# How many are affine connections of special types

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# Introduction

How big is an infinite well determined family of geometric objects? (pseudo-Riemannian metrics, affine connections,...)

To measure an infinite family of real analytic geometric objects we use

- a finite family of arbitrary functions of $k$ variables,
- a family of arbitrary functions of less variables,
- modulo another family of arbitrary functions of less variables.

The last family of functions corresponds to automorphisms of any geometric object from the given family.

# Introduction

In the real analytic case, the Cauchy-Kowalevski Theorem is the standard tool.


- Egorov, Yu.V., Shubin, M.A.: Foundations of the Classical Theory of Partial Differential Equations, Springer-Verlag, Berlin, 1998.
- Kowalevsky, S.: Zur theorie der partiellen differentialgleichungen, J. Reine Angew. Math. **80** (1875) 1–32.
- Petrovsky, I.G.: Lectures on Partial Differential Equations, Dover Publications, Inc., New York, 1991.

# An example

How many real analytic Riemannian metrics in dimension 3?

- Every such metric can be put locally into a diagonal form

Eisenhart, L.P.: Fields of parallel vectors in a Riemannian geometry, Trans. Amer. Math. Soc. **27** (4) (1925) 563–573.

Kowalski, O., Sekizawa, M.: Diagonalization of three-dimensional pseudo-Riemannian metrics, J. Geom. Phys. **74** (2013), 251–255.

- All coordinate transformations preserving diagonal form of the given metric depend on 3 arbitrary functions of two variables.
- Hence all Riemannian metrics in dimension 3 can be locally described by 3 arbitrary functions of 3 variables modulo 3 arbitrary functions of 2 variables.

# Overview of the results

An immediate question arise if we can determine the number of other basic geometric objects, namely the affine connections, in an arbitrary dimension $n$. We shall be occupied with real analytic connections in arbitrary dimension $n$.

- We give an alternative proof of the existence of a system of pre-semigeodesic coordinates.
- We describe the class of affine connections using
  $$\frac{n(n^2-1) \text{ functions of } n \text{ variables}}{\text{modulo } 2n \text{ functions of } n-1 \text{ variables.}}$$
- We describe the class of torsion-free affine connections using
  $$\frac{n(n-1)(n+2)/2 \text{ functions of } n \text{ variables}}{\text{modulo } 2n \text{ functions of } n-1 \text{ variables.}}$$

A well known fact from Riemannian geometry is that a Riemannian connection has symmetric Ricci form.

# Overview of the results

- We prove that the class of all affine connections with *skew-symmetric* Ricci form depends on $\underline{n(2n^2 - n - 3)/2 \text{ functions of } n \text{ variables}}$ and $n(n+1)/2$ functions of $n-1$ variables, modulo $2n$ functions of $n-1$ variables.
- Class of connections with *symmetric* Ricci form depends on $\underline{n(2n^2 - n - 1)/2 \text{ functions of } n \text{ variables}}$ and $n(n-1)/2$ functions of $n-1$ variables, modulo $2n$ functions of $n-1$ variables.
- Class of all torsion-free affine connections with *skew-symmetric* Ricci form depends on $\underline{n(n^2 - 3)/2 \text{ functions of } n \text{ variables}}$ and $n(n+1)/2$ functions of $n-1$ variables, modulo $2n$ functions of $n-1$ variables.
- Class of torsion-free connections with *symmetric* Ricci form depends on $\underline{(n^3 + n^2 - 4n + 2)/2 \text{ functions of } n \text{ variables}}$ modulo $2n$ functions of $n-1$ variables.

# Overview of the results

- All equiaffine connections depends on $\underline{n^3 - 2n + 1 \text{ functions of } n \text{ variables}}$ modulo a constant and modulo $2n$ functions of $n-1$ variables.
- Equiaffine connections with *skew-symmetric* Ricci form depends on $\underline{(2n^3 - n^2 - 5n + 2)/2 \text{ functions of } n \text{ variables}}$ and $n(n+1)/2$ functions of $n-1$ variables, modulo a constant and modulo $2n$ functions of $n-1$ variables.
- Equiaffine connections with *symmetric* Ricci form depends on $\underline{(2n^3 - n^2 - 3n + 2)/2 \text{ functions of } n \text{ variables}}$ and $n(n-1)/2$ functions of $n-1$ variables, modulo a constant and modulo $2n$ functions of $n-1$ variables.

# The Cauchy-Kowalevski Theorem of order 1

Consider a system of PDEs for unknown functions $U^1(x^1,\ldots,x^n),\ldots,U^N(x^1,\ldots,x^n)$ on $\mathcal{U}\subset\mathbb{R}^n$ and of the form

$$
\begin{array}{rcl}
\dfrac{\partial U^1}{\partial x^1} & = & H^1(x^1,\ldots,x^n,U^1,\ldots,U^N,\dfrac{\partial U^1}{\partial x^2},..,\dfrac{\partial U^1}{\partial x^n},\ldots,\dfrac{\partial U^N}{\partial x^2},..,\dfrac{\partial U^N}{\partial x^n}) \\
\dfrac{\partial U^2}{\partial x^1} & = & H^2(x^1,\ldots,x^n,U^1,\ldots,U^N,\dfrac{\partial U^1}{\partial x^2},..,\dfrac{\partial U^1}{\partial x^n},\ldots,\dfrac{\partial U^N}{\partial x^2},..,\dfrac{\partial U^N}{\partial x^n}) \\
 & \cdots & \\
\dfrac{\partial U^N}{\partial x^1} & = & H^N(x^1,\ldots,x^n,U^1,\ldots,U^N,\dfrac{\partial U^1}{\partial x^2},..,\dfrac{\partial U^1}{\partial x^n},\ldots,\dfrac{\partial U^N}{\partial x^2},..,\dfrac{\partial U^N}{\partial x^n}
\end{array}
$$

where $H^i, i=1,\ldots,N$, are real analytic functions of all variables in a neighborhood of

$$(x_0^1,\ldots,x_0^n,a^1,\ldots,a^N,a_2^1,\ldots,a_n^1,\ldots,a_2^N,\ldots,a_n^N),$$

where $x_0^i, a^i, a_j^i$ are arbitrary constants.

# The Cauchy-Kowalevski Theorem of order 1

Further, let the functions $\varphi^1(x^2, \dots, x^n), \dots, \varphi^N(x^2, \dots, x^n)$ be real analytic in a neighborhood of $(x_0^2, \dots, x_0^n)$ and satisfy

$$\begin{aligned} \varphi^j(x_0^2, .., x_0^n) &= a^j, \qquad j = 1, \dots, N, \\ \Big(\frac{\partial \varphi^1}{\partial x^2}, .., \frac{\partial \varphi^1}{\partial x^n}, .., \frac{\partial \varphi^N}{\partial x^2}, .., \frac{\partial \varphi^N}{\partial x^n}\Big)(x_0^2, .., x_0^n) &= (a_2^1, .., a_n^1, .., a_2^N, .., a_n^N). \end{aligned}$$

Then the system has a unique solution
$(U^1(x^1, \dots, x^n), \dots, U^N(x^1, \dots, x^n))$
which is real analytic around $(x_0^1, \dots, x_0^n)$, and satisfies

$$U^i(x_0^1, x^2, \dots, x^n) = \varphi^i(x^2, \dots, x^n), \qquad i = 1, \dots, N.$$

# The Cauchy-Kowalevski Theorem of order 2

The basic assumptions about the system of PDEs are analogous:
The left-hand sides are the second derivatives

$$\frac{\partial^2 U^1}{(\partial x^1)^2}, \dots, \frac{\partial^2 U^N}{(\partial x^1)^2}$$

and the right-hand sides $H^1, \dots, H^N$ involve, as arguments, the original coordinates, the unknown functions $U^1, \dots, U^N$, their first derivatives and their second derivatives except the derivatives written on the left-hand sides:

$$H^i(x^j, U^j, \frac{\partial U^j}{\partial x^k}, \frac{\partial^2 U^j}{\partial x^k \partial x^l}),$$

$$j = 1, \dots, N, \quad k = 1, \dots, n, \quad l = 2, \dots, n.$$

# The Cauchy-Kowalevski Theorem of order 2

There exist locally a unique $n$-tuple $(U^1, \ldots, U^N)$ of real analytic functions which is a solution of the new PDE system, and satisfies the initial conditions

$$\begin{aligned} U^i(x_0^1, x^2, \ldots, x^n) &= \varphi_0^i(x^2, \ldots, x^n), \\ \frac{\partial U^i}{\partial x^1}(x_0^1, x^2, \ldots, x^n) &= \varphi_1^i(x^2, \ldots, x^n). \end{aligned}$$

The general solution then depends
on $2N$ arbitrary functions $\varphi_0^i, \varphi_1^i$ of $n-1$ variables.
See [1], [2] and [3] for the general case and more details.

# Transformation of the connection

We work locally with the spaces $\mathbb{R}[u^1, \ldots, u^n]$, or $\mathbb{R}[x^1, \ldots, x^n]$.
We will use the notation $\mathbf{u} = (u^1, \ldots, u^n)$ and $\mathbf{x} = (x^1, \ldots, x^n)$.
For a diffeomorphism $f\colon \mathbb{R}[\mathbf{u}] \to \mathbb{R}[\mathbf{x}]$, we write $x^k = f^k(u^l)$, or $\mathbf{x} = \mathbf{x}(\mathbf{u})$ for short.

We start with the standard formula for the transformation of the connection, which is

$$\bar{\Gamma}^h_{ij}(\mathbf{u}) = \big(\Gamma^\gamma_{\alpha\beta}(\mathbf{x}(\mathbf{u}))\frac{\partial f^\alpha}{\partial u^i}\frac{\partial f^\beta}{\partial u^j} + \frac{\partial^2 f^\gamma}{\partial u^i \partial u^j}\big) G^h_\gamma. \tag{1}$$

# Transformation of the connection

**Lemma**

*For any affine connection determined by $\Gamma^h_{ij}(\mathbf{x})$, there exist a local transformation of coordinates determined by $\mathbf{x} = f(\mathbf{u})$ such that the connection in new coordinates satisfies*
*$\bar{\Gamma}^h_{11}(\mathbf{u}) = 0$, for $h = 1, \ldots, n$. All such transformations depend on $2n$ arbitrary functions of $n-1$ variables.*

*Proof.* We consider the equations (1) with $\bar{\Gamma}^h_{11}(\mathbf{u}) = 0$, which are

$$0 = \big(\Gamma^\gamma_{\alpha\beta}(\mathbf{x}(\mathbf{u}))\frac{\partial f^\alpha}{\partial u^1}\frac{\partial f^\beta}{\partial u^1} + \frac{\partial^2 f^\gamma}{(\partial u^1)^2}\big) G^h_\gamma, \qquad h = 1, \ldots, n.$$

We multiply these equations by the Jacobi matrix $F^\gamma_h$
and we obtain the equivalent equations

$$\frac{\partial^2 f^\gamma}{(\partial u^1)^2} = -\Gamma^\gamma_{\alpha\beta}(\mathbf{x}(\mathbf{u}))\frac{\partial f^\alpha}{\partial u^1}\frac{\partial f^\beta}{\partial u^1}, \qquad \gamma = 1, \ldots, n.$$

On the right-hand sides, we have analytic functions depending on $f^1, \ldots, f^n$ and their first derivatives.

# Transformation of the connection

We choose arbitrary analytic functions $\varphi^i_\lambda(u^2, \dots, u^n)$, for $i = 1, \dots, n$ and $\lambda = 0, 1$.
According to the Cauchy-Kowalevski Theorem (of pure order 2), there exist unique functions $f^i(u^1, \dots, u^n)$ such that

$$\begin{aligned} f^i(u^1_0, u^2, \dots, u^n) &= \varphi^i_0(u^2, \dots, u^n), \\ \frac{\partial f^i}{\partial u^1}(u^1_0, u^2, \dots, u^n) &= \varphi^i_1(u^2, \dots, u^n). \end{aligned}$$

Obviously, determinant of the Jacobi matrix for these functions will be nonzero for the generic choice of the functions $\varphi^i_\lambda(u^2, \dots, u^n)$.

$\square$

# Connections with arbitrary torsion

**Theorem**

*All affine connections with torsion in dimension $n$ depend locally on $n(n^2-1)$ arbitrary functions of $n$ variables modulo $2n$ arbitrary functions of $(n-1)$ variables.*

*Proof.* In pre-semigeodesic coordinates,
we have $n^3 - n = n(n^2-1)$ functions.
The transformations into pre-semigeodesic coordinates
is uniquely determined up to the choice of $2n$ functions
$\varphi_0^i(u^2,\ldots,u^n), \varphi_1^i(u^2,\ldots,u^n)$ of $n-1$ variables. $\square$

# Connection with zero torsion

**Theorem**

*All affine connections without torsion in dimension $n$ depend locally on $\frac{n(n-1)(n+2)}{2}$ arbitrary functions of $n$ variables modulo $2n$ arbitrary functions of $(n-1)$ variables.*

*Proof.* In pre-semigeodesic coordinates,
we have $\frac{n^2(n+1)}{2} - n = \frac{n(n-1)(n+2)}{2}$ functions.
The transformations into pre-semigeodesic coordinates is uniquely determined up to the choice of $2n$ functions $\varphi_0^i(u^2, \ldots, u^n), \varphi_1^i(u^2, \ldots, u^n)$ of $n-1$ variables. $\square$

# Skew-symmetric Ricci tensor

$$
\begin{aligned}
(\Gamma^2_{12})_1 &= -\sum_{k=3}^{n}(\Gamma^k_{1k})_1 + \Lambda'_{11} + \Lambda_{11}, \\
(\Gamma^1_{ii})_1 &= \Lambda'_{ii} + \Lambda_{ii}, \qquad i > 1, \\
(\Gamma^1_{1i})_1 &= -\sum_{k=2}^{n}(\Gamma^k_{ik})_1 + \Lambda'_{1i} + \Lambda_{1i}, \qquad i > 1, \\
(\Gamma^1_{ij})_1 &= \Lambda'_{ij} + \Lambda_{ij}, \qquad 1 < i < j \le n,
\end{aligned}
$$

# Skew-symmetric Ricci tensor

**Theorem**

*The family of connections with torsion whose Ricci form is skew-symmetric depends locally, on $\frac{n(2n^2-n-3)}{2}$ functions of $n$ variables and $\frac{n(n+1)}{2}$ functions of $n-1$ variables, modulo $2n$ functions of $n-1$ variables.*

*Proof.*

- In pre-semigeodesic coordinates, the family of connections with torsion depends on $q(n) = n(n^2-1)$ functions.
- We have $p(n) = n(n+1)/2$ conditions for the skew-symmetry of the Ricci form.
- We choose the $q(n) - p(n) = n(2n^2 - n - 3)/2$ Christoffel symbols as arbitrary functions.

$\square$

# Skew-symmetric Ricci tensor - without torsion

**Theorem**

*The family of connections without torsion whose Ricci form is skew-symmetric depends locally, on $\frac{n(n^2-3)}{2}$ functions of $n$ variables and $\frac{n(n+1)}{2}$ functions of $n-1$ variables, modulo $2n$ functions of $n-1$ variables.*

*Proof.*

- In pre-semigeodesic coordinates, the family of torsion-free connections depends on $q(n) = n(n-1)(n+2)/2$ functions.
- We have $p(n) = n(n+1)/2$ conditions for the skew-symmetry of the Ricci form.
- The $q(n) - p(n)$ functions can be chosen arbitrarily.

$\square$

# Symmetric Ricci tensor

**Theorem**

*A family of connections with torsion whose Ricci form is symmetric depends locally on $\frac{n(2n^2-n-1)}{2}$ functions of $n$ variables and $\frac{n(n-1)}{2}$ functions of $n-1$ variables modulo $2n$ arbitrary functions of $n-1$ variables.*

*Proof.* In pre-semigeodesic coordinates, there are just $q(n) = n^3 - n = n(n^2 - 1)$ nontrivial Christoffel symbols.

There are $p(n) = n(n-1)/2$ conditions for the symmetry of the Ricci form.

We let the $p(n)$ Christoffel symbols $\Gamma^1_{ij}$, to be determined later and we fix arbitrarily the $q(n) - p(n) = n(2n^2 - n - 1)/2$ other Christoffel symbols. $\square$

# Symmetric Ricci tensor - without torsion

We introduce the notation

$$P_j = \sum_{k=1}^{n} \Gamma^k_{kj}, \qquad j = 1, \ldots, n \tag{2}$$

and we obtain the conditions

$$(P_i)_j - (P_j)_i = 0, \qquad 1 \leq i < j \leq n. \tag{3}$$

This means that there is (locally) a function $F(x^1, \ldots, x^n)$ (unique up to a constant), such that

$$\mathrm{d}F = \sum_{i=1}^{n} P_i \, \mathrm{d}x^i. \tag{4}$$

Conversely, for any choice of an arbitrary function $F(x^1, \ldots, x^n)$, the formula (4) gives a solution $(P_1, \ldots, P_n)$ of the system (3).

# Symmetric Ricci tensor - without torsion

**Theorem**

*A family of connections without torsion whose Ricci form is symmetric depends locally on $\frac{n^3+n^2-4n+2}{2}$ functions of $n$ variables modulo $2n$ arbitrary functions of $n-1$ variables.*

*Proof.* In pre-semigeodesic coordinates, there are just $q(n) = n(n-1)(n+2)/2$ nontrivial Christoffel symbols.

We let the $n$ Christoffel symbols $\Gamma^n_{in}$, $i = 1, \ldots, n$, to be determined later and we fix arbitrarily the $q(n) - n$ other Christoffel symbols. Let us choose further an arbitrary function $F(x^1, \ldots, x^n)$.

Then the functions $P_i$ are well-determined by (4) and the Christoffel symbols $\Gamma^n_{in}$ are uniquely calculated from the equalities (2).

Altogether, we can choose arbitrarily the $\underline{q(n) - n + 1 = (n^3 + n^2 - 4n + 2)/2 \text{ functions}}$ of $n$ variables. □

# Equiaffine connections

We consider a volume element $\omega = f(x^1, \ldots, x^n) \cdot dx^1 \wedge \cdots \wedge dx^n$.

$$\nabla \omega = 0.$$

In the coordinates, we obtain

$$f_{x^k} - f \cdot \sum_{i=1}^{n} \Gamma^i_{ki} \quad = \quad 0, \qquad k = 1, \ldots, n.$$

If we put $L(x^1, \ldots, x^n) = \log(f(x^1, \ldots, x^n))$, then these equations can be written in the form $f_{x^k} = f \cdot L_{x^k}$. We choose an arbitrary function $L(x^1, \ldots, x^n)$ and we want the conditions

$$L_{x^k} \quad = \quad \sum_{i=1}^{n} \Gamma^i_{ki}, \qquad k = 1, \ldots, n$$

to be satisfied.

### Theorem

*The family of equiaffine connections in dimension $n$ depends on $n^3 - 2n + 1$ functions of $n$ variables modulo a constant and modulo $2n$ functions of $n - 1$ variables.*

*Proof.* The family of all connections depends on $n(n^2 - 1)$ Christoffel symbols.

Out of them, $n$ Christoffel symbols are determined from the $n$ equations.
Hence, we choose arbitrarily the function $L$ and all Christoffel symbols except $\Gamma^n_{kn}$.

Altogether, we choose arbitrarily the $n(n^2 - 1) - n + 1 = n^3 - 2n + 1$ functions. □

# Equiaffine connections with torsion and with skew-symmetric Ricci tensor

**Theorem**

*The family of equiaffine connections in dimension $n$ which have skew-symmetric Ricci form depends on $\frac{2n^3-n^2-5n+2}{2}$ functions of $n$ variables and $\frac{n(n+1)}{2}$ functions of $n-1$ variables modulo a constant and modulo $2n$ functions of $n-1$ variables.*

*Proof.* We have started with the $n(n^2-1)$ Christoffel symbols in the pre-semigeodesic coordinates.

Out of them, $n$ were determined using the conditions for $\nabla\omega = 0$ and $n(n+1)/2$ of them were determined from the conditions for the skew-symmetry of the Ricci form.
Further, the function $L$ was chosen arbitrarily.

Altogether,
the $n(n^2-1) - n - n(n+1)/2 + 1 = (2n^3 - n^2 - 5n + 2)/2$ functions were chosen arbitrarily. □

# Equiaffine connections with torsion and with symmetric Ricci tensor

**Theorem**

*The family of equiaffine connections in dimension $n$ which have symmetric Ricci form depends on $\frac{2n^3-n^2-3n+2}{2}$ functions of $n$ variables and $\frac{n(n-1)}{2}$ functions of $n-1$ variables modulo a constant and modulo $2n$ functions of $n-1$ variables.*

*Proof.* We have started with the $n(n^2-1)$ Christoffel symbols in the pre-semigeodesic coordinates.

Out of them, $n$ were determined using the conditions for $\nabla\omega = 0$ and $n(n-1)/2$ of them were determined from the conditions for the symmetry of the Ricci form.
Further, the function $L$ was chosed arbitrarily.

Altogether,
the $n(n^2-1) - n - n(n-1)/2 + 1 = (2n^3 - n^2 - 3n + 2)/2$
functions were chosen arbitrarily. $\square$

# Conclusions

### Theorem

*The number of all equiaffine connections with torsion, or those with skew-symmetric Ricci tensor, or those with symmetric Ricci tensor, respectively, is asymptotically equal at infinity to the number of all affine connections with torsion.*

### Theorem

*The number of torsion free affine connections with skew-symmetric Ricci tensor, or those with symmetric Ricci tensor, respectively, is asymptotically equal at infinity to the number of all torsion free affine connections.*

# References


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