Locally conformally Berwald manifolds and compact quotients of reducible manifolds by homotheties

V. Matveev ¹  Y. Nikolayevsky ²

¹Institute of Mathematics, Friedrich-Schiller-Universität, Jena, Germany

²La Trobe University, Melbourne, Australia

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Starting point:

Theorem (Matveev-Troyanov, 2012)

A connected closed conformally flat non-Riemannian Finsler manifold is either a Bieberbach manifolds or a Hopf manifolds. In particular, it is finitely covered either by a torus $\mathbb{T}^n$ or by $S^{n-1} \times S^1$.

"Conformally flat" means "locally conformally Minkowski". We wanted to extend this to the "next simplest" class of Finsler manifolds – Berwald manifolds.

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1. $F_x(\lambda \cdot \xi) = \lambda \cdot F_x(\xi)$ for any $\lambda \geq 0$.
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This can be naturally generalised to the Cartesian product of any number of manifolds.

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Theorem (Szabó, 1981)

- (Metrisability) Any Berwald connection is a Levi-Civita connection of some Riemannian metric.

- (Local de Rham) Any Berwald manifold is locally the Cartesian product (in the sense of Example 3) of Riemannian manifolds, Minkowski spaces and symmetric spaces of rank \( \geq 2 \).

Any of these factors may be absent.

“Symmetric space” means that the space has the same reduced holonomy; those were completely classified.

In particular, if the reduced holonomy group is the whole \( \text{SO}(n) \), then the Berwald space is Riemannian.

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Dowód.

Theorem (Szabó, 1981)

- (Metrizability) Any Berwald connection is a Levi-Civita connection of some Riemannian metric.
- (local de Rham) Any Berwald manifold is locally the Cartesian product (in the sense of Example 3) of Riemannian manifolds, Minkowski spaces and symmetric spaces of rank $\geq 2$.

Any of these factors may be absent.
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Definition

Locally (globally) conformally Berwald.

Example

Consider a Minkowski metric $F$ on $\mathbb{R}^n$. And consider the mapping

$$\alpha : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n \setminus \{0\}, \quad x \mapsto qx,$$

where $q > 0, \ q \neq 1$. That mapping generates a free and discrete action of the group $\mathbb{Z}$ on $\mathbb{R}^n \setminus \{0\}$, with the quotient space $M = (\mathbb{R}^n \setminus \{0\})/\mathbb{Z}$ diffeomorphic to $S^{n-1} \times S^1$. The group $\mathbb{Z}$ acts by isometries of the metric $\frac{1}{\|x\|}F$ and hence induces a Finsler metric on $M$, which is locally conformally related to the Berwald (even Minkowski) metric $F$. But if $F$ is not-Riemannian, the resulting metric is not globally conformally Berwald, as conformally related Berwald metrics are either homothetic, or Riemannian [Vincze, 2006] (consider the lift to $\mathbb{R}^n \setminus \{0\}$).
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Question:
Is it so that the above example is the only nontrivial possible? In other words, is the following true: “let \((M, F)\) be a connected, closed, locally conformally Berwald Finsler manifold. Then, either \(F\) is globally conformally Berwald or is conformally flat (in which case a finite cover of \((M, F)\) is diffeomorphic to the direct product \(S^{n-1} \times S^1\) by [Matveev-Troyanov, 2012, from Fried, 1980])”?

True (Theorem; MN, 2015) if the Berwald connection
• is either complete,
• or has holonomy of a symmetric space of rank \(\geq 2\).

Is it still true when the holonomy is reducible? Equivalent to the following:

Conjecture (Belgun-Moroianu, 2014)

On a closed manifold, any reducible locally metric connection that preserves a conformal structure is either the Levi-Civita connection of a certain Riemannian metric, or is flat.
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We are given a closed Riemannian manifold $M$. It is locally conformally reducible (and the conformal factor is unique up to multiplication by a positive constant, by [Vincze, 2006]). Is it true that it is either globally conformally reducible or (locally) conformally flat?

If not, then the universal cover $\tilde{M}$ carries a Riemannian metric $g$

- which is incomplete;
- whose holonomy group is reducible;
- such that the fundamental group $G$ acts by homothecies (not all isometries) of $g$, with $\tilde{M}/G = M$, closed;
- (not that important) conformally equivalent to the lift of the initial metric.

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Is such $g$ flat?
How close one can get to a possible proof/counterexample?

We have the following de Rham-type decomposition theorem.

**Theorem (MN, 2015)**

Let $(\tilde{M}, g)$ be a connected, simply connected, non-complete, analytic Riemannian manifold with reducible holonomy. Suppose a group $G$ acts upon $(\tilde{M}, g)$ cocompactly and freely by homothecies. Then $(\tilde{M}, g)$ is the (global) Riemannian product of a Euclidean space $\mathbb{R}^k$ and an incomplete Riemannian manifold $N$.

**Proof, idea:**

- Local product structure: a finite collection of complementary orthogonal totally geodesic foliations on $(\tilde{M}, g)$.
- If the shortest incomplete geodesic doesn’t lie on a leaf, then the leaf must be flat.
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But in general the answer is “no” (MN, 2015, CRAS).
Let \( g \) be a 3-dimensional Lie algebra defined by

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\]

Its (simply connected) Lie group \( G \) is solvable and is the Lorentz group of motions of the Minkowski plane. The group \( G \) is isomorphic to \( \mathbb{R}^3 \) with the multiplication defined by

\[
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \left( \begin{array}{c} D \left( \begin{pmatrix} x' \\ y' \end{pmatrix} \right) + \begin{pmatrix} x \\ y \end{pmatrix} \right),
\]

where \( D = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix} \).

Another way to visualise \( G \) is as the group of matrices

\[
\begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R}.
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\[
\begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R}.
\]

The group \( G \) admits a compact quotient by a subgroup \( \Gamma \subset G \) such that \( G/\Gamma \) is diffeomorphic to the torus \( \mathbb{T}^3 \).
Consider a matrix $A \in \text{SL}(2, \mathbb{Z})$ with two different real eigenvalues $e^\lambda$ and $e^{-\lambda}$, e.g. $A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = T^{-1} \text{diag}(e^\lambda, e^{-\lambda}) T$ for some nonsingular $T$. Then changing the $xy$-coordinates by the transformation $T$ and the coordinate $z$, by $z \mapsto \lambda z$, we get the group law in $G$ written as

$$
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} A^z \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} \\ z + z' \end{pmatrix}.
$$

As $A \in \text{SL}(2, \mathbb{Z})$, the action of $A^m$, $m \in \mathbb{Z}$, on $\mathbb{R}^2$ preserves the integer lattice $\mathbb{Z}^2$. So the integer lattice $\Gamma = \mathbb{Z}^3$ is a subgroup of $G$, with a compact quotient diffeomorphic to the torus $\mathbb{T}^3$ (one can visualise that quotient as follows: we first take the torus $\mathbb{T}^2$, the quotient of the $xy$-plane by $\mathbb{Z}^2$, then multiply it by $[0, 1]$ and then identify the top and the bottom by the diffeomorphism of $\mathbb{T}^2$ defined by $A$).
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Left-invariant Riemannian metric on $G$: the vector fields $e^z \partial_x, e^{-z} \partial_y, \partial_z$ are left-invariant (they are $X, Y, Z$ we started with, respectively). Take them orthonormal. In coordinates $(x, y, z)$ we get the following metric on $\mathbb{R}^3$:

$$ds^2 = e^{-2z} dx^2 + e^{2z} dy^2 + dz^2.$$

The foliation $x = \text{const}$ is totally geodesic and $G$-invariant (and so also $\Gamma$-invariant) and its orthogonal 1-dimensional foliation is also $G$-invariant. Now multiply $ds^2$ by $e^{2z}$. The resulting metric

$$ds'{}^2 = dx^2 + e^{4z} dy^2 + e^{2z} dz^2$$

is a direct product of the line $\mathbb{R}^1$ and a noncomplete two-dimensional Riemannian manifold (the negative half of the $z$-axis has finite length). $\Gamma$ acts on it by homothecies, with $G/\Gamma = \mathbb{T}^3$. 


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