Interpolation procedure in filtered backprojection algorithm for the limited-angle tomography

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Abstract A new data completion procedure for the limited-angle tomography is proposed. The procedure is based on explicit integral interpolation formula for band-limited functions. The completed data can be used in standard filtered backprojection tomographical algorithm. The results of numerical simulation are presented.

1 Introduction

The limited-angle tomography and related problem of interpolation of band-limited function are of constant attention from early sixties till nowadays. It appears in many fields, including signal processing, medical imaging, geophysics, astronomy, electron microscopy. See [6, 5, 12, 8, 7] Mathematically it consists of reconstruction of a function f from its Radon transform $\hat{f}$ known for incomplete data of angles.

The problem is characterized with strong ill-posedness. The standard regularization procedure of minimizing the L^2 norm gives poor results [7]. So, different other approaches were developed for regularization, for instance [3, 11, 10, 9]. In this article we propose a new procedure of data completion, based on explicit integral formula of interpolation of a band-limited function [1]. Data, completed with our procedure, can be used in standard filtered backprojection tomographical algorithms.

To fix notations, let us introduce the standard parametrization (ω, p) on the set of lines on the plane:

$$L_{\omega,p} = \{\, \omega_1 x_1 + \omega_2 x_2 = p \,\},$$

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where $\omega = (\omega_1, \omega_2) = (\cos\varphi, \sin\varphi)$ is the unit normal vector of line $L_{\omega,p}$, and p is its distance from the origin.

The Radon transform is defined as follows:

$$\hat{f}(\omega, p) = \int_{L_{\omega,p}} f(x)\, ds = \int f\big(x_1^0 + \sin\varphi t, x_2^0 - \cos\varphi t\big)\, dt, \tag{1}$$

where $x^0 = (x_1^0, x_2^0)$ is a point on a line $L_{\omega,p}$, $x_1^0\omega_1 + x_2^0\omega_2 = p$. One can see that $\hat{f}(-\omega, -t) = -\hat{f}(\omega, t)$, so we restrict the domain of $\hat{f}$ to $\varphi \in [0, \pi]$.

The *limited angle problem* consists of reconstruction of a function $f(x)$ from its Radon transform $\hat{f}(\omega, p)$, given only for $\varphi \in [0, \frac{1}{2}(\pi - \varphi_0)] \cup [\frac{1}{2}(\pi + \varphi_0), \pi]$, v. Fig. 1, where cone of unknown normals is colored in gray.

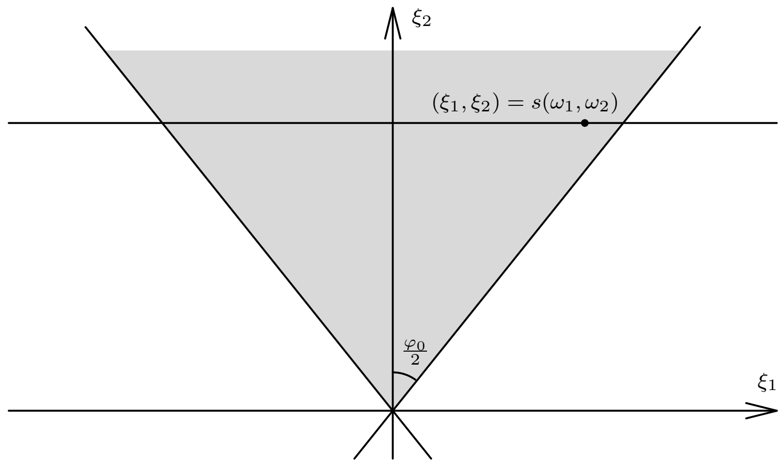

Fig. 1 Limited-angle problem. Unknown normals cone is shown in gray

We will use the following form of the Fourier transform and its inverse:

$$\tilde{f}(\xi) = (\sqrt{2\pi})^{-n} \int e^{-ix\xi} f(x)\, dx,$$
$$f(x) = (\sqrt{2\pi})^{-n} \int e^{ix\xi} \tilde{f}(\xi)\, d\xi,$$

where $x\xi = x_1\xi_1 + \cdots + x_n\xi_n$. Integration is performed over $\mathbb{R}^n$. We will consider only two-dimensional case, $n = 2$, but our results can be obviously generalized to higher dimensions.

Let us also recall the Fourier slice theorem ([2]):

$$\tilde{f}(s\omega) = (\sqrt{2\pi})^{-n} \int e^{-isp} \hat{f}(\omega, q)\, dq, \tag{2}$$

$$\hat{f}(\omega, p) = (\sqrt{2\pi})^{n-2} \int e^{isp} \tilde{f}(s\omega)\, ds, \tag{3}$$

We will assume that function f is supported into the unit ball, $\operatorname{supp} f \subset \mathbb{B} = \{\|x\|^2 \leq 1\}$.

The paper consists of four sections. In section 2 we obtain explicit analytical solution of the problem, based on the integral interpolation formula from [1]. In section 3 we apply this formula for reconstruction of the filtered Radon transform of f. In section 4 we discretize the obtained formula for using in the standard filtered backprojection reconstruction algorithm. Results of the numerical simulation are presented in section 5. We use a phantom similar to that of [7] for our simulations.

2 Analytical interpolation

Let us remind the theorem of integral interpolation of a band-limited function from [1].

Theorem 1 ([1]). *Let* $\psi \in L^2(\mathbb{R})$, $\operatorname{supp} \tilde{\psi} \subset [-1, 1]$. *Then for any* $\rho > 0$, $\xi \in (-\rho, \rho)$

$$\psi(\xi) = \frac{1}{\pi} e^{\sqrt{\rho^2 - \xi^2}} \int_{|\eta| > \rho} \frac{\sin \sqrt{\eta^2 - \rho^2}}{|\eta - \xi|} \psi(\eta)\, d\eta \tag{4}$$

Since function f is supported in the unit ball, the theorem 4 is applicable to $\psi(\xi) = \tilde{f}(\xi, \xi_2)$, where ξ_2 is fixed, i.e. $\tilde{f}$ is restricted onto the line $\xi_2 =$ const, Fig. 1. In that way it is possible to interpolate the Fourier transform to the whole plane. As a result one obtains the following theorem.

Theorem 2. *Let* $f \in L^2(\mathbb{R}^2)$, $\operatorname{supp} f \in \mathbb{B}$. *Then for any* $\omega = (\cos\varphi, \sin\varphi)$, *where* $\left|\frac{\pi}{2} - \varphi\right| < \frac{\varphi_0}{2}$, *the following interpolation formula is valid:*

$$\hat{f}(\omega_1, \omega_2, p) = \frac{1}{2\pi^2} \int ds \int\limits_{|\tau| > a} d\tau \int dq K_{\omega,p}(s, \tau, q) \hat{f}\left(\frac{\tau}{\sqrt{\tau^2 + \omega_2^2}}, \frac{\omega_2}{\sqrt{\tau^2 + \omega_2^2}}, q \right), \tag{5}$$

where $\omega = (\cos\varphi, \sin\varphi)$,

$$K_{\omega,p}(s, \tau, q) = \frac{\sin\left(s\sqrt{\tau^2 - a^2}\right)}{|\tau - \omega_1|} e^{is\left(p - q\sqrt{\tau^2 + \omega_2^2}\right) + |s|\sqrt{a^2 - \omega_1^2}},$$

$a = \omega_2 \tan(\varphi_0/2)$, φ_0 *characterizes the unknown angle range, Fig. 1*

Proof. To prove the theorem, one should start with the formula (3). Then substitute into it the interpolation formula (4). And, in turn, substitute the slice theorem formula (2) into the result. Finally, one should switch from η to τ coordinate with the following substitution $\eta = s\tau$ and take use of the equality $\rho = as$.

3 Filtration

The standard tomographical reconstruction algorithm consists of two steps: filtration and backprojection. We will interpolate the Radon transform of filtered function f_b, where

$$\tilde{f}_b(\xi) = \Phi_b(|\xi|)\tilde{f}(\xi),$$

parameter b characterizes resolution of reconstruction, [7]. Here $\Phi_b(s)$ is a filter function supported in $[-b, b]$.

Filtration implies that integration with respect to s in 5 is restricted only to finite interval $[-b, b]$, so the formula 5 will be transformed as follows:

Theorem 3. *Let* $f \in L^2(\mathbb{R}^2)$, $\operatorname{supp} f \in \mathbb{B}$. *Then for any* $\omega = (\cos\varphi, \sin\varphi)$ *where* $\left|\frac{\pi}{2} - \varphi\right| < \frac{\varphi_0}{2}$ *the following interpolation formula for the filtered Radon transform is valid:*

$$\hat{f}_b(\omega, p) = \frac{1}{\pi^2} \int_{|\tau|>a} d\tau \int dq K_{b,\omega,p}(\tau, q) \hat{f}\left(\frac{\tau}{\sqrt{\tau^2+\omega_2^2}}, \frac{\omega_2}{\sqrt{\tau^2+\omega_2^2}}, q\right), \quad (6)$$

where $\omega = (\cos\varphi, \sin\varphi)$,

$$K_{b,\omega,p}(\tau, q) = \int_0^b \Phi_b(s) e^{s\sqrt{a^2-\omega_1^2}} \cos s\left(p - q\sqrt{\tau^2+\omega_2^2}\right) \frac{\sin\left(s\sqrt{\tau^2-a^2}\right)}{|\tau-\omega_1|}\, ds, \quad (7)$$

$a = \omega_2 \tan(\varphi_0/2)$, φ_0 *characterizes the unknown angle range, Fig. 1*

4 Discrete interpolation

Now we will discretize the formula 6. Consider the standard parallel scanning scheme [7]. It means that the following data are given for $j = 0, \ldots P-1$, $l = -Q, \ldots, Q$:

$$g_{j,l} = \hat{f}(\cos\theta_j, \sin\theta_j, s_l), \text{ where } \theta_j = \pi j/P, s_l = hl, h = 1/Q.$$

Limited-angle restriction means that the data are given only for θ_j, satisfying $|\pi/2-\theta_j|>\varphi_0/2$, i.e. $j<P\frac{\pi-\varphi_0}{2\pi}$ and, $j>P\frac{\pi+\varphi_0}{2\pi}$.

Consider the following filer, parametrized by $\varepsilon\in[0,1]$ (cf. [7]):

$$\Phi_b(s)=\begin{cases}1-\varepsilon\frac{s}{b}, & 0\le s\le b,\\ 0, & s>b.\end{cases} \tag{8}$$

Integral in 7 can be calculated explicitly. So, the theorem 3 can be reformulated in the following way (cf. [1]):

Theorem 4. *Let $f\in L^2(\mathbb{R}^2)$, $\operatorname{supp} f\in\mathbb{B}$, filtration function be defined in (8). Then for any $\omega=(\cos\varphi,\sin\varphi)$ where $\left|\frac{\pi}{2}-\varphi\right|<\frac{\varphi_0}{2}$ the following interpolation formula for the filtered Radon transform is valid:*

$$\hat{f}_b(\omega,p)=\frac{1}{2\pi^2}\int_{|\tau|>a}d\tau\int\frac{dqK_{b,\omega,p}(\tau,q)}{|\tau-\omega_1|}\hat{f}\left(\frac{\tau}{\sqrt{\tau^2+\omega_2^2}},\frac{\omega_2}{\sqrt{\tau^2+\omega_2^2}},q\right), \tag{9}$$

where $\omega=(\cos\varphi,\sin\varphi)$,

$$K_{b,\omega,p}(\tau,q)=(1-\varepsilon)E_1(\alpha,\beta,\gamma)+\frac{\varepsilon}{b}E_2(\alpha,\beta,\gamma)\\ +\frac{\sin\nu_+}{\sqrt{\alpha^2+(\beta+\gamma)^2}}-\frac{\sin\nu_-}{\sqrt{\alpha^2+(\beta-\gamma)^2}}, \tag{10}$$

where

$$E_\lambda(\alpha,\beta,\gamma)=\frac{e^{\alpha b}\sin((\beta+\gamma)b-\lambda\nu_+)}{(\sqrt{\alpha^2+(\beta+\gamma)^2})^\lambda}-\frac{e^{\alpha b}\sin((\beta-\gamma)b-\lambda\nu_-)}{(\sqrt{\alpha^2+(\beta-\gamma)^2})^\lambda},$$

$\alpha=\sqrt{a^2-\omega_1^2}$, $\beta=p-q\sqrt{\tau^2+\omega_2^2}$, $\gamma=\sqrt{\tau^2-a^2}$, $\nu_\pm=\arctan((\beta\pm\gamma)/\alpha)$, $a=\omega_2\tan(\varphi_0/2)$, φ_0 *characterizes the unknown angle range, Fig. 1.*

Remark 1. The case $\varepsilon=0$ was considered in [1]. The present result has different form, more suitable for discretization.

Passing to discretization of this formula, let us make a substitution $\tau=\omega_2\cot\theta$ in the formula (9). It will be rewritten in the following way, which is opportune for direct discretization, using given data:

$$\hat{f}_b(\omega,p)=\frac{1}{2\pi^2}\int\limits_{|\frac{\pi}{2}-\theta|>\frac{\varphi_0}{2}}\int\frac{K_{b,\omega,p}(\omega_2\cot\theta,q)\,dqd\theta}{\sin\theta|\omega_2\cos\theta-\omega_1\sin\theta|}\hat{f}\left(\cos\theta,\sin\theta,q\right), \tag{11}$$

Note that the standard filtered backprojection algorithm reconstructs a function of essential band width b. So, the integral with respect to η in (4) should be restricted to finite segment $|\eta|<b$. As a corollary, the integral with

respect to τ in (9) should also be restricted to some finite segment. We use the segment $|\tau| < 1.5$. This threshold was found empirically.

Finally, the filtration kernel (10) has a large multiplier $e^{\alpha b}$, which increase computational errors. This multiplier depends on the direction ω. So, the filtration level could be chosen interdependently for each direction. We use $b = \ln(50)/\alpha$, so errors are increased up to 50 times..

Summarizing, we get the following modification of the standard filtered backprojection algorithm (cf. [7]):

1. For $j = 0, \dots, P-1$ compute $\psi_j = \frac{\pi j}{P}$
   a. if direction φ_j is given ($|\frac{\pi}{2} - \psi_j| > \frac{\varphi_0}{2}$), compute convolutions
   $$v_{j,k} = 1/Q \sum_{l=-Q}^{Q} w_b(s_k - s_l)\hat{f}(\cos\psi_j, \sin\psi_j, s_l), \quad k = -Q, \dots, Q$$
   where $w_b(s)$ is the filtration kernel corresponding to (7). Explicit expression of w_b can be found in [7].
   b. else (direction is missing) compute interpolated convolutions
   $$v_{j,k} = \frac{1}{2\pi QP} \sideset{}{'}\sum_{i=0}^{P-1} \sum_{l=-Q}^{Q} \frac{K_{b,\omega,p_k}(\omega_2 \cot\theta_i, s_l)}{\sin\theta_i |\omega_2 \cos\theta_i - \omega_1 \sin\theta_i|} \hat{f}(\cos\theta_i, \sin\theta_i, s_l),$$
   where $k = -Q, \dots, Q$, $\omega = (\cos\psi_j, \sin\psi_j)$, kernel K is given in (10), summation with respect to i is performed only for those i, that satisfy $1.5\omega_2 < |\tan\theta_i|$ and $\left|\frac{\pi}{2} - \theta_i\right| > \frac{\varphi_0}{2}$.
2. At each point $x = (x_1, x_2)$ of reconstruction interpolated discrete back-projections are computed:
   $$f_{bi}(x) = \frac{2\pi}{P} \sum_{j=0}^{P-1} \big((1-u)v_{j,k} + uv_{j,k+1}\big),$$
   where u and k for each pair of x and j are defined by the following formulas:
   $$s = x_1 \cos\theta_j + x_2 \sin\theta_j, k \le sQ < k+1, u = sQ - k.$$

Remark 2. The proposed algorithm can be used with any other filer $F_b(\xi)$ (such that $F_b(\xi) = 0$ for $|\xi| > b$). If we set in (7) $\varepsilon = 0$, composition of filtrations with (7) and with F_b is equivalent to filtration only with F_b. So, discrete interpolation should be followed by discrete filtration with F_b.

5 Numerical simulation

The results of numerical simulation are presented at the figure 2. We use a phantom similar to that of [7]. The original function has value 2 inside the ellipsoidal ring and 0.8 inside the smaller ellipse. The value of ε in filter (8) is equal to 1. There were 72 directions and 227 lines in each direction used.

In limited-angle problem we supposed that interval of 30° in unknown, i.e. 14 directional data are missing.

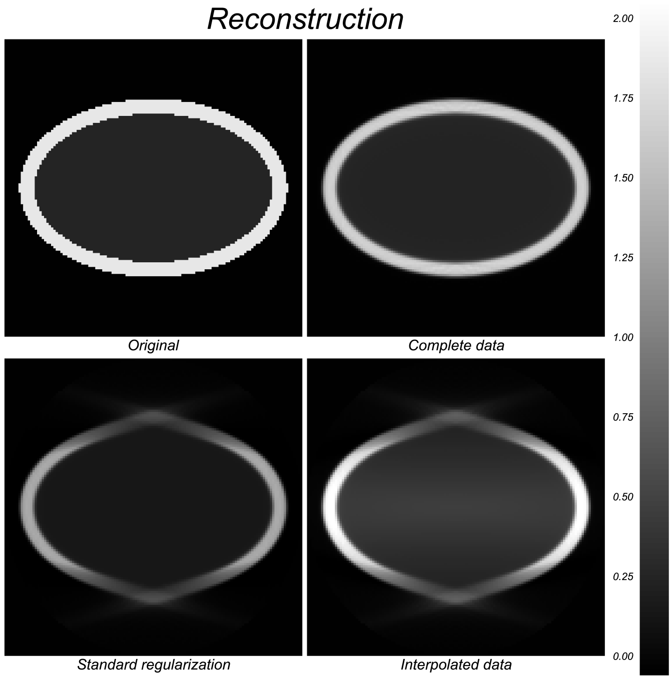

Fig. 2 Original function (top-left) and reconstruction: complete data (top-right), standard regularization data (bottom-left) an interpolated data (bottom-right). Unknown interval is 30°. The original function has value 2 inside the ellipsoidal ring and 0.8 inside the smaller ellipse.

Comparing errors of reconstruction, given at the figure 3, one can see significant improvement of reconstruction inside the object. At the same time outer artifact where weakened only slightly. Note that from tomographical point of view the interior of the object is the matter of interest.

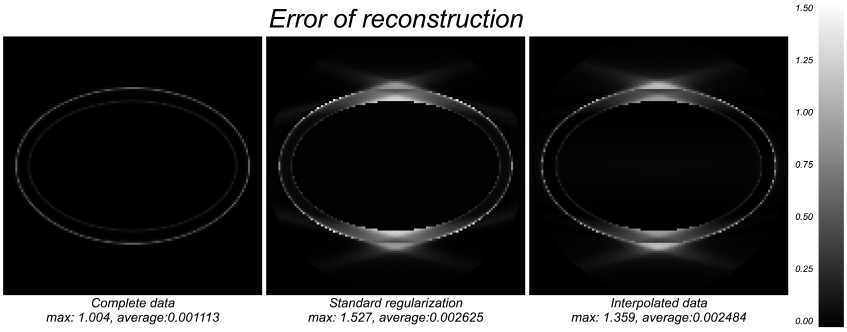

Fig. 3 Error of reconstructions (left-to-right): complete data, standard regularization date, interpolated data

6 Conclusions

We proposed a new universal data completion procedure for the limited-angle tomography. Our procedure can be used in the standard filtered backprojection reconstruction algorithm. The numerical simulation demonstrates that data completion allows to decrease some artifacts and improve reconstruction.

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