, where χ_i are projections of $\xi_i \parallel H_{\omega,p}$.
- The derivative of the Radon transform of f is known.
- Substituting it into the inversion formula 5, we obtain $\langle Rf(x), (\xi_1, \eta_1, \dots, \xi_m, \eta_m) \rangle$.

Example of explicit formula

- $f \in \mathcal{S}(S^1(\mathbb{R}^3)) \quad Rf = d^{\text{ext}} f$
- $Xf(\lambda, \xi) = Xf(\gamma_1(\lambda), \xi)$ and $Xf(\mu, \eta) = Xf(\gamma_2(\mu), \eta)$ are given for two curves
- $\forall v, u \in \mathbb{R}^3 \quad \left(\left\langle d^{\text{ext}} f, (u, v)\right\rangle\right)^{\wedge}(\omega, p) =$
$\frac{1}{2}\left(\frac{\partial}{\partial p}\langle\omega, v\rangle\left(\langle f, u\rangle\right)^{\wedge}(\omega, p) - \frac{\partial}{\partial p}\langle\omega, u\rangle\left(\langle f, v\rangle\right)^{\wedge}(\omega, p)\right) =$
$\frac{1}{2}\left(\frac{\partial}{\partial p}\langle\omega, v\rangle\left(\langle f, u''\rangle\right)^{\wedge}(\omega, p) - \frac{\partial}{\partial p}\langle\omega, u\rangle\left(\langle f, v''\rangle\right)^{\wedge}(\omega, p)\right)$
- Fix H_{ω_0, p_0}

Example of explicit formula—II



- $f = (f_1, f_2)$
- $f_1(x) = \langle f(x), \xi \rangle \frac{t}{a} - \langle f(x), \eta \rangle \frac{\tau}{a}$
- $f_2(x) = \langle f(x), \xi \rangle \left(\frac{1}{\sin\theta} - \frac{t}{a} \cot\theta \right) + \langle f(x), \eta \rangle \left(\frac{1}{\sin\upsilon} - \frac{\tau}{a} \cot\upsilon \right)$

Example of explicit formula—III

- $\frac{\partial^2 f_1^\wedge}{\partial p^2}(\omega, p) = \int_0^{2\pi} L^2 \frac{Xf(\lambda,\xi)}{a}\, d\theta - \int_0^{2\pi} L^2 \frac{Xf(\mu,\xi)}{a}\, d\upsilon$
- $\frac{\partial^2 f_2^\wedge}{\partial p^2}(\omega, p) = \frac{\partial}{\partial p} \int_0^{2\pi} L \frac{Xf(\lambda,\xi)}{\sin\theta}\, d\theta - \int_0^{2\pi} L \frac{\partial\lambda}{\partial p} \frac{\partial}{\partial\lambda} \frac{Xf(\lambda,\xi)}{\sin\theta}\, d\theta - \int_0^{2\pi} L^2 \frac{Xf(\lambda,\xi)\cot\theta}{a}\, d\theta + \frac{\partial}{\partial p} \int_0^{2\pi} L \frac{Xf(\mu,\eta)}{\sin\upsilon}\, d\upsilon - \int_0^{2\pi} L \frac{\partial\mu}{\partial p} \frac{\partial}{\partial\mu} \frac{Xf(\mu,\eta)}{\sin\upsilon}\, d\upsilon - \int_0^{2\pi} L^2 \frac{Xf(\mu,\eta)\cot\upsilon}{a}\, d\upsilon$
- $\frac{\partial}{\partial p} \left\langle d^{\text{ext}} f, (u,v) \right\rangle^\wedge (\omega, p)$ is known
- $\left\langle d^{\text{ext}} f, (u,v) \right\rangle = -\frac{1}{8\pi^2} \int_{\mathbb{S}^2} \frac{\partial^2}{\partial p^2} \left\langle d^{\text{ext}} f, (u,v) \right\rangle^\wedge (\omega, p)\, \Omega_2(\omega)$

The case of curves belonging to infinity

Definition 20 (cf. condition 10)**.** Fix a domain $B \subset \mathbb{R}^n$. Say that the curve γ of directions satisfies the *Kirillov-Tuy condition of order* m, if for almost any hyper-plane $H_{\omega,p}=\{\, x \in \mathbb{R}^n | \,\langle \omega, x \rangle = p \,\}$ intersecting the domain B, there is a set of generic directions $\gamma_1, \dots, \gamma_{\nu(m,n)} \in H_{\omega,p} \cap \gamma$.

Theorem 21. *Let a domain $B \subset \mathbb{R}^n$ and a curve of directions γ satisfying the Kirillov-Tuy condition of order m be given. Then, for any field $f \in \mathcal{S}(S^m)$, $\operatorname{supp} f \subset B$, the Saint-Venant operator $Rf(x)$ (solenoidal part of the tensor field ${}^s f(x)$), $x \in B$ can be reconstructed after the X-ray transform known for lines parallel to the directions curve γ with explicit formula.*

The case of lines tangent to a surface—I

Definition 22. Fix a domain $B \subset \mathbb{R}^3$. Say that the surface W, $W \cap B = \emptyset$ satisfies the *Kirillov-Tuy condition of order* m, if for almost any hyper-plane $H_{\omega,p}=\{\, x \in \mathbb{R}^3 |\, \langle \omega, x \rangle = p \,\}$ intersecting the domain B, there is a set of curves Γ_1, . . . , $\Gamma_{m+1} \subset H_{\omega,p} \cap W$ satisfying the following:

1. For almost every $x \in B \cap H_{\omega,p}$, for every $i = 1, \ldots, m+1$ there exit uniquely defined points $\gamma_i \in \Gamma_i$ such that vector $x - \gamma_i$ is tangent to Γ_i.
2. The set of vectors $\{\, x - \gamma_i \,\}$, $i = 1, \ldots, m+1$ is generic.

The case of lines tangent to a surface—II

Theorem 23. *Let a domain $B \subset \mathbb{R}^3$ and a surface W satisfying the Kirillov-Tuy condition of order m be given. Then, for any field $f \in \mathcal{S}(S^m)$, $\operatorname{supp} f \subset B$, the Saint-Venant operator $Rf(x)$ (solenoidal part of the tensor field ${}^s f(x)$), $x \in B$ can be reconstructed after the cone-beam transform with explicit formula.*

The case of lines tangent to a surface—III

Lemma 24. *Let ν be a volume form on the manifold X. Let a function $\psi(x)$, $x \in X$ be given. Suppose that in a neighborhood of p_0 the surface $\Psi_p = \{x \in X | \psi(x) = p\}$ is a smooth sub-manifold. Then for any smooth vector field V and function f:*

$$\int_{\psi(x)=p_0} V f \frac{\nu}{d\psi} = \frac{\partial}{\partial p} \int_{\psi(x)=p} f \langle d\psi, V\rangle \frac{\nu}{d\psi}\Big|_{p=p_0} - \int_{\psi(x)=p_0} \operatorname{div} V \cdot f \frac{\nu}{d\psi}.$$

- $\nu / d\psi$ is the Leray form: $d\psi \wedge \chi = \nu$,
- $\operatorname{div} V = (\nu(x))^{-1} \sum \partial/(\partial x^i)\big(\nu(x) V^i\big)$, where $\nu = \nu(x)\, dx$.

Hyperbolic space

- Klein model $\{\, y | \|y\| \le 1 \,\} \subset \mathbb{R}^n$
- $ds^2 = \frac{(1-y^2)\sum dy_i^2 + (\sum y_i dy_i)^2}{(1-y^2)^2}$
- $K = -1$
- geodesics are straight line segments
- $I_H f(x,\xi) = \int_{-\infty}^{\infty} \left\langle f(\gamma_{x,\xi}(t), \dot{\gamma}^m_{x,\xi}(t) \right\rangle dt$—hyperbolic X-ray transform
- $I_E f(x,\xi)$—Euclidean X-ray transform
- $I_H f(x,\xi) = (1-x^2)^{1-m} I_E \tilde{f}(x,\xi)$, where $\tilde{f}(y) = (1-y^2)^{m-1} f(y)$

Hyperbolic space—inner differentiation

- $d_H : C^\infty(S^m) \to C^\infty(S^{m+1})$,

  $(d_H f)_{i_1 \cdots_{m+1}} = \sigma(i_1, \dots, i_{m+1}) \nabla_{i_{m+1}} f_{i_1 \dots i_m}$
- $d_H f = (1 - y^2)^{-m} d_E \left((1 - y^2)^m f \right)$
- $d_E f = (1 - y^2)^m d_H \left((1 - y^2)^{-m} f \right)$

Hyperbolic space—kernel description

Theorem 25 (Sharafutdinov). *Let* $n \geq 2, m \geq 0$, $f \in C_0^\infty(S^m; \mathbb{R}^n)$.

$I_E f = 0 \iff \exists v \in C_0^\infty(S^{m-1}; \mathbb{R}^n)$ *such that* $\operatorname{supp} v \subset \operatorname{conv} \operatorname{supp} f$ *and* $f = d_E v$

Theorem 26. *Let* $n \geq 2, m \geq 0$, $f \in C_0^\infty(S^m; \mathbb{H}^n)$.

$I_E f = 0 \iff \exists v \in C_0^\infty(S^{m-1}; \mathbb{H}^n)$ *such that* $\operatorname{supp} v \subset \operatorname{conv} \operatorname{supp} f$ *and* $f = d_H v$

Kernel description (proof)

- $I_H f = 0 \iff (1-x^2)^{m-1} I_E(f(1-y^2)^{m-1}) = 0$
- $\exists \tilde{v} \in C_0^\infty(S^{m-1};\mathbb{R}^n)$, such that $\operatorname{supp}\tilde{v} \subset \operatorname{conv}\operatorname{supp} f$ and
  $f(1-y^2)^{m-1} = d_E(\tilde{v}) = (1-y^2)^{m-1} d_H\big((1-y^2)^{1-m}\tilde{v}\big)$
- $f = d_H\big((1-y^2)^{1-m}v\big) = d_H v$ for $v = (1-y^2)^{1-m}\tilde{v}$

Elliptic space

- Stereographic projection from a Sphere $\mathbb{S}^n$ onto $\mathbb{R}^n$
- $ds^2 = \frac{(1+y^2)\sum dy_i^2 + (\sum y_i dy_i)^2}{(1+y^2)^2}$
- $K = 1$
- geodesics are straight lines
- $I_S f(x,\xi) = \int_{-\infty}^{\infty} \left\langle f(\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}^m(t)\right\rangle dt$—elliptic X-ray transform
- $I_E f(x,\xi)$—Euclidean X-ray transform
- $I_S f(x,\xi) = (1+x^2)^{1-m} I_E \tilde{f}(x,\xi)$, where $\tilde{f}(y) = (1+y^2)^{m-1} f(y)$