Existence of Constant Scalar Curvature Sasaki Structures on Sasaki Join Manifolds.

Joint work with Charles Boyer, Hongnian Huang and Eveline Legendre

Christina Tønnesen-Friedman

Workshop on almost hermitian and contact geometry
The Palace in Bedlewo, Poland
October, 2015

Plan for this talk:

In this talk we look at the existence of constant scalar curvature Sasaki structures (cscS). We show that the modified Einstein–Hilbert functional detects the Sasaki-Futaki invariant. For certain so-called Sasaki join manifolds, we then apply this result to provide an **explicit, computable, necessary**, and **sufficient** condition for the existence of cscS within a certain sub cone of the Sasaki cone.

Plan:

1. Kähler geometry and Scalar Curvature
2. Sasakian geometry and Scalar Curvature
3. The join construction in Sasakian geometry and cscS metrics
4. The Einstein-Hilbert functional and Sasaki-Futaki invariant
5. Main Result

Kähler Geometry

Let N be a smooth compact manifold of real dimension $2d_N$.

- If J is a smooth bundle-morphism on the real tangent bundle, $J : TN \to TN$ such that $J^2 = -Id$ and $\forall X, Y \in TN$

$$J(\mathcal{L}_X Y) - \mathcal{L}_X JY = J(\mathcal{L}_{JX} JY - J\mathcal{L}_{JX} Y),$$

  then (N, J) is a **complex manifold** with **complex structure** J.
- A Riemannian metric g on (N, J) is said to be a **Hermitian Riemannian metric** if

$$\forall X, Y \in TN, \quad g(JX, JY) = g(X, Y)$$

- This implies that $\omega(X,Y) := g(JX,Y)$ is a $J-$ invariant $(\omega(JX,JY) = \omega(X,Y))$ non-degenerate $2-$ form on N.
- If $d\omega = 0$, then we say that (N,J,g,ω) is a **Kähler manifold** (or **Kähler structure**) with **Kähler form** ω and **Kähler metric** g.
- The second cohomology class $[\omega]$ is called the **Kähler class**.
- For fixed J, the subset in $H^2(N,\mathbb{R})$ consisting of Kähler classes is called the **Kähler cone**.

Scalar Curvature of Kähler metrics:

Given a Kähler structure (N, J, g, ω), the Riemannian metric g defines (via the unique Levi-Civita connection ∇)

- the **Riemann curvature tensor** $R : TN \otimes TN \otimes TN \to TN$
- and the trace thereoff, the **Ricci tensor** $r : TN \otimes TN \to C^{\infty}(N)$

- This gives us the **Ricci form**, $\rho(X, Y) = r(JX, Y)$.

- If $\rho = \lambda\omega$, where λ is some constant, then we say that (N, J, g, ω) is **Kähler-Einstein** (or just **KE**).
- the **scalar curvature**, $Scal \in C^\infty(N)$, where $Scal$ is the trace of the map $X \mapsto \tilde{r}(X)$ where $\forall X, Y \in TN, g(\tilde{r}(X), Y) = r(X, Y)$.
- If $Scal$ is a constant function, we say that (N, J, g, ω) is a constant scalar curvature Kähler metric (or just **CSC**).
- KE $\implies$ CSC (with $\lambda = \frac{Scal}{2d_N}$)
- Not all complex manifolds (N, J) admit CSC Kähler structures.

Admissible Kähler manifolds/orbifolds

- Special cases of the more general (admissible) constructions defined by/organized by Apostolov, Calderbank, Gauduchon, and T-F.
- Credit goes to **Calabi**, Koiso, Sakane, Simanca, Pedersen, Poon, Hwang, Singer, Guan, LeBrun, and others.

- Let ω_N be a primitive integral Kähler form of a CSC Kähler metric on (N, J).
- Let $\mathbb{1} \to N$ be the trivial complex line bundle.
- Let $n \in \mathbb{Z} \setminus \{0\}$.
- Let $L_n \to N$ be a holomorphic line bundle with $c_1(L_n) = [n\,\omega_N]$.
- Consider the total space of a projective bundle $S_n = \mathbb{P}(\mathbb{1} \oplus L_n) \to N$.
- Note that the fiber is $\mathbb{CP}^1$.
- S_n is called **admissible**, or an **admissible manifold**.

Admissible Kähler classes

- Let $D_1 = [\mathbb{1} \oplus 0]$ and $D_2 = [0 \oplus L_n]$ denote the "zero" and "infinity" sections of $S_n \to N$.
- Let r be a real number such that $0 < |r| < 1$, and such that $r\,n > 0$.
- A Kähler class on S_n, Ω, is **admissible** if (up to scale) $\Omega = \frac{2\pi n[\omega_N]}{r} + 2\pi PD(D_1 + D_2)$.
  ("PD" = Poincare Dual)
- In general, the **admissible cone** is a sub-cone of the Kähler cone.

- In each admissible class we can now construct **explicit** Kähler metrics g (called **admissible Kähler metrics**).
- We can generalize this construction to the log pair (S_n, Δ), where Δ denotes the branch divisor $\Delta = (1 - 1/m_1)D_1 + (1 - 1/m_2)D_2$.
- If $m = \gcd(m_1, m_2)$, then (S_n, Δ) is a fiber bundle over N with fiber $\mathbb{CP}^1[m_1/m, m_2/m]/\mathbb{Z}_m$.
- g is smooth on $S_n \setminus (D_1 \cup D_2)$ and has orbifold singularities along D_1 and D_2
- This gives enough flexibility to produce CSC examples.

Kähler orbifolds naturally lead to...

Sasakian Geometry:

Odd dimensional version of Kählerian geometry and special case of **contact structure**.

A Sasakian structure on a smooth manifold M of dimension $2n+1$ is defined by a quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ where

- η is **contact 1-form** defining a subbundle (contact bundle) in TM by $\mathcal{D} = \ker \eta$.
- ξ is the **Reeb vector field** of η $[\eta(\xi) = 1$ and $\xi \rfloor d\eta = 0]$
- Φ is an endomorphism field which annihilates ξ and satisfies $J = \Phi|_{\mathcal{D}}$ is a complex structure on the contact bundle $(d\eta(J\cdot, J\cdot) = d\eta(\cdot, \cdot))$

- $g := d\eta \circ (\Phi \otimes \mathbb{1}) + \eta \otimes \eta$ is a Riemannian metric
- ξ is a Killing vector field of g which generates a one dimensional foliation $\mathcal{F}_\xi$ of M whose transverse structure is Kähler.
- we let (g_T, ω_T) denote the transverse Kähler metric

- If ξ is **regular**, the transverse Kähler structure lives on a smooth manifold (quotient of regular foliation $\mathcal{F}_\xi$).
- If ξ is **quasi-regular**, the transverse Kähler structure has orbifold singularities (quotient of quasi-regular foliation $\mathcal{F}_\xi$).
- If not regular or quasi-regular we call it **irregular**... (that's most of them)

Transverse Homothety:

- If $\mathcal{S} = (\xi, \eta, \Phi, g)$ is a Sasakian structure, so is $\mathcal{S}_a = (a^{-1}\xi, a\eta, \Phi, g_a)$ for every $a \in \mathbb{R}^+$ with $g_a = ag + (a^2 - a)\eta \otimes \eta$.
- So Sasakian structures come in rays.

Deforming the Sasaki structure:

In its contact structure isotopy class:

- $$\eta \to \eta + d^c\phi, \quad \phi \quad \text{is basic}$$
- This corresponds to a deformation of the transverse Kähler form
$$\omega_T \to \omega_T + dd^c\phi$$
in its Kähler class in the regular/quasi-regular case.
- "Up to isotopy" means that the Sasaki structure might have to been deformed as above.

In the Sasaki Cone:

- Choose a maximal torus T^k, $0 \leq k \leq n+1$ in the Sasaki automorphism group
$$\mathfrak{Aut}(\mathcal{S}) = \{\phi \in \mathcal{D}\mathit{iff}(M) \,|\, \phi^*\eta = \eta,\, \phi^*J = J,\, \phi^*\xi = \xi,\, \phi^*g = g\}.$$
- The unreduced Sasaki cone is $\mathfrak{t}^+ = \{\xi' \in \mathfrak{t}_k \,|\, \eta(\xi') > 0\}$, where $\mathfrak{t}^k$ denotes the Lie algebra of T^k.
- Each element in $\mathfrak{t}^+$ determines a new Sasaki structure with the same underlying CR-structure.

Scalar Curvature of Sasaki metrics

- The scalar curvature of g behaves as follows
$$Scal = Scal_T - 2n$$
- $\mathcal{S} = (\xi, \eta, \Phi, g)$ has constant scalar curvature (CSC) if and only if the transverse Kähler structure has constant scalar curvature.
- $\mathcal{S} = (\xi, \eta, \Phi, g)$ has CSC iff its entire ray has CSC ("CSC ray").
- We will say that $\mathcal{S} = (\xi, \eta, \Phi, g)$
is CSC whenever it is CSC up to isotopy.

The Join Construction

- The join construction of Sasaki manifolds (Boyer, Galicki, Ornea) is the analogue of Kähler products.
- Given quasi-regular Sasakian manifolds $\pi_i : M_i \to \mathcal{Z}_i$. Let $L = \frac{1}{2l_1}\xi_1 - \frac{1}{2l_2}\xi_2$ be viewed as a vector field on $M_1 \times M_2$.
- Form (l_1, l_2)- join by taking the quotient by the action induced by L:

$$\begin{tikzcd}
M_1 \times M_2 \arrow[d, "\pi_{12}"] \arrow[dr, "\pi_L"] & \\
\mathcal{Z}_1 \times \mathcal{Z}_2 & M_1 \star_{l_1,l_2} M_2 \arrow[l, "\pi"]
\end{tikzcd}$$

- $M_1 \star_{l_1,l_2} M_2$ is a S^1-orbibundle (generalized Boothby-Wang fibration).
- $M_1 \star_{l_1,l_2} M_2$ has a natural quasi-regular Sasakian structure for all relatively prime positive integers l_1, l_2. Fixing l_1, l_2 fixes the contact orbifold. It is a smooth manifold iff $\gcd(\mu_1 l_2, \mu_2 l_1) = 1$, where μ_i is the order of the orbifold $\mathcal{Z}_i$.

Join with a weighted 3-sphere

- Take $\pi_2 : M_2 \to \mathcal{Z}_2$ to be the S^1-orbibundle
$$\pi_2 : S^3_{\mathbf{w}} \to \mathbb{CP}[\mathbf{w}]$$
determined by a weighted S^1-action on S^3 with weights $\mathbf{w} = (w_1, w_2)$ such that $w_1 \geq w_2$ are relative prime.
- $S^3_{\mathbf{w}}$ has an extremal Sasakian structure with transverse Kähler form $\omega_{\mathbf{w}}$ on $\mathbb{CP}[\mathbf{w}]$ satisfying $[\omega_{\mathbf{w}}] = \frac{[\omega_0]}{w_1 w_2}$, where ω_0 is the standard Fubini-Study volume form on $\mathbb{CP}^1$.
- Let $M_1 = M$ be a regular CSC Sasaki manifold whose quotient is a unit volume compact CSC Kähler manifold N with scalar curvature equal to $\frac{A}{4\pi}$.
- Assume $\gcd(l_2, l_1 w_1 w_2) = 1$ (equivalent with $\gcd(l_2, w_i) = 1$).

- $$\begin{tikzcd}[column sep=small]
M \times S^3_{\mathbf{w}} \arrow[dd, "\pi_{12}"] \arrow[dr, "\pi_L"] & \\
& M \star_{l_1,l_2} S^3_{\mathbf{w}} =: M_{l_1,l_2,\mathbf{w}} \arrow[dl, "\pi"] \\
N \times \mathbb{CP}[\mathbf{w}] &
\end{tikzcd}$$

The w-Sasaki cone

- The Lie algebra $\mathfrak{aut}(\mathcal{S}_{l_1,l_2,\mathbf{w}})$ of the automorphism group of the join satisfies $\mathfrak{aut}(\mathcal{S}_{l_1,l_2,\mathbf{w}}) = \mathfrak{aut}(\mathcal{S}_1) \oplus \mathfrak{aut}(\mathcal{S}_{\mathbf{w}})$, mod $(L_{l_1,l_2,\mathbf{w}} = \frac{1}{2l_1}\xi_1 - \frac{1}{2l_2}\xi_2)$, where $\mathcal{S}_1$ is the Sasakian structure on M, and $\mathcal{S}_{\mathbf{w}}$ is the Sasakian structure on $S^3_{\mathbf{w}}$.
- The unreduced Sasaki cone $\mathfrak{t}^+_{l_1,l_2,\mathbf{w}}$ of the join $M_{l_1,l_2,\mathbf{w}}$ thus has a 2-dimensional subcone $\mathfrak{t}^+_{\mathbf{w}}$ is called the **w**-Sasaki cone.
- $\mathfrak{t}^+_{\mathbf{w}}$ is inherited from the Sasaki cone on S^3

- Each Reeb vector field in $\mathfrak{t}^+_{\mathbf{w}}$ is determined by a choice of $(v_1, v_2) \in \mathbb{R}^+ \times \mathbb{R}^+$. Indeed, $\xi_{\mathbf{v}} = v_1 H_1 + v_2 H_2$, where H_i is the restriction to $M_{l_1, l_2, \mathbf{w}}$ of the rotation $z_i \mapsto e^{i\theta} z_i$ on S^3.
- The ray of $\xi_{\mathbf{v}}$ is quasi-regular iff $v_2/v_1 \in \mathbb{Q}$.
- $\mathfrak{t}^+_{\mathbf{w}}$ has a regular ray (given by $(v_1, v_2) = (1, 1)$) iff l_2 divides $w_1 - w_2$.

Motivating Question

- Does $\mathfrak{t}_{\mathbf{w}}^{+}$ have a CSC ray?

Key Proposition (Boyer, T-F)

Let $M_{l_1,l_2,\mathbf{w}} = M \star_{l_1,l_2} S^3_{\mathbf{w}}$ be the join as described above.
Let $\mathbf{v} = (v_1, v_2)$ be a weight vector with relatively prime integer components and let $\xi_{\mathbf{v}}$ be the corresponding Reeb vector field in the Sasaki cone $\mathfrak{t}^+_{\mathbf{w}}$.
Then the quotient of $M_{l_1,l_2,\mathbf{w}}$ by the flow of the Reeb vector field $\xi_{\mathbf{v}}$ is (S_n, Δ)
with $n = l_1\left(\frac{w_1v_2 - w_2v_1}{s}\right)$, where $s = \gcd(l_2, w_1v_2 - w_2v_1)$, and the branch divisor $\Delta = (1 - \frac{1}{m_1})D_1 + (1 - \frac{1}{m_2})D_2$, with ramification indices $m_i = v_i\frac{l_2}{s}$.

The Kähler class on the (quasi-regular) quotient

- is admissible up to scale (when $(v_1, v_2) \neq (w_1, w_2)$).
- We can determine exactly which one it is.
- So we can test it for containing an admissible CSC Kähler metrics.
- Hence we can test if the ray of $\xi_{\mathbf{v}}$ is admissible CSC (up to isotopy).
- By lifting the admissible construction to the Sasakian level (in a way so it depends smoothly on (v_1, v_2)), we can also handle the irregular rays.
- In fact,

Proposition (Boyer, T-F)

Consider a ray in the **w**-cone determined by a choice of $b = v_2/v_1 > 0$. Then the Sasakian structures of the ray has admissible CSC metrics (up to isotopy) if and only if $f_{CSC}(b) = 0$, where $f_{CSC}(b) = \frac{-f(b)}{(w_1 b - w_2)^3}$ and $f(b)$ is a polynomial given by:

$$
\begin{array}{ll}
- & (d_N+1)l_1 w_1^{2d_N+3} b^{2d_N+4} \\
+ & w_1^{2(d_N+1)} b^{2d_N+3}(Al_2 + l_1(d_N+1)w_2) \\
- & w_1^{d_N+2} w_2^{d_N} b^{d_N+3}((d_N+1)(A(d_N+1)l_2 - l_1((d_N+1)w_1 + (d_N+2)w_2))) \\
+ & w_1^{d_N+1} w_2^{d_N+1} b^{d_N+2}(2Ad_N(d_N+2)l_2 - (d_N+1)(2d_N+3)l_1(w_1+w_2)) \\
- & w_1^{d_N} w_2^{d_N+2} b^{d_N+1}(d_N+1)(A(d_N+1)l_2 - l_1((d_N+2)w_1 + (d_N+1)w_2)) \\
+ & w_2^{2(d_N+1)}(b(Al_2 + l_1(d_N+1)w_1)) \\
- & (d_N+1)l_1 w_2^{2d_N+3}.
\end{array}
$$

$f(b)$ has a root of order three at $b = w_2/w_1$ when $w_1 > w_2$ and order at least four when $w_1 = w_2 = 1$ (where the case of $b = w_2/w_1 = 1$ gives a product transverse CSC structure). Thus $f_{CSC}(b)$ is a polynomial of order $2d_N + 1$ with positive roots corresponding to the rays in the **w**-cone that admit admissible CSC metrics.

Theorem (Boyer, T-F)

- For each vector $\mathbf{w} = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ with relatively prime components satisfying $w_1 > w_2$ there exists a Reeb vector field $\xi_{\mathbf{v}}$ in the 2-dimensional $\mathbf{w}$-Sasaki cone on $M_{l_1,l_2,\mathbf{w}}$ such that the corresponding ray of Sasakian structures $\mathcal{S}_a = (a^{-1}\xi_{\mathbf{v}}, a\eta_{\mathbf{v}}, \Phi, g_a)$ has constant scalar curvature.
- Suppose in addition that the scalar curvature of N is positive. Then for sufficiently large l_2 there are at least three CSC rays in the $\mathbf{w}$-Sasaki cone of the join $M_{l_1,l_2,\mathbf{w}}$.

Remark: When $N = \mathbb{CP}^1$, $M_{l_1,l_2,\mathbf{w}}$ are S^3-bundles over S^2. These were treated by Boyer and Boyer, Pati, as well as by E. Legendre.

Motivating Question

- Within the **w**-Sasaki cone of $M_{l_1,l_2,\mathbf{w}}$, how restrictive was it to look for **admissible** CSC Sasaki structures?
- Collaborating with Eveline Legendre and Hongnian Huang, we found that....
- it is not restrictive at all!

The Sasaki-Futaki invariant

For a given Kähler class, the existence of a CSC Kähler representative is obstructed by the so-called Futaki Invariant (Futaki 1983).

Likewise, there is a Sasaki version of this invariant, the so-called Sasaki-Futaki Invariant or Transversal Futaki Invariant (Boyer, Galicki, Simanca, Futaki, Ono, Wang, 2008–09)

$$\mathbf{F}_\xi : \text{Lie algebra of transverse holomorphic vector fields} \longrightarrow \mathbb{R}$$

- The vanishing of $\mathbf{F}_\xi$ is a necessary, but in general not sufficient, condition for the existence of a CSC Sasaki structure in the space of Sasaki structures with fixed Reeb vector field ξ and fixed transverse holomorphic structure.
- In particular, $\mathbf{F}_\xi$ is an obstruction to the existence of CSC Sasaki structure in the isotopy class of a given Sasaki structure.

The (modified) Einstein-Hilbert functional

For a given Sasaki structure, let $\mathbf{V}_\xi$ denoted the volume of the Sasaki metric and let $\mathbf{S}_\xi$ denoted the total transversal scalar curvature.
We define the Einstein–Hilbert functional

$$\mathbf{H}(\xi) = \frac{\mathbf{S}_\xi^{n+1}}{\mathbf{V}_\xi^{n}} \tag{1}$$

as a functional on the Sasaki cone.

- Note that $\mathbf{H}$ is homogeneous since the rescaling $\xi \mapsto \frac{1}{\lambda}\xi$ gives

$$\lambda^{n+1} dv_g \text{ and } \frac{1}{\lambda} Scal_T .$$

- Note also that $\mathbf{H}(\xi)$ only depends on the isotopy class of the Sasaki structure.

A Lemma linking the SF-invariant and the EH functional (Boyer, Huang, Legendre, T-F)

Given $a \in T_\xi \mathfrak{t}^+$, we have

$$d\mathbf{H}_\xi(a) = \frac{n(n+1)\mathbf{S}_\xi^n}{\mathbf{V}_\xi^n}\mathbf{F}_\xi(\Phi(a)).$$

The EH-functional on $M_{l_1,l_2,\mathbf{w}}$

On the manifolds $M_{l_1,l_2,\mathbf{w}}$ with $b = v_2/v_1 \neq w_2/w_1$ - up to an overall positive constant rescale that does not depend on (v_1, v_2) - the Einstein-Hilbert functional restricted to the **w**-cone takes the form

$$\mathbf{H}(b) \quad = \quad \frac{\left(l_1 w_1^{d_N+1} b^{d_N+2} + (l_2 A - l_1 w_2) w_1^{d_N} b^{d_N+1} + (l_1 w_1 - l_2 A) w_2^{d_N} b - l_1 w_2^{d_N+1}\right)^{d_N+2}}{(w_1 b - w_2)\left(w_1^{d_N+1} b^{d_N+2} - w_2^{d_N+1} b\right)^{d_N+1}}.$$

Furthermore, we have the boundary behavior

$$\lim_{b\to 0} \mathbf{H}(b) = +\infty, \qquad \lim_{b\to +\infty} \mathbf{H}(b) = +\infty.$$

Note that $\mathbf{H}(b) = \mathbf{H}(H_1 + bH_2) = \mathbf{H}(v_1 H_1 + v_2 H_2)$.

Derivative of $\mathbf{H}(b)$

If we consider $a = H_2$, then $d\mathbf{H}_{\xi_{\mathbf{v}}}(a)$ is just $H'(b)$.

We calculate that

$$\begin{aligned}
\mathbf{H}'(b) &= \frac{\left(v_1^{d_N+1} b^{d_N+1} \int_{M_{l_1,l_2,\mathbf{w}}} Scal^T \, dv_{g_{\mathbf{v}}}\right)^{d_N+1} f_{CSC}(b)}{b^{d_N+2}\left(v_1^{d_N+2} b^{d_N+1} \int_{M_{l_1,l_2,\mathbf{w}}} dv_{g_{\mathbf{v}}}\right)^{d_N+2}} \\
&= \frac{\left(\int_{M_{l_1,l_2,\mathbf{w}}} Scal^T \, dv_{g_{\mathbf{v}}}\right)^{d_N+1} f_{CSC}(b)}{(bv_1)^{2d_N+3}\left(\int_{M_{l_1,l_2,\mathbf{w}}} dv_{g_{\mathbf{v}}}\right)^{d_N+2}}. \\
&= \frac{(\mathbf{S}_{\xi_{\mathbf{v}}})^{d_N+1} f_{CSC}(b)}{(bv_1)^{2d_N+3}(\mathbf{V}_{\xi_{\mathbf{v}}})^{d_N+2}},
\end{aligned}$$

which with $n = d_N + 1$ equals

$$\frac{\mathbf{S}_{\xi_{\mathbf{v}}}^{n} f_{CSC}(b)}{(bv_1)^{2n+1}\mathbf{V}_{\xi_{\mathbf{v}}}^{n+1}}$$

The Link between the SF-invariant and $f_{CSC}(b)$

So, on $M_{l_1,l_2,\mathbf{w}}$ in the $\mathbf{w}$-Sasaki cone, for $a = H_2$, up to an overall positive constant rescale that does not depend on (v_1, v_2), we have

$$\frac{n(n+1)\mathbf{S}^n_{\xi_{\mathbf{v}}}}{\mathbf{V}^n_{\xi_{\mathbf{v}}}}\mathbf{F}_\xi(\Phi(a)) = d\mathbf{H}_{\xi_{\mathbf{v}}}(a) = \frac{\mathbf{S}^n_{\xi_{\mathbf{v}}} f_{CSC}(b)}{(bv_1)^{2n+1}\mathbf{V}^{n+1}_{\xi_{\mathbf{v}}}}$$

Since, as one may check, $\mathbf{S}^n_{\xi_{\mathbf{v}}}$ is a smooth rational function with only isolated zeroes and f_{CSC} as well as the Sasaki-Futaki invariant varies smoothly in the Sasaki-cone, we conclude that, up to a positive multiple,

$f_{CSC}(b)$ **represents the value of the Sasaki-Futaki invariant $\mathbf{F}_{\xi_{\mathbf{v}}}$ in a specific direction transversal to the rays.**

Thus in this case, **the vanishing of the Sasaki-Futaki invariant actually does imply the existence of cscS metrics for the given ray**.

In cases where the **w**-cone is the entire Sasaki cone we have that the existence of cscS metrics is equivalent to the vanishing of $f_{CSC}(b)$ and hence with the vanishing of $\mathbf{F}_{\xi}$.

Main Result (Boyer, Huang, Legendre, T-F)

Consider a ray in the **w**-cone of $M_{l_1,l_2,\mathbf{w}}$ determined by a choice of $b = v_2/v_1 > 0$. Then the Sasakian structures of the ray has ~~admissible~~ CSC metrics (up to isotopy) if and only if $f_{CSC}(b) = 0$, where $f_{CSC}(b) = \frac{-f(b)}{(w_1 b - w_2)^3}$ and $f(b)$ is a polynomial given by:

$$bla \quad bla \quad bla$$

$f(b)$ has a root of order three at $b = w_2/w_1$ when $w_1 > w_2$ and order at least four when $w_1 = w_2 = 1$ (where the case of $b = w_2/w_1 = 1$ gives a product transverse CSC structure). Thus $f_{CSC}(b)$ is a polynomial of order $2d_N + 1$ with positive roots corresponding to the rays in the **w**-cone that admit ~~admissible~~ CSC metrics.

Concluding Remarks

- Is it possible to prove the main result directly without using the EH-functional?
- Maybe, but a lot of "useful facts" available in Kähler geometry are still "under construction" for the Sasaki case.
- The EH-functional is also related to a stability notion.
- Next Question: Within the **w**-Sasaki cone of $M_{l_1,l_2,\mathbf{w}}$, how restrictive is it to look for **admissible** extremal Sasaki structures?

Thank You For Your Attention

References

- **Apostolov, Calderbank, Gauduchon, and T-F.** Hamiltonian 2-forms in Kähler geometry, III *Extremal metrics and stability*, Inventiones Mathematicae 173 (2008), 547–601.
  For the "admissible construction" of Kähler metrics

- **Boyer and Galicki** Sasakian geometry, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2008.
- Other papers by **Boyer** et all.
  For the "join" of Sasaki structures

- **Boyer and T.-F.** The Sasaki Join, Hamiltonian 2-forms, and Constant Scalar Curvature (to appear in JGA, 2015) and references therein to our previous papers.
- **Boyer, Huang, Legendre, and T.-F.** The Einstein-Hilbert functional and the Sasaki-Futaki Invariant, to appear in IMRN.
  For the details and proofs behind the statements in this talk.