C-projective structures with large symmetry

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based on joint work with

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Let ∇ be a linear connection on a smooth connected almost complex manifold (M^{2n}, J) of $\mathbb{C}$-dimension $n \geq 2$. We will assume it is a complex connection $\nabla J = 0$. Every almost complex structure projects to a complex connection: $\nabla \mapsto \frac{1}{2}(\nabla - J\nabla J)$.

The torsion $T_\nabla \in \Omega^2(M) \otimes \mathcal{D}(M)$ has the total complex-antilinear part $T_\nabla^{--} \in \Omega^{0,2}(M) \otimes \mathcal{D}(M)$ equal to $\frac{1}{4}N_J$, where

$$N_J(X,Y) = [JX, JY] - J[JX,Y] - J[X,JY] - [X,Y]$$

is the Nijenhuis tensor of J. In particular, for non-integrable J, the complex connection ∇ is never symmetric. However the other parts of T_∇ can be set to zero by a choice of complex connection. There always exist minimal connections ∇ characterized by $T_\nabla = T_\nabla^{--}$.

Recall that two real connections are projectively equivalent if their (unparametrized) geodesics γ given by $\nabla_{\dot\gamma}\dot\gamma \in \mathbb{R}\cdot\dot\gamma$ are the same. Thus equivalence $\nabla \sim \bar\nabla$ means $\nabla_X X - \bar\nabla_X X \in \mathbb{R}\cdot X$ $\forall X \in \mathcal{D}(M)$, and any connection is projectively equivalent to a symmetric one: $\nabla \simeq \nabla - \frac{1}{2}T_\nabla$.

The complex case is different. A J-planar curve $\gamma : \mathbb{R} \supset I \to M$ is given by the differential equation $\nabla_{\dot\gamma}\dot\gamma \in \mathbb{C}\dot\gamma = \langle \dot\gamma, J\dot\gamma\rangle_{\mathbb{R}}$. Reparametrization does not change this property. The class of unparametrized J-planar curves is encoded by 1 function φ given in terms of decomposition $\nabla_{\dot\gamma}\dot\gamma = \alpha\dot\gamma + \beta J\dot\gamma$ by $\varphi = (\alpha + \nabla_{\dot\gamma})(\beta^{-1})$. Geodesics correspond to $\beta = 0$. J-planar curves, as complex analogs of geodesics, are of certain interest in Hermitian geometry.

Two pairs (J, ∇) with $\nabla J = 0$ on the same manifold M are called c-projectively equivalent if they share the same class of J-planar curves. It is easy to show that the almost complex structure J is restored up to sign by the c-projective equivalence, so we fix it.

Definition

Two complex connections on almost complex background (M, J) are equivalent $\nabla \sim \bar\nabla$ if they have the same J-planar curves, i.e. $\nabla_X X - \bar\nabla_X X \in \mathbb{C} \cdot X\ \forall X \in \mathcal{D}(M)$. An equivalence class $(J, [\nabla])$ of complex connections is called a c-projective structure.

Let us reformulate this definition tensorially in the case of minimal connections (the general case requires additional normalization).

The above equivalence relation on the space of all minimal complex connections writes $\nabla \sim \bar{\nabla} = \nabla + \mathrm{Id} \odot \Psi - J \odot J^* \Psi$ for some 1-form $\Psi \in \Omega^1(M)$. In other words, $\bar{\nabla}$ is c-projectively equivalent to ∇ if and only if (notice that $T_\nabla = T_{\bar{\nabla}}$)

$$\bar{\nabla}_X Y = \nabla_X Y + \Psi(X)Y + \Psi(Y)X - \Psi(JX)JY - \Psi(JY)JX$$

A vector field v is called a c-projective symmetry, if its local flow Φ_t^v preserves the class of J-planar curves. Equivalently we have: $(\Phi_t^v)^* J = J$, $[(\Phi_t^v)^* \nabla] = [\nabla]$. The first equation can be re-written as $L_v J = 0$. The second equation can be re-written as $L_v[\nabla] = 0$, or in local coordinates, with the connection ∇ given by the Christoffel symbols Γ^i_{jk}, so:

$$\Omega^i_{jk} - \phi_{(j}\delta^i_{k)} + \phi_\alpha J^\alpha{}_{(j} J^i{}_{k)} = 0,$$

where $\Omega^i_{jk} = L_v(\Gamma)^i_{jk}$ and $\phi_j = \frac{1}{2(n+1)}\Omega^i_{ji}$.

The space of c-projective vector fields forms a Lie algebra, denoted $\mathfrak{cp}(\nabla, J)$. The maximal dimension of this algebra is equal to $2(n^2+2n)$, and this bound is achieved only if the structure is flat, i.e. c-projectively locally equivalent to $\mathbb{C}P^n$ equipped with the standard complex structure J_{can} and the class of the Levi-Civita connection ∇^{FS} of the Fubini-Studi metric. Indeed, the group of c-projective automorphisms of $(\mathbb{C}P^n, J_{\text{can}}, [\nabla^{\text{FS}}])$ is $\mathrm{PSL}(n+1,\mathbb{C})$, so $\mathfrak{cp}(\nabla^{\text{FS}}, J_{\text{can}}) = \mathfrak{sl}(n+1,\mathbb{C})$.

To explain this upper bound let me briefly recall the (real) projective situation. This is a Cartan geometry of type G/P, where $G = \mathrm{SL}(n+1,\mathbb{R})$ and P is the 1st parabolic subgroup (stabilizer of a line). The maximal dimension of the symmetry of the Cartan geometry is $\dim(G) = n^2+2n$, and this bound is achieved only if the structure is flat, i.e. projectively locally equivalent to $\mathbb{R}P^n$ equipped with the class of ∇^{LC}.

Similarly, c-projective structures are described in the context of Cartan geometry of type G/P, where $G = \mathrm{SL}(n+1,\mathbb{C})$ and P is the 1st parabolic subgroup (stabilizer of a complex line).

For many geometric structures the natural (and often nontrivial) problem is to compute the next possible/realizable dimension, the so-called submaximal dimension, of the algebra of symmetries. Namely, this is the maximal dimension of a non-flat geometry. For the algebra of (usual) projective vector fields the question was settled by A.Tresse for $n = 2$ and I.Egorov for $n > 2$. For c-projective vector fields the answer is as follows.

Theorem (BK & V.Matveev & D.The 2015)

Consider a c-projective structure $(J, [\nabla])$ on M. If it is not everywhere flat, then $\dim \mathfrak{cp}(\nabla, J)$ is bounded from above by

$$\mathfrak{S} = \begin{cases} 2n^2 - 2n + 4, & n \neq 3, \\ 18, & n = 3. \end{cases}$$

and this estimate is sharp (realizable).

The sub-maximal dimensional bound $2n^2 - 2n + 4$ is realizable for both non-minimal and minimal complex connections.

Background: Cartan and parabolic geometries

Recall that a Cartan geometry of type G/P is given by a Lie group G, its subgroup P, a principal bundle $\mathcal{G} \to M$ with the structure group P and a 1-form (Cartan connection) $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ such that

- $\omega : T_u\mathcal{G} \xrightarrow{\sim} \mathfrak{g}$, $u \in P$,
- $R_p^*\omega = \mathrm{Ad}_p^{-1}\omega$, $p \in P$,
- $\omega(X_v) = v$, $v \in \mathfrak{p}$.

Parabolic geometries are Cartan geometries with G a semi-simple Lie group and P a parabolic subgroup (notice play G vs. $\mathfrak{g}$).

Examples

Model G/P	Underlying (curved) geometry
$SO(p+1,q+1)/P_1$	sign (p,q) conformal structure
$SL_{m+2}/P_{1,2}$	2nd ord ODE system in m dep vars
G_2/P_1	$(2,3,5)$-distributions
$SL_{m+1}/P_{1,m}$	Lagrangian contact structures
$SU(p+1,q+1)/P_{1,m}$	CR-structures with (p,q) Levi signature

A parabolic subgroup $P \subset G$ defines the gradation

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k,$$

and $\mathfrak{p} = \oplus_{i \geq 0} \mathfrak{g}_i$. Denote also $\mathfrak{m} = \oplus_{i<0} \mathfrak{g}_i$. The curvature of the geometry is fully encoded into its harmonic part $\kappa_H \in H^2_+(\mathfrak{m}, \mathfrak{g})$, (normalization is given by the Kostant co-differential: $\partial^* \kappa = 0$).

For any $\mathfrak{a}_0 \subset \mathfrak{g}_0$ the Tanaka prolongation $\mathrm{pr}(\mathfrak{m}, \mathfrak{a}_0)$ is the maximal graded Lie algebra containing $\mathfrak{g}_- \oplus \mathfrak{a}_0$ as the non-positive part. The symmetry algebra $\mathcal{S}$ has a filtration determined by a point $u \in \mathcal{G}$ such that the associated graded Lie algebra $\mathfrak{s} \subset \mathfrak{g}$. We have

Lemma

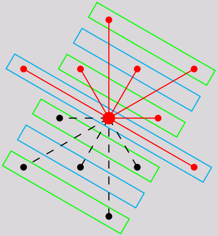
The embedding $\mathfrak{s}_i \subset \mathfrak{g}_i$ at a regular point satisfies: $[\mathfrak{s}_i, \mathfrak{g}_{-1}] \subset \mathfrak{s}_{i-1}$.

In the flat case K.Yamaguchi's prolongation theorem states that all but two parabolic G/P type geometries for a (complex) simple Lie group G and parabolic P are obtained via a reduction of $\mathfrak{g}_0$ and Tanaka prolongation. The two exceptional structures of types A_n/P_1 and C_n/P_1 are obtained by a higher order reduction.

Example ($(2,3,5)$-distributions)

Any such Δ can be described as Monge eqn $z' = f(x,z,y,y',y'')$.
$M:(x,z,y,p,q)$, $\Delta = \{\partial_q, \partial_x + p\partial_y + q\partial_p + f\partial_z\}$, $f_{qq} \neq 0$.
$\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3}$ with dims $(2,1,2)$, and $\mathfrak{g}_0 = \mathfrak{gl}_2$.

Same as G_2/P_1 data:



$\Leftrightarrow$

$$Lie(G_2) = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3}^{\mathfrak{p}_1}$$

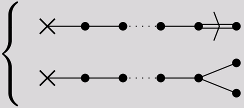
Yamaguchi $pr(\mathfrak{m}, \mathfrak{g}_0) = Lie(G_2)$.
Any $(2,3,5)$-dist. $= (G_2, P_1)$-type geom.

Example (Conformal geometry)

Let $(M,[\mu])$ be sig. (p,q) conformal mfld, $n=p+q$. Here, $\Delta = TM$, $\mathfrak{m}=\mathfrak{g}_{-1}$, and $\mathfrak{g}_0=\mathfrak{co}(\mathfrak{g}_{-1})$.

Same as $SO_{p+1,q+1}/P_1$ data: if $g=\begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{p,q} & 0 \\ 1 & 0 & 0 \end{pmatrix}$, then

$$\mathfrak{so}_{p+1,q+1} = \left(\begin{array}{c|c|c} 0 & 1 & \cdot \\ \hline -1 & 0 & 1 \\ \hline \cdot & -1 & 0 \end{array}\right) \quad \Leftrightarrow$$



$(n$ odd$)$;

$(n$ even$)$.

$$\mathfrak{so}_{p+1,q+1} = \mathfrak{g}_{-1} \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{p}_1}$$

Yamaguchi $pr(\mathfrak{m},\mathfrak{g}_0)=\mathfrak{so}_{p+1,q+1}$.

Any conformal geometry $=(SO_{p+1,q+1},P_1)$-type geom.

Example (Projective structure: type $A_n/P_1 = SL(n+1)/P_1$)

To obtain the real projective geometry consider the algebra $\mathfrak{D}_\infty(\mathbb{R}^n)$ of formal vector fields on $V = \mathbb{R}^n$, with gradation $\mathfrak{g}_{-1} = V$, $\mathfrak{g}_0 = V^* \otimes V = \mathbb{R} \oplus \mathfrak{sl}(V)$, $\mathfrak{g}_1 = S^2 V^* \otimes V$, ...

The $\mathfrak{g}_0$-module decomposition into irreducibles $\mathfrak{g}_1 = \mathfrak{g}_1' \oplus \mathfrak{g}_1''$ has components $\mathfrak{g}_1' = (S^2V^* \otimes V)_0 = \mathrm{Ker}(q : S^2V^* \otimes V \to V^*)$ and $\mathfrak{g}_1'' = V^* \xrightarrow{i} S^2V^* \otimes V$, $i(p)(v,w) = p(v)w + p(w)v$.

Prolongation of the first is $\mathfrak{g}' = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1' \oplus \mathfrak{g}_2' \oplus \dots$, where $\mathfrak{g}_k' = \mathrm{Ker}(q : S^{k+1}V^* \otimes V \to S^kV^*)$. This is the gradation of the algebra $\mathfrak{S}\mathfrak{D}_\infty(\mathbb{R}^n) = \{\xi \in \mathfrak{D}_\infty(\mathbb{R}^n) : \mathrm{div}(\xi) = \mathrm{const}\}$.

The other reduction has the trivial prolongation and we get $\mathfrak{g}'' = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1'' = \mathfrak{sl}(n+1)$ – the grading of $SL(n+1,\mathbb{R})/P_1$.

C-projective structure is obtained similarly by a complex space W: $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = W \oplus \mathfrak{gl}_{\mathbb{C}}(W) \oplus W^*$ (real reductions). This parabolic geometry has type $SL(n+1,\mathbb{C})_{\mathbb{R}}/P_1$.

The gap problem

Question: What is the symmetry bound of a non-flat geometry?

Often there is a gap between maximal and submaximal symmetry dimensions, i.e. $\exists$ forbidden dimensions.

Example (Riemannian geometry in dim $= n$)

n	max	submax	Ref
2	3	1	*Lie* (1882), *Darboux* (1894)
3	6	4	*Bianchi* (1898), *Ricci* (1898)
4	10	8	*Egorov* (1955)
≥ 5	$\binom{n+1}{2}$	$\binom{n}{2}+1$	*Wang* (1947)

For other signatures the result is the same, except the 4D case

NB: We restrict to symmetry algebras, i.e. study the geometric structures locally. Globally some other bounds can be achieved, e.g. for $n = 2$ there is a global model with 2 symmetries (flat $\mathbb{T}^2$).

Old and new results on the parabolic gap phenomenon

Geometry	*Max*	*Submax*	*Citation*
scalar 2nd order ODE mod point / proj 2D str	8	3	Tresse (1896)
$(2,3,5)$-distributions	14	7	Cartan (1910)
proj $(n+1)$-dim str	n^2+4n+3	n^2+4	Egorov (1951)
scalar 3rd order ODE mod contact	10	5	Wafo Soh, Qu Mahomed (2002)

Geometry	Max	Submax
Sign (p,q) conf geom $n=p+q,\quad p,q\geq 2$	$\binom{n+2}{2}$	$\binom{n-1}{2}+6$
Systems 2nd ord ODE	$(m+2)^2-1$	m^2+5
Lagrangian contact str	m^2+2m	$(m-1)^2+4,\ m\geq 3$
Contact projective str	$m(2m+1)$	$2m^2-5m+8+\delta_m^2$
Exotic parabolic contact structure of type E_8/P_8	248	147

General dim bound for regular normal parabolic geometries

$$\phi \in H^2_+(\mathfrak{m},\mathfrak{g}),\ \mathfrak{a}_0^\phi = \mathfrak{ann}(\phi) \subset \mathfrak{g}_0,\ \mathfrak{a}^\phi = \mathrm{pr}(\mathfrak{m},\mathfrak{a}_0^\phi) = \mathfrak{m} \oplus \sum_{i\geq 0} \mathfrak{a}_i^\phi$$

Theorem (BK & D.The 2013)

For G/P parabolic geometry: $\dim(\mathfrak{inf}(\mathcal{G},\omega)) \leq \inf_{x\in M} \dim(\mathfrak{a}^{\kappa_H(x)})$.

Algorithm to compute the sub-maximal bound:

- Compute by Kostant BBW: $H^2_+(\mathfrak{m},\mathfrak{g}) = \bigoplus_i \mathbb{V}_i$;
- Find lwv $v_i \in \mathbb{V}_i$ in $\mathfrak{g}_0$-irreps;
- Compute the annihilator $\mathfrak{a}_0^{v_i} = \mathfrak{ann}(v_i) \subset \mathfrak{g}_0$;
- Prolong $\mathfrak{a}_0^{v_i} \rightsquigarrow \mathfrak{a}^{v_i} = \mathfrak{m} \oplus \mathfrak{a}_0^{v_i} \oplus \mathfrak{a}_1^{v_i} \oplus \dots$;
- Most cases are prolongation rigid, exceptions are classified;
- Most cases have the universal bound sharp (excepts classified):

$$\mathfrak{U} = \max_i \dim(\mathfrak{a}^{v_i}).$$

Finite type structures have finite-dimensional symmetry algebra and the symmetry transformations form a Lie group. In particular, this is true for all Cartan geometries. Beyond the realm of Cartan geometries the gap phenomenon is more complicated.

Almost complex structures have infinite type. The most symmetric structure is integrable: the symmetry of the complex structure J_0 on $\mathbb{C}^n$ is parametrized by $2n$ functions of n arguments. In the next (submaximal) case the symmetry is parametrized by $n-2+\delta_n^2+\delta_n^3$ functions of $(n-1)$ arguments.

Example: $n=2$. Consider $\mathbb{C}^2(z,w)$ with the almost complex structure

$$J\partial_z = i\partial_z + w\partial_{\bar{w}}, \quad J\partial_w = i\partial_w.$$

This structure is non-integrable $N_J(\partial_z,\partial_w) = -2i\partial_{\bar{w}}$, and it has the following infinite transformation pseudogroup of symmetries:

$$(z,w) \mapsto (e^{2ir}z + c, e^{-ir}(w+\zeta(z))),$$

where $r \in \mathbb{R}$, $c \in \mathbb{C}$ and $\zeta_{\bar{z}} = \frac{i}{2}\bar{\zeta}$ (reducible to the Laplace eqn).

C-projective structures: curvature types

In the case of c-projective geometry we compute by Kostant's BBW theorem that the curvature module $H^2_+(\mathfrak{m},\mathfrak{g})$ over $\mathfrak{g}_0=\mathfrak{gl}(n,\mathbb{C})_\mathbb{R}$ has 3 irreducible components (real: in complexification there are 6):

$$H^2_+(\mathfrak{g}_-,\mathfrak{g})=\mathbb{V}_I\oplus\mathbb{V}_{II}\oplus\mathbb{V}_{III},$$

where $\mathbb{V}_I=\Lambda^{2,0}\mathfrak{g}_-^*\otimes_\mathbb{C}\mathfrak{sl}(\mathfrak{g}_-,\mathbb{C})$ for $n>2$ and $\mathbb{V}_I=\Lambda^{2,0}\mathfrak{g}_-^*\otimes_\mathbb{C}\mathfrak{g}_-^*$ for $n=2$; $\mathbb{V}_{II}=\Lambda^{1,1}\mathfrak{g}_-^*\otimes_\mathbb{C}\mathfrak{sl}(\mathfrak{g}_-,\mathbb{C})$; $\mathbb{V}_{III}=\Lambda^{0,2}\mathfrak{g}_-^*\otimes_\mathbb{C}\mathfrak{g}_-$.
The harmonic curvature splits in accordance to the above into irreducible components $\kappa_H=\kappa_\mathrm{I}+\kappa_\mathrm{II}+\kappa_\mathrm{III}$ and

- κ_I is the $(2,0)$-part of Weyl projective curvature of ∇, $n>2$;
- κ_I is the $(2,0)$-part of the Liouville tensor when $n=2$;
- κ_II is the $(1,1)$-part of Weyl projective curvature tensor of ∇;
- κ_III is $\frac{1}{4}N_J$ (torsion of a minimal complex connection ∇).

We remark that on a complex background (M,J) ($\kappa_\mathrm{III}=0$):

- A holomorphic connection exists in $[\nabla]\Leftrightarrow\kappa_\mathrm{II}=0$;
- $\kappa_\mathrm{I}=0$ is a necessary condition for $(M,J,[\nabla])$ to be (pseudo-) Kähler metrizable (Calderbank-Eastwood-Matveev-Neusser).

Main result: submaximal dimensions

Step 1. Prove prolongation-rigidity: for any $\phi \in \mathbb{V}$ we have $\mathrm{pr}(\mathfrak{m}, \mathfrak{a}_0^\phi)_+ = 0$. Thus $\mathfrak{a}^\phi = \mathfrak{g}_- \oplus \mathfrak{a}_0^\phi$ and to maximize this choose the lowest weight vector in $\mathbb{V}$.

Step 2. Compute the lowest weight in every $\mathfrak{g}_0$-module $\mathbb{V}_{\mathrm{I}}, \mathbb{V}_{\mathrm{II}}, \mathbb{V}_{\mathrm{III}}$ and determine the bound in every type of the curvature.

Theorem (BK & V.Matveev & D.The 2015)

SubMax	$n=2$	$n=3$	$n=4$	$n=5$	$\dots$	
Type I	*6*	*16*	*26*	*40*	$\dots$	$2n^2-4n+10$
Type II	*8*	*16*	*28*	*44*	$\dots$	$2n^2-2n+4$
Type III	*8*	*18*	*28*	*42*	$\dots$	$2n^2-4n+12$

The middle line (winning submax) is the only candidate and it is metrizable c-proj structure (in contrast with the real case!).

To finish the proof we need Step 3: Realization.

Model - Type I

This is the holomorphic version of the Egorov's (symm) connection. So $J=i$ and in complex coordinates the only non-trivial Christoffel symbols are (conjugate equations are shortened to +Cc):

$$\Gamma^1_{23} = z^2 \quad (\text{+Cc: } \Gamma^{\bar 1}_{\bar 2\bar 3} = \overline{z^2}).$$

The harmonic curvature $\kappa_H = \kappa_{\rm I} \neq 0$ corresponds to

$$W_\nabla = dz^2 \wedge dz^3 \otimes z^2 \partial_{z^1} \quad (\text{+Cc}).$$

The c-projective symmetries are $2\cdot(n^2-2n+5)$ vector fields (both real and imaginary parts shall be counted):

$$\partial_{z^1},\ \partial_{z^3},\ \dots,\ \partial_{z^n},\ z^i\partial_{z^j}\ (i\geq 2,\ j\neq 2,3),$$
$$2z^1\partial_{z^1}+z^2\partial_{z^2},\ z^1\partial_{z^1}+z^3\partial_{z^3},\ z^2z^3\partial_{z^1}-\partial_{z^2},\ (z^2)^3\partial_{z^1}-3z^2\partial_{z^3}.$$

For $n=2$ the model and $2\cdot 3=6$ symmetries come from Tresse:

$$\Gamma^1_{22} = -\Gamma^1_{11} = \frac{1}{2z^1} \quad (\text{+Cc}).$$
$$\partial_{z^2},\ z^1\partial_{z^1}+z^2\partial_{z^2},\ z^1z^2\partial_{z^1}+\tfrac12(z^2)^2\partial_{z^2}.$$

Model - Type II

Consider the complex connection ∇ with respect to the standard complex structure $J=i$ on $\mathbb{C}^n$ given in the complex coordinates by

$$\Gamma^2_{11}=\overline{z^1}\quad (+\text{Cc: } \Gamma^{\bar 2}_{\bar 1\bar 1}=z^1).$$

Its curvature has pure type II, $\kappa_H=\kappa_{\rm II}\neq 0$:

$$W_\nabla = dz^1\wedge d\overline{z^1}\otimes z^1\partial_{z^2}\quad (+\text{Cc}).$$

The c-projective symmetries are found by straightforward computation to be real and imaginary parts of the following (linearly independent) complex-valued vector fields:

$$\partial_{z^2},\ \dots,\ \partial_{z^n},\quad z^i\partial_{z^j}\quad (i\neq 2,\ j>1),$$
$$z^1\partial_{z^1}+2z^2\partial_{z^2}+\overline{z^2}\partial_{\overline{z^2}},\quad \partial_{z^1}-\tfrac12(\overline{z^1})^2\partial_{\overline{z^2}}.$$

Since the totality of these $2\cdot(n^2-n+2)$ coincides with the universal upper bound, these are all symmetries, and so the above $(J,[\nabla])$ is a sub-maximal c-projective structure of curvature type II.

Model - Type III, $n > 2$

Consider a sub-maximal symmetric almost complex structure on $\mathbb{C}^n$ found in BK'2014 (there are two, only one works):

$$J\partial_{z^1} = i\partial_{z^1} + z^2\partial_{\overline{z^3}},\ J\partial_{z^2} = i\partial_{z^2},\ \dots, J\partial_{z^n} = i\partial_{z^n}.$$

Let $\tilde{\nabla} = d$ be the trivial connection, and $\nabla = \frac{1}{2}(\tilde{\nabla} - J\tilde{\nabla}J)$ the corresponding complex connection. Its nontrivial Christoffel symbols are (our convention: $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$):

$$\Gamma^{\bar{3}}_{21} = \frac{i}{2}\ \ (+\text{Cc: } \Gamma^{3}_{\bar{2}\bar{1}} = -\frac{i}{2}).$$

Thus the torsion $T_\nabla \neq 0$ is as required, and the curvature $R_\nabla = 0$. Consequently, the curvature $\kappa_H = \kappa_{\mathrm{III}} \neq 0$ has type III. There are $2 \cdot (n^2 - 2n + 6)$ c-projective symmetries:

$$\partial_{z^1},\quad \partial_{z^3},\ \dots,\ \partial_{z^n},\quad z^i\partial_{z^j}\quad (i \neq 3,\ j > 2),$$

$$z^1\partial_{z^1} + \overline{z^3}\partial_{\overline{z^3}},\quad z^2\partial_{z^2} + \overline{z^3}\partial_{\overline{z^3}},\quad \partial_{z^2} + \tfrac{z^1}{2i}(\partial_{z^3} + \partial_{\overline{z^3}}),$$

$$z^1\partial_{z^2} - \tfrac{i}{4}(z^1)^2\partial_{\overline{z^3}},\quad z^2\partial_{z^1} - \tfrac{i}{4}(z^2)^2\partial_{\overline{z^3}}.$$

Model - Type III, $n = 2$

This is an exceptional case, for which the abstract realization fails. The local model is expressed in real coordinates (x, y, p, q), frame $e_1 = \partial_x$, $e_2 = \partial_y$, $e_3 = \partial_p$, $e_4 = \partial_q - \frac{3y}{2p}\,\partial_x - \frac{5x}{2p}\,\partial_y$ and dueal co-frame $\theta_1 = dx + \frac{3y}{2p}\,dq$, $\theta_2 = dy + \frac{5x}{2p}\,dq$, $\theta_3 = dp$, $\theta_4 = dq$ so:

$$Je_1 = e_2,\ Je_2 = -e_1,\ Je_3 = e_4,\ Je_4 = -e_3.$$

$$\nabla e_1 = \frac{1}{2p}\,e_2 \otimes \theta_4,\ \nabla e_3 = -\frac{1}{p}\,(e_1 \otimes \theta_1 - e_2 \otimes \theta_2 + e_3 \otimes \theta_3 + e_4 \otimes \theta_4) - \frac{1}{4p^2}\,\bigl(e_1 \otimes (3x\theta_3 + 3y\theta_4) + e_2 \otimes (3y\theta_3 + 13x\theta_4)\bigr).$$

The torsion T_∇ represents $\kappa_{\mathrm{III}} \neq 0$. The curvature $R_\nabla \neq 0$, yet $\kappa_H = \kappa_{\mathrm{III}}$. The 8 symmetries are:

$$x\partial_x + y\partial_y,\ p^{-3/2}\partial_y,\ p\partial_p + q\partial_q,\ \partial_q,$$

$$p\,(y\partial_x - x\partial_y) - 2pq\,\partial_p + (p^2 - q^2)\partial_q,\ (p^{1/2} + p^{-3/2}q^2)(p\partial_x - q\partial_y),$$

$$(p^{1/2} + p^{-3/2}q^2)\partial_y + 2p^{-3/2}q(q\partial_y - p\partial_x),\ p^{-3/2}(q\partial_y - \tfrac{1}{3}p\partial_x).$$

Model - Type IV

If the connection is not minimal, then Kostant's normalization fails $\partial^* \kappa_1 \neq 0$, but it is possible to construct the canonical Cartan connection by a different normalization:

$$T_\nabla = T^{--} + T^{-+}_{\text{traceless}}.$$

In the case when $T^{-+}_{\text{traceless}} \neq 0$ the submaximal bound is again $2(n^2 - n + 2)$ as for type II. In complex notations choose $J = i$ and the nontrivial Christoffels:

$$\Gamma^2_{1\bar{1}} = \Gamma^{\bar{2}}_{\bar{1}1} = 1.$$

Its torsion $T_\nabla = (\partial_{z^2} - \partial_{\overline{z^2}}) \otimes dz^1 \wedge d\overline{z^1} = \mathbb{T}^{-+}_{\text{traceless}}$, while $T^{--}_\nabla = 0$ and $R_\nabla = 0$. The totality of $2(n^2 - n + 2)$ c-projective symmetries are real and imaginary parts of the following vector fields:

$$\partial_{z^i}, \quad z^i \partial_{z^j} \ (i \neq 2, j \neq 1), \quad z^1 \partial_{z^1} + z^2 \partial_{z^2} + \overline{z^2} \partial_{\overline{z^2}}.$$

Metrizability

An important problem in projective geometry is to decide if a given projective connection is metrizable. In the c-projective case, the corresponding problem is to determine if a representative of $[\nabla]$ is given by a Levi-Civita connection of a pseudo-Kähler structure (g, J). For such structures we can also compute the submaximal symmetry dimension.

Theorem (BK & V.Matveev & D.The 2015)

For the Levi-Civita connection ∇^g of a Kähler structure (g, J), which is not of constant holomorphic sectional curvature, $\dim \mathfrak{cp}(\nabla^g, J) \leq 2n^2 - 2n + 3$, the bound is realized by $M^{2n} = \mathbb{C}P^1 \times \mathbb{C}^{n-1}$ with its natural Kähler form and connection. In the pseudo-Kähler case we have: $\dim \mathfrak{cp}(\nabla^g, J) \leq 2n^2 - 2n + 4$, the estimate is sharp in any signature $(2p, 2(n-p))$, $0 < p < n$.

Thus the submaximal symmetry dimension $\mathfrak{S}_0 = 2n^2 - 2n + 4$ for complex c-projective structures realizes on a pseudo-Kähler metric.

To see the claim for the pseudo-Kähler metric consider

$$g = |z_1|^2\, dz_1\, d\bar{z}_1 + dz_1\, d\bar{z}_2 + d\bar{z}_1\, dz_2 + \sum\nolimits_3^n \epsilon_k\, dz_k\, d\bar{z}_k.$$

Its Levi-Civita connection coincides with the sub-maximal ∇ of type II, whence the latter is metrizable.

To see the drop of the sub-maximal dimension in the Kähler case assume first there are no essential c-projective transformations, i.e. for some metric g with Levi-Civita connection in the c-proj class:

$$\mathfrak{cp}(\nabla^g, J) = \mathfrak{aff}(g, J).$$

For $x \in M$ with $R_{\nabla^g}(x) \neq 0$ by the Ambrose-Singer theorem the holonomy algebra $\mathcal{H}_x \subset \mathfrak{u}(n)$ is nontrivial. From the de Rham decomposition theorem $T_x M = \oplus_{k=0}^m \Pi_k$, where Π_i are $\mathcal{H}_x$-invariant and $\Pi_0 = \operatorname{ann}(\mathcal{H}_x)$. Then

$$\mathfrak{aff}(g, J) \supset \tilde{\mathfrak{a}}_0(x) = \{\operatorname{diag}(A, c_1, \dots, c_m)\},$$

where $A \in \mathfrak{gl}(\Pi_0, J)$, $c_k \in \mathbb{R}$. Consequently we obtain

$$\dim \mathfrak{aff}(g, J) \leq 2n + \tilde{\mathfrak{a}}_0(x) \leq 2n + \max_{r<n}(2r^2 + n - r) \leq 2n^2 - 2n + 3.$$

Finally assume there is an essential c-projective symmetry.
Two pseudo-Kähler metrics g and $\tilde{g}$ on a complex manifold (M, J) are c-projectively equivalent ($\nabla^g \simeq \nabla^{\tilde{g}}$) if for (1,1)-tensor

$$A = \tilde{g}^{-1} g \cdot \left| \frac{\det(\tilde{g})}{\det(g)} \right|^{1/2(n+1)} : TM \to TM,$$

and $\omega(X, Y) = g(JX, Y)$, $\tau_A = \frac{1}{4}\operatorname{tr}(A)$, $v_A = \operatorname{grad}_g \tau_A$, it holds:

$$(\nabla_X A)Y = g(X, Y)v_A + Y(\tau_A)X + \omega(X, Y)Jv_A - JY(\tau_A)JX.$$

Degree of mobility $D(g, J)$ is dimension of the solution space of this finite type overdetermined PDE system on A. Denote $\mathfrak{i}(g, J)$ resp. $\mathfrak{h}(g, J)$ the algebra of J-holomorphic infinitesimal isometries/homotheties of g. Note that $\mathfrak{h}(g, J) = \mathfrak{i}(g, J)$ if the symmetry acts transitively. By Domashev-Mikeš - Matveev-Roseman

$$\dim \mathfrak{cp}(\nabla^g, J) \leq \dim \mathfrak{h}(g, J) + D(g, J) - 1,$$

$$D(g, J) \leq (n+1)^2, \quad D(g, J)_{\text{sub.max}} = (n-1)^2 + 1 = n^2 - 2n + 2.$$

Now the claim follows from

$$\dim \mathfrak{i}(g, J) = n^2 + 2n, \quad \dim \mathfrak{i}(g, J)_{\text{sub.max}} \leq n^2 + 2.$$