Locally conformally Berwald manifolds and compact quotients of reducible manifolds by homotheties

V.Matveev [1] Y. Nikolayevsky [2]

[1]Institute of Mathematics, Friedrich-Schiller-Universität, Jena, Germany

[2]La Trobe University, Melbourne, Australia

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Starting point:

Theorem (Matveev-Troyanov, 2012)

A connected closed conformally flat non-Riemannian Finsler manifold is either a Bieberbach manifolds or a Hopf manifolds. In particular, it is finitely covered either by a torus $\mathbb{T}^n$ or by $S^{n-1} \times S^1$.

"Conformally flat" means "locally conformally Minkowski".
We wanted to extend this to the "next simplest" class of Finsler manifolds – Berwald manifolds.

Question: is the following true?

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Definition

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1. $F_x(\lambda \cdot \xi) = \lambda \cdot F_x(\xi)$ for any $\lambda \geqslant 0$.
2. $F_x(\xi + \eta) \leqslant F_x(\xi) + F_x(\eta)$.
3. $F_x(\xi) = 0 \iff \xi = 0$.

Examples:

- Riemannian: $F_x(\xi) = \sqrt{g_{ij}(x)\xi^i\xi^j}$.
- Minkowski: take a Minkowski norm F_0 on $\mathbb{R}^n$ and define the Finsler metric on $\mathbb{R}^n$ by $F_x(\xi) := F_0(\xi)$ for $x \in \mathbb{R}^n$; plays the same role in Finsler settings as the Euclidean metric in Riemannian settings.

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A Finsler metric F is *Berwald*, if there exists a torsion-free affine connection ∇ on M whose parallel transport preserves F.

Examples:

1. Riemannian: ∇ is the Levi-Civita connection.
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$$F((x_1, x_2), (\xi_1, \xi_2)) = N(F_1(x_1, \xi_1), F_2(x_2, \xi_2)).$$

   This can be naturally generalised to the Cartesian product of any number of manifolds.

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Theorem (Szabó, 1981)

- *(Metrisability) Any Berwald connection is a Levi-Civita connection of some Riemannian metric.*
- *(local de Rham) Any Berwald manifold is locally the Cartesian product (in the sense of Example 3) of Riemannian manifolds, Minkowski spaces and symmetric spaces of rank $\geqslant 2$.*

Any of these factors may be absent.
"Symmetric space" means that the space has the same reduced holonomy; those were completely classified.
In particular, if the reduced holonomy group is the whole $\mathrm{SO}(n)$, then the Berwald space is Riemannian.

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Idea: Binet-Legendre metric (introduced in Centore, 1999; used in Matveev-Troyanov, 2012).

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Idea: Binet-Legendre metric (introduced in Centore, 1999; used in Matveev-Troyanov, 2012).

Theorem (Szabó, 1981)

- *(Metrisability) Any Berwald connection is a Levi-Civita connection of some Riemannian metric.*
- *(local de Rham) Any Berwald manifold is locally the Cartesian product (in the sense of Example 3) of Riemannian manifolds, Minkowski spaces and symmetric spaces of rank $\geqslant 2$.*

Any of these factors may be absent.
"Symmetric space" means that the space has the same reduced holonomy; those were completely classified.
In particular, if the reduced holonomy group is the whole $\mathrm{SO}(n)$, then the Berwald space is Riemannian.

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Definition

Locally (globally) conformally Berwald.

Example

Consider a Minkowski metric F on $\mathbb{R}^n$. And consider the mapping

$$\alpha : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}, \quad x \mapsto qx,$$

where $q > 0,\ q \neq 1$. That mapping generates a free and discrete action of the group $\mathbb{Z}$ on $\mathbb{R}^n \setminus \{0\}$, with the quotient space $M = (\mathbb{R}^n \setminus \{0\})/\mathbb{Z}$ diffeomorphic to $S^{n-1} \times S^1$. The group $\mathbb{Z}$ acts by isometries of the metric $\frac{1}{\|x\|}F$ and hence induces a Finsler metric on M, which is locally conformally related to the Berwald (even Minkowski) metric F. But if F is not-Riemannian, the resulting metric is not globally conformally Berwald, as conformally related Berwald metrics are either homothetic, or Riemannian [Vincze, 2006] (consider the lift to $\mathbb{R}^n \setminus \{0\}$).

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Question:

Is it so that the above example is the only nontrivial possible? In other words, is the following true: "let (M, F) be a connected, closed, locally conformally Berwald Finsler manifold. Then, either F is globally conformally Berwald or is conformally flat (in which case a finite cover of (M, F) is diffeomorphic to the direct product $S^{n-1} \times S^1$ by [Matveev-Troyanov, 2012, from Fried, 1980])"?

True (Theorem; MN, 2015) if the Berwald connection

- is either complete,
- or has holonomy of a symmetric space of rank $\geqslant 2$.

Is it still true when the holonomy is reducible? Equivalent to the following:

Conjecture (Belgun-Moroianu, 2014)

On a closed manifold, any reducible locally metric connection that preserves a conformal structure is either the Levi-Civita connection of a certain Riemannian metric, or is flat.

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Riemannianisation:

We are given a closed Riemannian manifold M. It is locally conformally reducible (and the conformal factor is unique up to multiplication by a positive constant, by [Vincze, 2006]). Is it true that it is either globally conformally reducible or (locally) conformally flat?

If not, then the universal cover $\tilde{M}$ carries a Riemannian metric g

- which is incomplete;
- whose holonomy group is reducible;
- such that the fundamental group G acts by homothecies (not all isometries) of g, with $\tilde{M}/G = M$, closed;
- (not that important) conformally equivalent to the lift of the initial metric.

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Is such g flat?

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How close one can get to a possible proof/counterexample?

We have the following de Rham-type decomposition theorem.

Theorem (MN, 2015)

Let $(\tilde{M}, g)$ be a connected, simply connected, noncomplete, analytic Riemannian manifold with reducible holonomy. Suppose a group G acts upon $(\tilde{M}, g)$ cocompactly and freely by homothecies. Then $(\tilde{M}, g)$ is the (global) Riemannian product of a Euclidean space $\mathbb{R}^k$ and an incomplete Riemannian manifold N.

Proof, idea:

- Local product structure: a finite collection of complementary orthogonal totally geodesic foliations on $(\tilde{M}, g)$.
- If the shortest incomplete geodesic doesn't lie on a leaf, then the leaf must be flat.
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But in general the answer is "no" (MN, 2015, CRAS).
Let $\mathfrak{g}$ be a 3-dimensional Lie algebra defined by

$$[Z, X] = X, \quad [Z, Y] = -Y, \quad [X, Y] = 0.$$

Its (simply connected) Lie group G is solvable and is the Lorentz group of motions of the Minkowski plane. The group G is isomorphic to $\mathbb{R}^3$ with the multiplication defined by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} D \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} \\ z + z' \end{pmatrix}, \qquad \text{where } D = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix}.$$

Another way to visualise G is as the group of matrices

$$\begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix}, \qquad x, y, z \in \mathbb{R}.$$

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Consider a matrix $A \in \mathrm{SL}(2,\mathbb{Z})$ with two different real eigenvalues e^{λ} and $e^{-\lambda}$, e.g. $A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = T^{-1}\mathrm{diag}(e^{\lambda}, e^{-\lambda})T$ for some nonsingular T. Then changing the xy-coordinates by the transformation T and the coordinate z, by $z \mapsto \lambda z$, we get the the group law in G written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} A^z \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} \\ z + z' \end{pmatrix}.$$

As $A \in \mathrm{SL}(2,\mathbb{Z})$, the action of A^m, $m \in \mathbb{Z}$, on $\mathbb{R}^2$ preserves the integer lattice $\mathbb{Z}^2$. So the integer lattice $\Gamma = \mathbb{Z}^3$ is a subgroup of G, with a compact quotient diffeomorphic to the torus $\mathbb{T}^3$ (one can visulaise that quotient as follows: we first take the torus $\mathbb{T}^2$, the quotient of the xy-plane by $\mathbb{Z}^2$, then multiply it by $[0,1]$ and then identify the top and the bottom by the diffeomorphism of $\mathbb{T}^2$ defined by A).

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Left-invariant Riemannian metric on G: the vector fields $e^z\partial_x, e^{-z}\partial_y, \partial_z$ are left-invariant (they are X, Y, Z we started with, respectively). Take them orthonormal. In coordinates (x, y, z) we get the following metric on $\mathbb{R}^3$:

$$ds^2 = e^{-2z}dx^2 + e^{2z}dy^2 + dz^2.$$

The foliation $x = \text{const}$ is totally geodesic and G-invariant (and so also Γ-invariant) and its orthogonal 1-dimensional foliation is also G-invariant. Now multiply ds^2 by e^{2z}. The resulting metric

$$ds'^2 = dx^2 + e^{4z}dy^2 + e^{2z}dz^2$$

is a direct product of the line $\mathbb{R}^1$ and a noncomplete two-dimensional Riemannian manifold (the negative half of the z-axis has finite length). Γ acts on it by homothecies, with $G/\Gamma = \mathbb{T}^3$.

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