Formality of cosymplectic and Sasakian manifolds

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Joint works with Giovanni Bazzoni, Indranil Biswas, Vicente Muñoz and Aleksy Tralle:
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- N compact cokähler manifold $\Longrightarrow N \times S^1$ compact Kähler manifold and so formal $\Longrightarrow N$ formal.

- Sasakian manifolds
  - All higher Massey products are trivial. But formality is not an obstruction to the existence of Sasakian structures even for simply-connected Sasakian manifolds.

- Cosymplectic (also called almost cosymplectic) manifolds, i.e. the odd-dimensional counterpart to symplectic manifolds:

  There exist compact non-formal cosymplectic manifolds of dimension $(2n+1) \geq 3$.

Contents

Differential graded commutative algebras (DGAs)

$(\mathcal{A}, d_{\mathcal{A}})$ is a DGA (or differential algebra for short) over $\mathbb{R}$ if

- $\mathcal{A}$ is a graded commutative algebra over $\mathbb{R}$, i.e.

$$\mathcal{A} = \bigoplus_{i\geq 0} \mathcal{A}^i \qquad (\mathcal{A}^i \text{ subspace of elements of degree } i)$$

$$\mathcal{A}^p \times \mathcal{A}^q \xrightarrow{\ \cdot\ } \mathcal{A}^{p+q} \quad \text{commutative in the graded sense}$$

$$x \cdot y \;=\; (-1)^{pq}\, y \cdot x, \qquad x \in \mathcal{A}^p,\ y \in \mathcal{A}^q,$$

- $d_{\mathcal{A}} : \mathcal{A}^* \longrightarrow \mathcal{A}^{*+1}$ differential of degree $+1$

$\mathbb{R}$-linear, $\qquad {d_{\mathcal{A}}}^2 = 0,$

$$d_{\mathcal{A}}(x \cdot y) \;=\; (d_{\mathcal{A}}x) \cdot y + (-1)^p\, x \cdot (d_{\mathcal{A}}y)$$

EXAMPLES. M differentiable manifold:

$(\Omega^*(M), d)$ de Rham complex of differential forms on M.

$(H^*(M), 0)$ de Rham cohomology algebra of M with differential $= 0$.

Minimal differential algebras

$(\mathcal{A}, d_{\mathcal{A}})$ is a minimal differential algebra if

- $\mathcal{A} = \bigwedge V =$ Symmetric $\left(V^{2k}\right) \otimes$ Exterior $\left(V^{2k-1}\right)$,

  $V = \oplus V^i$

- V has a basis $\{a_1, a_2, \ldots\}$ such that
  - in each degree, the number of generators is finite;
  - if $i < j$, then $|a_i| \leq |a_j|$, where $|a_i| = \deg(a_i)$;
  - $d_{\mathcal{A}} a_j \in \bigwedge(a_1, \ldots, a_{j-1})$, i.e. $d_{\mathcal{A}} a_j$ is expressed in terms of the preceding a_i $(i < j)$.

- In general, $(\Omega^*(M), d)$ and $(H^*(M), d = 0)$ are non-minimals: they are non-free algebras.

Minimal models

- A minimal differential algebra $(\bigwedge V, d_V)$ is a minimal model of a differentiable manifold M if there is a quasi-isomorphism $\rho : (\bigwedge V, d_V) \longrightarrow (\Omega^*(M), d)$, that is, there is

$$\rho : (\bigwedge V, d_V) \longrightarrow (\Omega^*(M), d) \qquad \text{morphism of DGAs}$$
$$\rho^* : H^*(\bigwedge V, d_V) \xrightarrow{\cong} H^*(M)$$

- A DGA $(\mathcal{B}, d_{\mathcal{B}})$ is a model of M, with minimal model $(\bigwedge V, d_V)$, if there is $\nu : (\bigwedge V, d_V) \longrightarrow (\mathcal{B}, d_{\mathcal{B}})$ quasi $\cong$. So,

$$(\mathcal{B}, d_{\mathcal{B}}) \xleftarrow{\nu} (\bigwedge V, d_V) \xrightarrow{\rho} (\Omega^*(M), d)$$

where ρ and ν are quasi $\cong$.

- (D. Sullivan) If M is simply connected, and $(\bigwedge V, d_V)$ is the minimal model of $M \Longrightarrow ((\pi_i(M) \otimes \mathbb{R}))^* \cong V^i$

Formal manifolds

A differentiable manifold M, with minimal model $(\bigwedge V, d_V)$, is formal if $(H^*(M), 0)$ is a model of M, that is,

$$\exists\, \psi : \left(\bigwedge V, d_V\right) \longrightarrow (H^*(M), 0) \qquad \text{quasi} \cong$$

$$(H^*(M), 0) \xleftarrow{\psi} (\bigwedge V, d_V) \xrightarrow{\rho} (\Omega^*(M), d)$$

- If M has a Riemannian metric for which all wedge products of harmonic forms are harmonic, then M is formal. In this case, M is said to be geometrically formal.
- M simply connected compact manifold, $\dim M \leq 6 \Longrightarrow M$ is formal.
- M connected and compact orientable manifold, $\dim M \leq 4$ and with $b_1(M) = 1 \Longrightarrow M$ is formal.

Triple Massey products

Consider M a differentiable manifold, and

$a = [\alpha] \in H^p(M), \quad b = [\beta] \in H^q(M), \quad c = [\gamma] \in H^r(M).$

The (triple) Massey product $\langle a, b, c\rangle$ is defined if $a \cup b = 0 = b \cup c$, i.e.

$$\begin{aligned} \alpha \wedge \beta &= d\mu, & \mu &\in \Omega^{p+q-1}(M), \\ \beta \wedge \gamma &= d\nu, & \nu &\in \Omega^{q+r-1}(M). \end{aligned}$$

Then, $d\big(\alpha \wedge \nu + (-1)^{p+1}\mu \wedge \gamma\big) = 0$, and

$$\begin{aligned} \langle a, b, c\rangle &= [\alpha \wedge \nu + (-1)^{p+1}\mu \wedge \gamma] \\ &\in \frac{H^{p+q+r-1}(M)}{a \cup H^{q+r-1}(M) + c \cup H^{p+q-1}(M)}. \end{aligned}$$

Higher Massey products

$$a_i \in H^{p_i}(M), \quad a_i = [\alpha_i], \quad 1 \le i \le t, \quad t \ge 4.$$

The higher Massey product $\langle a_1, a_2, \cdots, a_{t-1}, a_t \rangle$ is defined if

$$\langle a_i, a_{i+1}, a_{i+2}, \cdots, a_j \rangle = 0, \qquad 1 \le i < j \le t, \qquad (i,j) \ne (1,t),$$

i.e., there are $\alpha_{i,j} \in \Omega^*(M)$, $1 \le i \le j \le t$, $(i,j) \ne (1,t)$ s. t.

$$\alpha_{i,i} = \alpha_i,$$
$$d\,\alpha_{i,i+1} = (-1)^{|\alpha_{i,i}|}\alpha_{i,i} \wedge \alpha_{i+1,i+1}, \qquad 1 \le i \le t-1,$$
$$d\,\alpha_{i,i+2} = (-1)^{|\alpha_{i,i}|}\alpha_{i,i} \wedge \alpha_{i+1,i+2} + (-1)^{|\alpha_{i,i+1}|}\alpha_{i,i+1} \wedge \alpha_{i+2,i+2}, \cdots$$
$$d\,\alpha_{i,j} = \sum_{k=i}^{j-1}(-1)^{|\alpha_{i,k}|}\alpha_{i,k} \wedge \alpha_{k+1,j}.$$

Then, $\langle a_1, a_2, \cdots, a_t \rangle$ is the set of cohomology classes:

$$\left\{ \left[\sum_{k=1}^{t-1}(-1)^{|\alpha_{1,k}|}\alpha_{1,k} \wedge \alpha_{k+1,t} \right] \right\} \subset H^{p_1+\ldots+p_t-(t-2)}(M),$$

and $\langle a_1, a_2, \cdots, a_t \rangle$ is trivial if $0 \in \langle a_1, a_2, \cdots, a_t \rangle$.

- M formal $\Longrightarrow$ all Massey products of M are trivial

Cosymplectic (or almost cosymplectic) manifolds

N is a cosymplectic manifold if N is a differentiable manifold, $\dim N = 2n + 1$, admitting a

- cosymplectic structure, that is, a pair (η, F) of differential forms, where $\eta \in \Omega^1(N)$ and $F \in \Omega^2(N)$ such that

$$\boxed{d\eta = 0 = dF, \qquad \eta \wedge F^n \neq 0 \quad \text{at every } p \in N}$$

- N is orientable; and if N is compact $\Longrightarrow$ $b_{2i}(N)$ and $b_1(N) \neq 0$, and so N non-simply connected.

- Examples: (M, ω) symplectic manifold $\Longrightarrow$ $(N = M \times \mathbb{R},\ \eta,\ F)$ cosymplectic with

$$\boxed{\eta = dt, \quad F = \omega}$$

  $(N,\ \eta,\ F)$ cosymplectic $\Longrightarrow$ $(N \times \mathbb{R},\ \omega = F + \eta \wedge\ dt)$ symplectic

Cosymplectic manifolds

(N, η, F) cosymplectic manifold. Then,

- the distribution $H = \ker(\eta)$ is integrable since $d\eta = 0$; and
- $\eta \wedge F^n$ volume form on N $\Longrightarrow$ there exists a nowhere vanishing vector field ξ (Reeb vector field) on N given by

$$\boxed{\eta(\xi) = 1, \quad \imath_\xi(F) = 0 \iff \imath_\xi\left(\eta \wedge F^n\right) = F^n}$$

So,

$$\boxed{TN = H \oplus \langle \xi \rangle}$$

and $H = \ker(\eta)$ is also a symplectic distribution.

- If (g_H, J) is an almost Hermitian structure on H with Kähler form F, we have

$$g = g_H + \eta^2 \quad \text{Riemannian metric on } N,$$

such that $TN = H \oplus \langle \xi \rangle$ is orthogonal, $g(\xi, \xi) = 1$, and

$$\phi : TN \to TN, \qquad \phi_H = J, \qquad \phi(\xi) = 0.$$

Almost contact metric structures

Let N be a differentiable manifold, $\dim N = 2n+1$. An almost contact metric structure (η, ξ, ϕ, g) on N:

- $\eta \in \Omega^1(N)$, and a nowhere vanishing vector field ξ on N s. t.

$$\boxed{\eta(\xi) = 1}$$

- $\phi : TN \to TN$ is an endomorphism satisfying

$$\boxed{\phi^2 = -\mathrm{Id} + \eta \otimes \xi} \qquad \phi(\xi) = 0;$$

- g Riemannian metric such that, for X, Y vector fields on N,

$$\boxed{g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)}$$

Thus, the decomposition $TN = H \oplus \langle \xi \rangle$, with $H = \ker(\eta)$, is orthogonal.

- The fundamental 2-form F of (η, ξ, ϕ, g) is defined by

$$F(X, Y) = g(\phi(X), Y),$$

and it satisfies $F(X, Y) = F(\phi(X), \phi(Y))$ and $\eta \wedge F^n \neq 0$.

Cokähler manifolds

An almost contact metric structure (η, ξ, ϕ, g) on N is cokähler if

- (η, F) cosymplectic $(d\eta = 0 = dF)$; and
- $N_\phi = 0$, where N_ϕ is the Nijenhuis tensor of ϕ given by

$$N_\phi(X, Y) = \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + [\phi X, \phi Y],$$

for all vector fields X, Y on N. So (η, ξ, ϕ, g) is normal, i.e. $N_\phi + d\eta \otimes \xi = 0$.

- (N, η, ξ, ϕ, g) is a cokähler manifold $\Longrightarrow$ $(M = N \times \mathbb{R}, h, J)$ $(M = N \times S^1)$ is Kähler, with $h = g + (dt)^2$, $J(\xi) = \partial_t$, and $J(X) = \phi(X)$, for X vector field on N.

  So, if N is compact, then

  - $b_1(N) =$ odd, $\qquad b_{2i}(N) \neq 0$;
  - N is formal $\quad (M = N \times S^1$ formal $\Longrightarrow N$ formal).

Sasakian manifolds

(η, ξ, ϕ, g) is a Sasakian structure on N if

- (η, ξ, ϕ, g) is contact metric, i.e., $F = d\eta$, so η is a contact form $\big(\eta \wedge (d\eta)^n \neq 0\big)$.
- (η, ξ, ϕ, g) normal i.e., the Nijenhuis tensor N_ϕ satisfies

$$N_\phi + d\eta \otimes \xi = 0.$$

or, equivalently,

(N, g) Sasakian $\Longleftrightarrow$ $\big(M = N \times \mathbb{R}^+,\, g^c = t^2\, g + (dt)^2\big)$ Kähler

- (N, η, ξ, ϕ, g) is Sasakian manifold $\Longrightarrow$ ξ is a Killing vector field, $\mathcal{L}_\xi g = 0$.
- (N, η, ξ, ϕ, g) compact Sasakian manifold, $\dim N = 2n + 1$, $\Longrightarrow$ $b_{2i+1}(N)$ are even for $1 \leq (2i + 1) \leq n$.
- (η, ξ, ϕ, g) is K-contact if it is contact and ξ is a Killing vector field.

The mapping torus of a diffeomorphism

M differentiable manifold, and $\varphi : M \to M$ diffeomorphism. The mapping torus M_φ of φ is the manifold, $\dim M_\varphi = (\dim M) + 1$,

$$\boxed{M_\varphi = \frac{M \times [0,1]}{(x,0) \sim (\varphi(x),1)} = \frac{M \times \mathbb{R}}{\mathbb{Z}}}$$

where the action of $\mathbb{Z}$ on $M \times \mathbb{R}$ is given by the diffeomorphism

$$\begin{array}{ccc} M \times \mathbb{R} & \longrightarrow & M \times \mathbb{R} \\ (x,t) & \rightsquigarrow & (\varphi(x), t+1) \end{array}$$

- $\varphi = Id \Longrightarrow M_\varphi = M \times S^1$. In general,

$$\boxed{\begin{array}{ccc} M \hookrightarrow & M_\varphi \xrightarrow{\pi} S^1 \\ & (x,t) \rightsquigarrow e^{2\pi\, it} \end{array}}$$

LOCALLY TRIVIAL FIBER BUNDLE OVER S^1 AND FIBER M

Symplectic mapping tori

(M,ω) symplectic manifold, $\dim M = 2n$, and
$\varphi : (M,\omega) \to (M,\omega)$ symplectomorphism, i.e.,

- $\varphi : M \to M$ diffeomorphism such that $\boxed{\varphi^*\omega = \omega}$

The mapping torus M_φ (of a symplectomorphism φ) is a cosymplectic manifold

(M,ω) symplectic $\implies$ $(M\times\mathbb{R},\, dt,\, \omega)$ cosymplectic manifold. If $\varphi : (M,\omega) \to (M,\omega)$ is a symplectomorphism. Then, we have

$$M\times\mathbb{R} \longrightarrow M_\varphi = \frac{M\times\mathbb{R}}{\mathbb{Z}}$$

The forms dt and ω on $M\times\mathbb{R}$ are $\mathbb{Z}$-invariant. So, they induce a closed 1-form $\eta \in \Omega^1(M_\varphi)$ and a closed 2-form $F \in \Omega^2(M_\varphi)$ s. t.

$$\boxed{d\eta = 0 = dF, \qquad \eta \wedge F^n \neq 0}$$

Not all cosymplectic manifolds are symplectic mapping tori.
Example: $\mathbb{R}^{2n+1}$.

A theorem of Li

Theorem (H. Li, Asian J. Math, 12, 2008)

If N is a compact cosymplectic manifold $\implies$ N is the mapping torus of a symplectomorphism of a compact symplectic manifod, i.e. there exist a compact symplectic manifold (M, ω) and a symplectomorphism $\varphi : (M, \omega) \to (M, \omega)$ such that

$$N = M_\varphi$$

- If (N, η, F) compact cosymplectic $\implies 0 \neq [\eta] \in H^1(N)$. Thus, N is a mapping torus.

Theorem (D. Tischler, Topology 9, 1970.)

A compact manifold is a mapping torus if and only if it admits a non-vanishing closed 1-form.

Cohomology of a mapping torus

M differentiable manifold, and $\varphi : M \to M$ diffeomorphism.

$$M \times \mathbb{R} \longrightarrow M_\varphi = \frac{M \times \mathbb{R}}{\mathbb{Z}}$$

Mayer-Vietoris sequence implies that for any $p \geq 0$,

$$\boxed{\begin{aligned} H^p(M_\varphi) \cong\ & \ker\left(\varphi^* - Id : H^p(M) \longrightarrow H^p(M)\right) \\ & \oplus [dt] \wedge \frac{H^{p-1}(M)}{\operatorname{Im}\left(\varphi^* - Id : H^{p-1}(M) \longrightarrow H^{p-1}(M)\right)} \end{aligned}}$$

Non-formal mapping tori

M compact oriented manifold, and
$\varphi : M \longrightarrow M$ an orientation-preserving diffeomorphism.

$$M \times \mathbb{R} \longrightarrow M_\varphi = \frac{M \times \mathbb{R}}{\mathbb{Z}}$$

$\eta \in \Omega^1(M_\varphi)$ closed 1-form on M_φ induced by $dt \in \Omega^1(\mathbb{R})$.

Theorem (G. Bazzoni, –, V. Muñoz, Trans. AMS 367, 2015)

If for some $p > 0$, the map

$$\varphi^* : H^p(M) \longrightarrow H^p(M)$$

has the eigenvalue $\lambda = 1$ with multiplicity $r = 2$ (*), then M_φ has a non-trivial Massey product, and so it is non-formal.

* $K^p = K_1^p = \ker(\varphi^* - I) \subsetneq K_2^p = \ker(\varphi^* - I)^2 = K_r^p \subset H^p(M), \qquad r \geq 2$

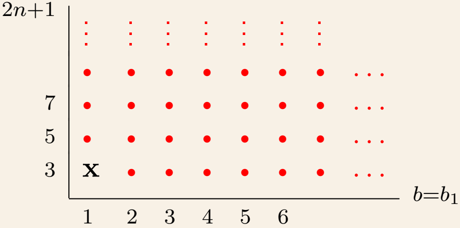
$[\beta] \in (\ker(\varphi^* - I)^2) - \ker(\varphi^* - I)$ and $[\alpha] = (\varphi^* - Id)[\beta]$
Since $[\alpha] \in \ker(\varphi^* - Id) \cap \mathrm{Im}(\varphi^* - Id)$, the Massey product $\langle [\eta], [\eta], [\widetilde{\alpha}] \rangle$ is defined and it is non-trivial.

Geography of non-formal symplectic mapping tori

Theorem (G. Bazzoni, –, V. Muñoz, Trans. AMS 367, 2015)

For every pair $(2n+1, b) \neq (3, 1)$, with $n, b \geq 1$, there is a compact non-formal cosymplectic manifold of dimension $2n+1$ and with first Betti number $b_1 = b$.

If $(2n+1, b) = (3, 1)$ all are formal but not necessarily cokähler.



Examples 3-dim. non-formal with $b_1 = 2k$

Σ_k symplectic surface of genus $k \geq 1$, $H^1(\Sigma_k) = \langle \xi_1, \ldots, \xi_{2k} \rangle$.
Take

$$\varphi : \Sigma_k \longrightarrow \Sigma_k \quad \text{symplectomorphism}$$

$$\varphi^* : H^1(\Sigma_k) \longrightarrow H^1(\Sigma_k)$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ with respect to the basis } \langle \xi_i \rangle.$$

Now,

$$\boxed{\varphi^*(\xi_1) = \xi_1 + \xi_2, \qquad \varphi^*(\xi_i) = \xi_i, \; 2 \leq i \leq k}$$

So, $b_1((\Sigma_k)_\varphi) = 2k$ since $H^1((\Sigma_k)_\varphi) = \langle [\eta], \widetilde{\xi}_2, \ldots, \widetilde{\xi}_{2k} \rangle$, and

$$\boxed{\xi_2 \in \ker(\varphi^* - Id) \cap \operatorname{Im}(\varphi^* - Id)}$$

$$\Longrightarrow \langle [\eta], [\eta], \widetilde{\xi}_2 \rangle \neq 0 \Rightarrow (\Sigma_k)_\varphi \; \underline{\text{NON- FORMAL}}$$

Examples 3-dim. non-formal with $b_1 = 2k - 1$

Σ_k symplectic surface of genus $k \geq 2$, $H^1(\Sigma_k) = \langle \xi_1 \ldots \xi_{2k} \rangle$.
Consider

$$\psi : \Sigma_k \longrightarrow \Sigma_k \quad \text{symplectomorphism}$$

$$\psi^* : H^1(\Sigma_k) \longrightarrow H^1(\Sigma_k)$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with respect to the basis $\langle \xi_i \rangle$.

$$\boxed{\begin{array}{ll} \psi^*(\xi_1) = \xi_1 + \xi_2, & \psi^*(\xi_2) = \xi_2. \\ \psi^*(\xi_3) = \xi_3 + \xi_4, & \psi^*(\xi_i) = \xi_i, \quad 4 \leq i \leq 2k. \end{array}}$$

$$H^1((\Sigma_k)_\psi) = \langle [\eta], \widetilde{\xi}_2, \widetilde{\xi}_4, \ldots, \widetilde{\xi}_{2k} \rangle, \quad b_1 = 2k - 1.$$

Example 5-dim. non-formal with $b_1 = 1$

$\mathbb{T}^4$: 4-torus, $H^1(\mathbb{T}^4) = \langle [e^i],\ 1 \leq i \leq 4 \rangle$,
$\omega = e^1 \wedge e^2 + e^3 \wedge e^4$ symplectic form. Define

$$\varphi : \mathbb{T}^4 \longrightarrow \mathbb{T}^4 \quad \text{symplectomorphism}$$

$$\varphi^* : H^1\left(\mathbb{T}^4\right) \longrightarrow H^1\left(\mathbb{T}^4\right)$$

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Now $\varphi^* : H^1\left(\mathbb{T}^4\right)) \longrightarrow\ H^1\left(\mathbb{T}^4\right)$ has no the eigenvalue $\lambda = 1$.
Thus, $H^1\left((\mathbb{T}^4)_\varphi)\right) = \langle [\eta] \rangle$, i.e., $b_1\left(\mathbb{T}^4_\varphi\right) = 1$.

A non-trivial Massey product on $(\mathbb{T}^4)_\varphi$

$$\varphi^*[e^1 \wedge e^3] = [e^1 \wedge e^3] - [e^1 \wedge e^4],$$
$$\varphi^*[e^2 \wedge e^3] = [e^2 \wedge e^3] - [e^2 \wedge e^4],$$
$$\varphi^*[e^i \wedge e^j] = [e^i \wedge e^j] \text{ otherwise.}$$

- $[e^1 \wedge e^4] \in \ker(\varphi^* - Id) \cap \mathrm{Im}(\varphi^* - Id)$

  $\Longrightarrow$ $\mathbb{T}^4_\varphi$ is non-formal, since it has a non-zero Massey product

$$\boxed{\langle [\eta], [\eta], \widetilde{[e^1 \wedge e^4]} \rangle \neq 0}$$

The basic forms and the basic cohomology

(N, η, ξ, ϕ, g) almost contact metric manifold, $\dim N = 2n+1$, $\mathcal{F}_\xi$ characteristic foliation. A k-form $\alpha \in \Omega^k(N)$ on N is basic if

$$\boxed{\iota_\xi \alpha = 0 = \iota_\xi(d\alpha) \iff \iota_\xi \alpha = 0 = \mathcal{L}_\xi \alpha}$$

$$\Omega^k_B(N) = \Big\{\alpha \in \Omega^k(N) \mid \alpha \text{ basic } k\text{-form}\Big\}.$$

- $\eta \notin \Omega^1_B(N)$ since $\iota_\xi \eta = 1$;
- η contact form (so $\iota_\xi d\eta = 0$) $\Longrightarrow d\eta \in \Omega^2_B(N)$;
- $\Omega^0_B(N) = \{f \in \Omega^0(N) \mid \iota_\xi\, df = \xi(f) = 0\}$.

If $(x, y_1, \dots, y_{2n})$ is a local system of foliated coordinates on an open $U \subset N$, $\xi = \frac{\partial}{\partial x}$, the local expression of a basic k-form α is

$$\alpha_{|U} = \sum f_{i_1 \cdots i_k}(y_1, \dots, y_{2n})\, d\, y_{i_1} \wedge \dots \wedge d\, y_{i_k}.$$

- $\Omega^k_B(N) = 0, \qquad k \geq 2n+1 = \dim N.$

The basic cohomology

- $\alpha \wedge \beta \in \Omega_B^{k+r}(N)$ and $d\,\alpha \in \Omega_B^{k+1}(N)$, for $\alpha \in \Omega_B^k(N)$ and $\beta \in \Omega_B^r(N)$. So, $\Omega_B^*(N) = \bigoplus_{k=0}^{2n} \Omega_B^k(N)$,

$$(\Omega_B^*(N),\, d) \hookrightarrow (\Omega^*(N),\, d).$$

The cohomology of $(\Omega_B^*(N), d)$ is the basic cohomology of (N, η, ξ, ϕ, g),

$$\boxed{H_B^k(N) = \frac{\ker\left(d:\ \Omega_B^k(N) \longrightarrow \Omega_B^{k+1}(N)\right)}{d\,(\Omega_B^{k-1}(N))}, \qquad 0 \le k \le 2n}$$

- $H_B^k(N) = 0,$ $\qquad k \geq 2n+1 = \dim N.$
- $H_B^0(N) \cong \mathbb{R} \cong H^0(N),$ $\qquad H_B^1(N) \hookrightarrow H^1(N)$. So, if N is compact $\Longrightarrow \dim H_B^1(N) \le \dim H^1(N) < \infty.$
But, for $k \geq 2$, $H_B^k(N)$ may be infinite dimensional.
- N compact, η contact $\Longrightarrow$ $0 \neq [(d\eta)^k]_B \in H_B^{2k}(N)$, $k \geq 1$.

The basic cohomology of K-contact manifolds

Theorem (A. El Kacimi and G. Hector, Ann. Inst. Fourier 36, 1986; F. Kamber and Ph. Tondeur, Math. Ann. 277, 1987; A. El Kacimi, Compositio Math. 73, 1990.)

(N, η, ξ, ϕ, g) compact K-contact manifold, $\dim N = 2n+1$, then

i) $\dim H^k_B(N) < \infty$, $\quad 0 \leq k \leq 2n$;

ii) $H^1_B(N) \cong H^1(N)$, $\quad H^{2n}_B(N) = \langle [(d\eta)^n]_B \rangle$;

iii) If (N, η, ξ, ϕ, g) is a compact Sasakian manifold, then

$$H^*_B(N) = \bigoplus_{k=0}^{2n} H^k_B(N)$$

satisfies the hard Lefschetz property with respect to $\omega = [d\eta]_B \in H^2_B(N)$, that is, for $k \leq n$, the map

$$\begin{array}{rccl} L^{n-k}_{\omega}\colon & H^k_B(N) & \longrightarrow & H^{2n-k}_B(N) \\ & \alpha & \longrightarrow & \alpha \cup \omega^{n-k} \end{array}$$

isomorphism

A model for a compact Sasakian manifold

Theorem (A.M. Tievsky, PhD Thesis, MIT, 2008)

(N, η, ξ, ϕ, g) compact Sasakian manifold, $\dim N = 2n+1$.
Then, a model for N is the DGA

$$\Big(H_B^*(N) \otimes \bigwedge(x),\, d\Big),$$

$|x| = 1$, $\qquad d\Big(H_B^*(N)\Big) = 0$, $\qquad d\,x = \omega\,(= [d\eta]_B)$.

- $|x| = 1 \Longrightarrow x \cdot x = 0$;
- $\gamma \in H_B^*(N) \otimes \bigwedge(x)$ and $|\gamma| = p \Longrightarrow \gamma = \alpha + \beta \cdot x$, $\alpha \in H_B^p(N)$, $\beta \in H_B^{p-1}(N)$. So, $d\,\gamma = (-1)^{|\beta|}\,\beta \cdot \omega$;
- If $a = [\alpha + \beta \cdot x] \in H^p\Big(H_B^*(N) \otimes \bigwedge(x),\, d\Big)$, then
    - $a = [\alpha] \in H^p\Big(H_B^*(N) \otimes \bigwedge(x),\, d\Big)$ if $p \leq n$;
    - $a = [\beta \cdot x] \in H^p\Big(H_B^*(N) \otimes \bigwedge(x), d\Big)$ if $p > n$.

Higher order Massey products on Sasakian manifolds

Theorem (I. Biswas, –, V. Muñoz, A. Tralle: arXiv: 1402.6861)

If N is a compact Sasakian manifold, $\dim N = 2n+1$, then all the higher Massey products on N are trivial.

Proof. A model of (N, η, ξ, ϕ, g) is $\Big(H_B^*(N) \otimes \bigwedge(x),\, d\Big)$, where

$$|x| = 1, \qquad d\Big(H_B^*(N)\Big) = 0, \qquad d\,x = \omega\,(= [d\eta]_B).$$

To prove this theorem is equivalent to prove that the higher Massey products

$$\langle a_1, \cdots, a_t \rangle, \quad a_i = [\alpha_i + \beta_i \cdot x] \in H^{p_i}\Big(H_B^*(N) \otimes \bigwedge(x),\, d\Big),\ t \geq 4,$$

are trivial. But $\dim N = 2n+1$ implies that there is at most one cohomology class a_j with $|a_i| = p_i \geq n+1$, that is,

- $a_i = [\alpha_i]$ if $|a_i| = p_i \leq n$, for $1 \leq i \leq t$; or
- there is only one cohomology class a_j, $1 \leq j \leq t$, s. t. $p_j \geq n+1$, i.e.

$$a_j = [\beta_j \cdot x] \quad \text{but} \quad a_i = [\alpha_i] \text{ for } i \neq j$$

Suppose that it is defined the quadruple Massey product

$$\langle a_1, a_2, a_3, a_4\rangle \subset H^{p_1+\ldots+p_4-2}\Big(H_B^*(N)\otimes\bigwedge(x),\, d\Big),\ \ \deg(a_i)\le n,$$

So, $a_i = [\alpha_i]$, where $\alpha_i \in H_B^{p_i}(N)$, $1\le i\le 4$. Take $b\in\langle a_1,a_2,a_3,a_4\rangle$. Then, there are

$$\gamma_{i,j} = \alpha_{i,j} + \beta_{i,j}\cdot x \in \Big(H_B^*(N)\otimes\bigwedge(x), d\Big),$$

$1\le i\le j\le 4$ and $(i,j)\ne(1,4)$, such that

$$\begin{aligned}
\gamma_{i,i} &= \alpha_{i,i} = \alpha_i, && 1\le i\le 4,\\
d\,\gamma_{i,i+1} &= (-1)^{|\gamma_{i,i}|}\gamma_{i,i}\cdot\gamma_{i+1,i+1}, && 1\le i\le 3,\\
d\,\gamma_{1,3} &= (-1)^{|\gamma_{1,1}|}\gamma_{1,1}\cdot\gamma_{2,3} + (-1)^{|\gamma_{1,2}|}\gamma_{1,2}\cdot\gamma_{3,3},\\
d\,\gamma_{2,4} &= (-1)^{|\gamma_{2,2}|}\gamma_{2,2}\cdot\gamma_{3,4} + (-1)^{|\gamma_{2,3}|}\gamma_{2,3}\cdot\gamma_{4,4}.
\end{aligned}$$

$$\boxed{b = (-1)^{|\gamma_{1,1}|}\gamma_{1,1}\cdot\gamma_{2,4} + (-1)^{|\gamma_{1,2}|}\gamma_{1,2}\cdot\gamma_{3,4} + (-1)^{|\gamma_{1,3}|}\gamma_{1,3}\cdot\gamma_{4,4}}$$

$$
\begin{aligned}
\gamma_{i,i} = \alpha_{i,i} =\ & \alpha_i, && 1 \le i \le 4,\\
(-1)^{|\beta_{i,i+1}|}\,\beta_{i,i+1}\cdot\omega = (-1)^{|\alpha_{i,i}|}\,&\alpha_{i,i}\cdot\alpha_{i+1,i+1}, && 1 \le i \le 3,\\
(-1)^{|\beta_{1,3}|}\,\beta_{1,3}\cdot\omega = (-1)^{|\alpha_{1,1}|}\,&\alpha_{1,1}\cdot\alpha_{2,3} + (-1)^{|\alpha_{1,2}|}\,\alpha_{1,2}\cdot\alpha_{3,3},\\
(-1)^{|\beta_{2,4}|}\,\beta_{2,4}\cdot\omega = (-1)^{|\alpha_{2,2}|}\,&\alpha_{2,2}\cdot\alpha_{3,4} + (-1)^{|\alpha_{2,3}|}\,\alpha_{2,3}\cdot\alpha_{4,4},\\
0 = (-1)^{|\alpha_{1,1}|}\,&\alpha_{1,1}\cdot\beta_{2,3} - (-1)^{|\beta_{1,2}|+|\alpha_{3,3}|}\,\beta_{1,2}\cdot\alpha_{3,3},\\
0 = (-1)^{|\alpha_{2,2}|}\,&\alpha_{2,2}\cdot\beta_{3,4} - (-1)^{|\beta_{2,3}|+|\alpha_{4,4}|}\,\beta_{2,3}\cdot\alpha_{4,4}.
\end{aligned}
$$

Now, for $1 \le i \le j \le 4$ and $(i,j) \neq (1,4)$, we consider the new elements $\widetilde{\gamma}_{ij} \in \Big(H_B^*(N) \otimes \bigwedge(x), d\Big)$ given by

$$
\widetilde{\gamma}_{ij} = \widetilde{\alpha}_{ij} + \widetilde{\beta}_{ij}\cdot x \in \Big(H_B^*(N) \otimes \bigwedge(x), d\Big)
$$

$$
\widetilde{\gamma}_{i,i} = \alpha_i\,, \qquad \widetilde{\gamma}_{i,i+1} = \beta_{i,i+1}\cdot x\,, \qquad \widetilde{\gamma}_{i,j} = 0 \ \text{ for } \ j \ge i+2\,.
$$

$$
\boxed{\widetilde{b} = (-1)^{|\widetilde{\gamma}_{1,1}|}\,\widetilde{\gamma}_{1,1}\cdot\widetilde{\gamma}_{2,4} + (-1)^{|\widetilde{\gamma}_{1,2}|}\,\widetilde{\gamma}_{1,2}\cdot\widetilde{\gamma}_{3,4} + (-1)^{|\widetilde{\gamma}_{1,3}|}\,\widetilde{\gamma}_{1,3}\cdot\widetilde{\gamma}_{4,4} = 0}
$$

(M, g, J) compact Kähler manifold, $\dim M = 2n$, with Kähler form ω such that $[\omega] \in H^2(M, \mathbb{Z})$. Take the principal S^1-bundle corresponding to $[\omega]$, that is,

$$S^1 \hookrightarrow N^{2n+1} \xrightarrow{\pi} M^{2n}, \qquad d\eta = \pi^*\omega\,.$$

Then, η is a contact form on N. Moreover, there is a Sasakian structure (η, ξ, ϕ, G) on N (with contact form η) given by

- ξ is the vector field on N such that

$$\boxed{\eta(\xi) = 1, \qquad \imath_\xi(d\eta) = 0}$$

  So,

$$TN = H \oplus \langle \xi \rangle, \qquad H = \ker(\eta)$$

- ϕ and G are given by

$$\boxed{\phi\,(X) \;=\; \big(J(\pi_* X)\,\big)^h, \qquad G \;=\; \pi^*(g) + (\eta)^2}$$

Non-simply connected non-formal Sasakian mfds

Generalized Heisenberg nilmanifold N^{2n+1}. Take the $2n$-dimensional torus T^{2n} with Kähler form

$$\omega = \xi_1 \wedge \xi_2 + \ldots + \xi_{2n-1} \wedge \xi_{2n},$$

where $[\xi_1], [\xi_2], \ldots, [\xi_{2n}]$ are the generators of $H^1(T^{2n}, \mathbb{Z})$. Take the principal S^1-bundle corresponding to $[\omega] \in H^2(T^{2n}, \mathbb{Z})$,

$$S^1 \hookrightarrow N \xrightarrow{\pi} T^{2n}, \qquad d\eta = \pi^*(\omega),$$

Then, N is a non-simply connected Sasakian manifold which is non-formal. In fact, the minimal model of N is

$$\Big(\bigwedge(a_1, \ldots, a_{2n}, x), d\Big), \qquad |a_i| = |x| = 1,$$

$$da_i = 0, \qquad dx = a_1 \cdot a_2 + a_3 \cdot a_4 + \ldots + a_{2n-1} \cdot a_{2n}.$$

Now,

$$a_1 \cdot a_1 = 0, \quad a_1 \cdot (a_2 \cdot a_3 \cdot a_5 \cdot \ldots \cdot a_{2n-1}) = d(x \cdot a_3 \cdot a_5 \cdot \ldots \cdot a_{2n-1}).$$

$$\boxed{\langle a_1, a_1, a_2 \cdot a_3 \cdot a_5 \cdot \ldots \cdot a_{2n-1} \rangle \neq 0}$$

Simply connected compact non-formal Sasakian

$S^2 \times S^2 \times S^2$ simply connected compact Kähler manifold with Kähler form

$$\omega = \omega_1 + \omega_2 + \omega_3,$$

where $[\omega_1], [\omega_2], [\omega_3]$ are the generators of the cohomology $H^2(S^2, \mathbb{Z})$ of each of the S^2-factors. Consider the principal S^1-bundle

$$S^1 \hookrightarrow N \xrightarrow{\pi} S^2 \times S^2 \times S^2, \qquad d\eta = \pi^*(\omega).$$

Then, N is a simply connected compact non-formal Sasakian manifold.

If $(x, y) \in S^2 \times S^2$, the restriction to each $(x, y) \times S^2$ is the circle bundle with Euler class equal to $[\omega_3]$, i.e it is the Hopf bundle

$$S^1 \hookrightarrow S^3 \longrightarrow S^2.$$

Varying $(x, y) \in S^2 \times S^2$, we have that N is the S^3-bundle over $S^2 \times S^2$

$$S^3 \hookrightarrow N \longrightarrow S^2 \times S^2.$$

A minimal model of $S^2 \times S^2$ is the differential algebra

$$\left(\bigwedge(a,b,x,y),\, d\right), \qquad |a| = |b| = 2, \qquad |x| = |y| = 3,$$
$$da = db = 0, \qquad dx = a^2, \qquad dy = b^2.$$

Thus, a minimal model of the total space of the fiber bundle $S^3 \hookrightarrow N^7 \longrightarrow S^2 \times S^2$ is the differential algebra

$$\left(\bigwedge(a,b,x,y,z),\, d\right), \qquad |a| = |b| = 2, \qquad |x| = |y| = |z| = 3,$$
$$da = db = 0, \qquad dx = a^2, \quad dy = b^2, \quad dz = a\, b.$$

Now $a^2 = dx, \quad ab = dz$, which implies that it is defined the triple Massey product

$$\boxed{\langle a, a, b\rangle \neq 0}$$

Simply connected K-contact non-Sasakian manifolds

Theorem (I. Biswas, –, V. Muñoz, A. Tralle: arXiv: 1402.6861)

Let (M, ω) be a simply connected compact symplectic manifold, $\dim M = 2n$, with $0 \notin \langle a_1, a_2, a_3, a_4 \rangle \subset H^*(M)$. Then, there exists a sphere bundle

$$S^{2k+1} \hookrightarrow N \longrightarrow M^{2n}, \qquad (k+1) > n,$$

such that the total space N is K-contact, but N does not admit any Sasakian structure.

<u>Proof</u>. $S^1 \hookrightarrow P^{2n+1} \longrightarrow M^{2n}$ principal S^1-bundle corresponding to $[\omega] \in H^2(M, \mathbb{Z})$, and (S^{2k+1}, ν) compact K-contact manifold. Then, the associated S^{2k+1}-bundle

$$S^{2k+1} \hookrightarrow N = P \times_{S^1} S^{2k+1} \longrightarrow M$$

is such that N has a K-contact structure. The model of N is $\left((\bigwedge V_M, d_V) \otimes \bigwedge(z), D\right)$, where $\deg(z) = 2k+1$, $D(V_M) = d_V$, and $D(z) = 0$ since $2k+2 > 2n = \dim M$.

THANK YOU VERY MUCH!!