Workshop on almost hermitian and contact geometry

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Killing spinors with torsion parallel under the characteristic connection

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Plan of the talk: We will speak about twistor and Killing spinors with respect to a metric connection with (totally) skew-symmetric torsion $0 \neq T \in \Lambda^3(T^*M)$.

• Twistor spinors: Were introduced by R. Penrose and his collaborators, in General Relativity as solutions of a conformal invariant spinorial field equation (twistor equation).

• During 80s: Systematic investigation of twistor spinors from the view point of conformal geometry (Baum, Lichnerowicz, Friedrich).

• Why twistor spinors with *torsion*?

(eigenvalue estimates of $\not{D}$ by a twistor operator) [Agricola-Becker-B.-Kim'12]

$$\not{D} = D^g + \frac{1}{4}T.$$

Dirac operator and twistor operator

- Let (M^n, g) be connected oriented Riemannian **spin** manifold, Σ the spinor bundle.

$\longrightarrow$ Dirac operator $D^g := \mu \circ \nabla^g : \Gamma(\Sigma) \overset{\nabla^g}{\to} \Gamma(TM \otimes \Sigma) \overset{\mu}{\to} \Gamma(\Sigma)$  P. Dirac (1902–1984)

$$\boxed{D^g(\varphi) = \Sigma_i\, e_i \cdot \nabla^g_{e_i}\varphi}$$

$\longrightarrow$ Penrose or twistor operator

$$P^g := p \circ \nabla^g : \Gamma(\Sigma) \overset{\nabla^g}{\to} \Gamma(TM \otimes \Sigma) \overset{p}{\to} \Gamma(\ker \mu)$$

$p : TM \otimes \Sigma \to \ker \mu \subset TM \otimes \Sigma$ the orthogonal projection onto the kernel of the Clifford multiplication.

Locally $p(X \otimes \varphi) = X \otimes \varphi + \frac{1}{n} \Sigma_{i=1}^n\, e_i \otimes e_i \cdot X \cdot \varphi$, thus

$$\boxed{P^g(\varphi) = \Sigma_{i=1}^n\, e_i \otimes \{\nabla^g_{e_i}\varphi + \tfrac{1}{n} e_i \cdot D^g\varphi\}}$$

Def. A spinor field $\varphi \in \Gamma(\Sigma)$ is called

Killing spinor $\Leftrightarrow \nabla^g_X \varphi = \kappa X \cdot \varphi$

twistor spinor $\Leftrightarrow \varphi \in \mathrm{Ker}(P^g) \Leftrightarrow \nabla^g_X \varphi + \frac{1}{n} X \cdot D^g(\varphi) = 0$

1) $D^g(\varphi) = -n\kappa\varphi$, so $\boxed{\varphi \in \mathrm{Ker}(P^g)}$ automatically.

$\longrightarrow$ *$\varphi \in \Gamma(\Sigma)$ is a Killing spinor iff $\varphi \in \mathrm{Ker}(P^g)$ and φ is a D^g-eigenspinor.*

2) $X^\varphi := \Sigma_i\, i\langle \varphi, e_i \cdot \varphi \rangle$ *Killing vector field* of (M^n, g)

3) $\exists\ \varphi \in \mathcal{K}(M^n, g)_\kappa\ \kappa \in \mathbb{R}\backslash\{0\} \Rightarrow (M^n, g)$ compact Einstein with positive Scal^g,

$$\boxed{\mathrm{Ric}^g(X) = 4(n-1)\kappa^2 X, \quad \mathrm{Scal}^g = 4n(n-1)\kappa^2.}$$

- Th. Friedrich's inequality (1980): (M^n, g) compact + spin. Then, the first eigenvalue of D^g satisfies

$$\boxed{\boxed{\lambda^2 \geqslant \frac{n}{4(n-1)} \mathrm{Scal}^g_{\min}}}$$

$\longrightarrow$ equality case: φ must be a real KS: $\nabla^g_X \varphi = \mp \frac{1}{2}\sqrt{\frac{\mathrm{Scal}^g_{\min}}{n(n-1)}} X \cdot \varphi = \kappa X \cdot \varphi$.

Elementary properties of twistor spinors

(due to H. Baum '81, A. Lichnerowicz '87-'88 and Th. Friedrich '89)

Let $\boxed{\varphi \in \mathrm{Ker}(P^g)}$. Then,

(1) For a conformal change $\tilde{g} = e^{2u} g$, then

$$(TM, g) \cong (TM, \tilde{g}), X \mapsto \widetilde{X} := e^{-u} X, \quad (\Sigma, g) \to (\widetilde{\Sigma}, \tilde{g}), \varphi \to \widetilde{\varphi}$$

$$D^{\tilde{g}} = e^{-\frac{n-1}{2}u} \circ D^g \circ e^{\frac{n-1}{2}u}, \quad P^{\tilde{g}} = e^{\frac{u}{2}} \circ P^g \circ e^{-\frac{u}{2}}$$

$$\varphi \in \mathrm{Ker}(P^g) \Leftrightarrow e^{\frac{u}{2}} \tilde{\varphi} \in \mathrm{Ker}(P^{\tilde{g}})$$

(2) $(D^g)^2(\varphi) = \frac{n}{4(n-1)} \mathrm{Scal}^g \varphi$.

(3) $\nabla^g_X(D^g(\varphi)) = \frac{n}{2(n-2)}\big[- \mathrm{Ric}^g(X) + \frac{\mathrm{Scal}^g}{2(n-1)} X\big] \cdot \varphi = \frac{n}{2} \mathrm{Sch}^g(X) \cdot \varphi$.

Consequences

$\Rightarrow$ $\varphi \in \Gamma(\Sigma)$ is a twistor spinor iff $\varphi \oplus D^g(\varphi) \in \Gamma(E)$ is **parallel** with respect the covariant derivative on $E = \Sigma \oplus \Sigma$ given by

$$\nabla^E_X = \begin{pmatrix} \nabla^g_X & (1/n)X \\ -\frac{n}{2}\,\mathrm{Sch}^g(X) & \nabla^g_X \end{pmatrix}.$$

$\Rightarrow$ Any twistor spinor $\varphi \in \mathrm{Ker}(P^g)$ is defined by its values $\varphi(p), D^g(\varphi(p))$ at some point $p \in M$.

$\Rightarrow$ $\dim \mathrm{Ker}(P^g) \leqslant 2^{[\frac{n}{2}]+1} = 2(\dim \Delta_n)$.

$\Rightarrow$ Zeros of $\varphi \in \mathrm{Ker}(P^g)$ are **isolated**.

Generalization to metric connections with skew-torsion

- $\{\nabla^s : s \in \mathbb{R}\}$, totally skew-symmetric torsion $4sT$, $T \in \wedge^3 M$:

$$g(\nabla^s_X Y, Z) = g(\nabla^g_X Y, Z) + 2sT(X, Y, Z), \quad s \in \mathbb{R}.$$

Special members:

- $s = 0 \longrightarrow$ L-C connection $(T \equiv 0)$
- $s = 1/4 \longrightarrow \nabla^{1/4} \equiv \nabla^c := \nabla^g + \frac{1}{2}T$ (**characteristic connection**)

- General Strategy: Given a G-structure $(G \subsetneq \mathrm{SO}(n))$ replace the L-C connection ∇^g with a metric connection preserving the G-structure.

$\Longrightarrow$ If the torsion form $T \in \Lambda^3(T^*M)$ ($\Leftrightarrow$ same geodesics with ∇^g), then we get the characteristic connection ∇^c:

$$g(\nabla^c_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2}T(X, Y, Z).$$

- If G is contained in the stabiliser of a generic spinor field then $\exists$ a ∇^c-parallel spinor: $\nabla^c \varphi = 0$
  - $n = 5 \Rightarrow G = \mathrm{SU}(2)$
  - $n = 6 \Rightarrow G = \mathrm{SU}(3)$
  - $n = 7 \Rightarrow G = \mathrm{G}_2$
  - $n = 8 \Rightarrow G = \mathrm{Spin}(7)$

The lift of ∇^s and ∇^c into Σ:

$$\nabla^s_X\varphi = \nabla^g_X\varphi + s(X\lrcorner T)\cdot\varphi, \quad \nabla^c_X\varphi = \nabla^g_X\varphi + \tfrac{1}{4}(X\lrcorner T)\cdot\varphi.$$

- $D^s = \mu\circ\nabla^s$ $\boxed{D^s(\varphi) = \Sigma_i\, e_i\cdot\nabla^s_{e_i}\varphi = D^g(\varphi) + 3sT\cdot\varphi}$

- $P^s = p\circ\nabla^s$ $\boxed{P^s(\varphi) = \Sigma_{i=1}^n\, e_i\otimes\{\nabla^s_{e_i}\varphi + \tfrac{1}{n}e_i\cdot D^s\varphi\}}$

Def. [Agricola-Becker-Bender-Kim'12], [Agricola-Höll'13]

Killing spinor with torsion (KsT) $\Leftrightarrow$ $\nabla^s_X\varphi = \zeta X\cdot\varphi$, for some $\zeta\in\mathbb{R}\backslash\{0\}$ and $s\in\mathbb{R}\backslash\{0\}$.

Twistor spinor with torsion (TsT) $\Leftrightarrow$ $\varphi\in\mathrm{Ker}(P^s)$ for some $s\neq 0$ $\Leftrightarrow$

$$\nabla^s_X\varphi + \frac{1}{n}X\cdot D^s(\varphi) = 0.$$

The interest in TsT and in some special cases KsT is due to the fact that they realize (under the condition $\nabla^c T = 0$) the equality case for eigenvalue estimates of the **cubic Dirac operator** [Agricola-Becker-B.-Kim'12]

$$\not{D} = D^g + \frac{1}{4}T$$

i.e. the Dirac operator associated to the connection with torsion $T/3$.

From now on we assume (M^n, g, T)*:* $\boxed{\nabla^c T = 0}$.

Consequences

- $dT = 2\sigma_T$ where $\sigma_T = \frac{1}{2}\Sigma_i (e_i \lrcorner T) \wedge (e_i \lrcorner T)$.
- $\delta^s T = 0 \; \forall \; s \in \mathbb{R} \Rightarrow \boxed{\mathrm{Ric}^s = \mathrm{Ric}^g - 4s^2 S}$ $\quad S(X,Y) = \Sigma_i \, g(T(X,e_i), T(Y,e_i))$

$\rightarrow$ Set: $\boxed{\mathcal{D}^s(\varphi) := \Sigma_i (e_i \lrcorner T) \cdot \nabla^s_{e_i} \varphi.}$

(Generalized) Schröndinger-Lichnerowicz formulas

$$(D^g)^2 = \Delta^g + \frac{1}{4}\mathrm{Scal}^g,$$ [Scrönd.'62], [Lichn.'63]

$$\not{D}^2 = \Delta^{1/4} + \frac{1}{4}\mathrm{Scal}^g + \frac{1}{4} dT - \frac{1}{8}\|T\|^2$$ [Bis.'89], [Ko.'99], [I.-D.'00], [Agric.'02]

$$= \Delta^{1/4} + \frac{1}{4}\mathrm{Scal}^g - \frac{1}{4} T^2 + \frac{1}{8}\|T\|^2$$

$$(D^c)^2 = \Delta^{1/4} + \frac{1}{2} dT - \mathcal{D}^{1/4} + \frac{1}{4}\mathrm{Scal}^c$$ [Friedr.-Iv.'01], [Agric.'02]

$$(D^s)^2 = \Delta^s + s(3-4s) dT - 4s\mathcal{D}^s + \frac{1}{4}\mathrm{Scal}^c$$ [F.-Iv.'01], [Agric.'02], [A.-F.'03]

$$(D^{s/3})^2 = \Delta^s - sT^2 + \frac{1}{4}\mathrm{Scal}^g + (s - 2s^2)\|T\|^2$$ [F.-Iv. '01], [A.-Friedr.'03]

The anticommutaror of D^s and T (in the case $\boxed{\nabla^c T = 0}$)

$$D^s \cdot T + T \cdot D^s = (1-4s)dT - 2\mathcal{D}^s \Rightarrow D^c \cdot T + T \cdot D^c = -2\mathcal{D}^s$$ [Friedr.-Iv.'01]

$$(D^{s/3})^2 \cdot T = T \cdot (D^{s/3})^2 \Rightarrow \quad \not{D}^2 \cdot T = T \cdot \not{D}^2$$ [Agric.-Friedr.'03]

- Σ decomposes into a direct sum of T-eigenbundles preserved by ∇^c: [Agric.-Friedr.'03]

$$\Sigma = \bigoplus_{\gamma \in \mathrm{Spec}(T)} \Sigma_\gamma, \quad \nabla^c \Sigma_\gamma \subset \Sigma_\gamma,\ \forall \gamma \in \mathrm{Spec}(T), \quad \Gamma(\Sigma) = \bigoplus_{\gamma \in \mathrm{Spec}(T)} \Gamma(\Sigma_\gamma)$$

Universal estimate $\boxed{\lambda_1(\not{D}^2|_{\Sigma_\gamma}) \geqslant \frac{1}{4}\mathrm{Scal}^g_{\min} + \frac{1}{8}\|T\|^2 - \frac{1}{4}\gamma^2 := \beta_{\mathrm{univ}}(\gamma)}$

- *equality case:* iff φ is $\boxed{\nabla^c\text{-parallel}}$ and $\mathrm{Scal}^g =$ constant.

- **Twistorial estimate** [Agric.-Becker-B.-Kim'12].

$$\boxed{\lambda_1(\not{D}^2|_{\Sigma_\gamma}) \geqslant \frac{n}{4(n-1)}\mathrm{Scal}^g_{\min} + \frac{n(n-5)}{8(n-3)^2}\|T\|^2 + \frac{n(4-n)}{4(n-3)^2}\gamma^2 := \beta_{\mathrm{tw}}(\gamma)}$$

- *equality case:* iff φ is $\boxed{\text{TsT}}$ for $s = (n-1)/4(n-3)$ and $\mathrm{Scal}^g =$ constant.

Twistor spinors with torsion

Lem.1 [Chr-2015] For any twistor spinor $\varphi \in \mathrm{Ker}(P^s)$ and for any vector field X the following relations hold:

$$\begin{aligned} -\frac{1}{2}\mathrm{Ric}^s(X)\cdot\varphi &= -\frac{8s}{n}(X\lrcorner T)\cdot D^s(\varphi) + \frac{n-2}{n}\nabla^s_X(D^s(\varphi)) \\ &\quad -\frac{1}{n}X\cdot(D^s)^2(\varphi) - s(3-4s)(X\lrcorner\sigma_T)\cdot\varphi. \\ \frac{1}{2}\mathrm{Scal}^s\,\varphi &= -\frac{24s}{n}T\cdot D^s(\varphi) + \frac{2(n-1)}{n}(D^s)^2(\varphi) \\ &\quad -4s(3-4s)\sigma_T\cdot\varphi. \end{aligned}$$

Hints: [Becker-Bender's PhD '12], [Agric.-Becker-B.-Kim '12] $(s = 1/4)$

$$\sum_i e_i\cdot R^s(X,e_i)\varphi = -\frac{1}{2}\mathrm{Ric}^s(X)\cdot\varphi + s(3-4s)(X\lrcorner\sigma_T)\cdot\varphi.$$

Remark : For a TsT $\varphi \in \mathrm{Ker}(P^s)$ it holds that

$$\boxed{\mathcal{D}^s(\varphi) = -\tfrac{3}{n}T\cdot D^s(\varphi), \quad \Delta^s(\varphi) = \tfrac{1}{n}(D^s)^2(\varphi)}$$

...Lemma 1 implies that

$$\nabla^s_X(D^s(\varphi)) = \frac{n}{2}\,\mathrm{Sch}^s(X)\cdot\varphi + \frac{sn}{(n-1)(n-2)}\Big[\Big(\frac{8(n-1)}{n}(X\lrcorner T) + \frac{12}{n}X\cdot T\Big)\cdot D^s(\varphi) + (3-4s)\big(X\cdot dT + (n-1)(X\lrcorner\sigma_T)\big)\cdot\varphi\Big].$$

Lem.2 The mapping $\nabla^{s,E}:\Gamma(E)\to\Gamma(T^*M\otimes E)$ given by

$$\nabla^{s,E}_X(\Phi) = \Big(\nabla^s_X\varphi_1 + \frac{1}{n}X\cdot\varphi_2\Big)\oplus\Big(-\frac{n}{2}\,\mathrm{Sch}^s(X)\cdot\varphi_1 - \frac{sn(3-4s)}{(n-1)(n-2)}\big[X\cdot dT + (n-1)(X\lrcorner\sigma_T)\big]\cdot\varphi_1 - \frac{sn}{(n-1)(n-2)}\big[\frac{8(n-1)}{n}(X\lrcorner T) + \frac{12}{n}X\cdot T\big]\cdot\varphi_2 + \nabla^s_X\varphi_2\Big),$$

where $X\in\Gamma(TM)$ and $\Phi:=\varphi_1\oplus\varphi_2\in\Gamma(\Sigma\oplus\Sigma)$, defines a covariant derivative on the vector bundle $E=\Sigma\oplus\Sigma$.

• Special member: $s=0 \longrightarrow \nabla^{0,E}\equiv\nabla^E$.

Thm.1 [Chr'15] Let (M^n, g, T) $(n \geqslant 3)$ a connected Riemannian spin manifold with $\nabla^c T = 0$. Then, any twistor spinor with torsion $\varphi \in \mathrm{Ker}(P^s)$ satisfies the equation

$$\nabla_X^{s,E}(\varphi \oplus D^s(\varphi)) = 0.$$

Conversely, if $(\varphi \oplus \psi) \in \Gamma(E)$ is $\nabla^{s,E}$-parallel, then φ is a twistor spinor with torsion such that $D^s(\varphi) = \psi$.

Cor. Let (M^n, g, T) $(n \geqslant 3)$ a connected Riemannian spin manifold with $\nabla^c T = 0$. Then,

a) $\dim \mathrm{Ker}(P^s) \leqslant 2^{[\frac{n}{2}]+1} = 2(\dim \Delta_n)$.

b) If φ and $D^s(\varphi)$ vanish at some point $p \in M$ and $\varphi \in \mathrm{Ker}(P^s)$, then $\varphi \equiv 0$.

Prop.1 [Chr.'15] Let (M^n, g, T) $(n \geqslant 3)$ be a connected Riemannian spin manifold with $\nabla^c T = 0$. Then, any zero point of $0 \neq \varphi \in \mathrm{Ker}(P^s)$ is isolated, i.e. the zero-set of φ is discrete.

Hints: Compute the Hessian Hess^{∇^s} of the function $|\varphi|^2$ in $p \in M$.

$$\mathrm{Hess}_p^{\nabla^s}(|\varphi|^2)(X, Y) = \frac{2}{n^2}\Big[\big(Y \cdot D^s(\varphi), X \cdot D^s(\varphi)\big)\Big]_p = \frac{2}{n^2}(|D^s(\varphi)|^2)_p g_p(X, Y).$$

- So, if $(D^s(\varphi))_p \neq 0$, then p is a non-degenerate critical point of $|\varphi|^2$ and thus an isolated zero point of φ. If $(D^s(\varphi))_p = 0$, then φ must be trivial.

∇^c-parallel spinors and characteristic spinors

- ∇^c-parallel spinor: $\nabla^c\varphi = 0$
- characteristic, or ∇^c-harmonic spinor: $D^c(\varphi) = 0$

Known results (1) [Friedr.-Iv.'01] Let $\varphi_0 \in \mathrm{Ker}(\nabla^c)$. Then,

$$\begin{aligned} \mathrm{Scal}^c\,\varphi_0 &= -2dT\cdot\varphi_0 = -4\sigma_T\cdot\varphi_0, \quad (*) \\ \mathrm{Ric}^c(X)\cdot\varphi_0 &= \frac{1}{2}(X\lrcorner dT + \nabla^c_X T)\cdot\varphi_0. \end{aligned}$$

(2) The scalar curvatures $\mathrm{Scal}^c, \mathrm{Scal}^g$ are **constant**. If in addition $\varphi \in \Sigma_\gamma$ for some $\gamma \neq 0$, i.e. $T\cdot\varphi = \gamma\varphi$, $\Rightarrow$

$$\mathrm{Scal}^g = 2\gamma^2 - \frac{1}{2}\|T\|^2, \quad \mathrm{Scal}^c = 2(\gamma^2 - \|T\|^2), \quad \mathrm{Ric}^c(X) = \frac{1}{2}(X\lrcorner dT)\cdot\varphi_0 = (X\lrcorner\sigma_T)\cdot\varphi_0.$$

(3) In the presence of a ∇^c-parallel spinor $\varphi \in \Sigma_\gamma$ [Agric.-Becker-B.-Kim. '12]

$$\beta_{\mathrm{tw}}(\gamma) \leqslant \beta_{\mathrm{univ}}(\gamma).$$

For $n \leqslant 8$ [Agric.-Becker-B.-Kim. '12]

$$0 \leqslant 2n\|T\|^2 + (n-9)\gamma^2, \qquad \mathrm{Scal}^g \leqslant \frac{9(n-1)}{2(9-n)}\|T\|^2.$$

- *equality case:* iff **universal estimate** coincides with **twistorial estimate**.

• It is an interesting question to check if these twistor spinors with torsion are also some **kind** of *Killing spinors* and what the geometric inclusions are when the two estimates coincide, if any.

$\Longrightarrow$ In general, the twistor equation with torsion cannot be reduced to a Killing spinor equation

• **Do there exist exceptions? YES!**

• $n = 3 \Rightarrow (\mathrm{S}^3, g_{\mathrm{can}})$ (new)

• $n = 6 \Rightarrow$ nearly Kähler mnfds. [Agric.-Becker-B.-Kim. '12]

• $n = 7 \Rightarrow$ nearly parallel G_2-mnfds. (new)

Prop.2 [Chr.'15] Let (M^n, g, ∇^c) be a compact connected Riemannian spin manifold with $\nabla^c T = 0$ and positive scalar curvature, carrying a non-trivial spinor field $0 \neq \varphi_0 \in \Gamma(\Sigma)$ such that

$$\boxed{\nabla^c \varphi_0 = 0, \quad T \cdot \varphi_0 = \gamma \varphi_0.}$$

Then, φ_0 is a real Killing spinor (with respect to g) if and only if

$$\gamma^2 = \frac{4n}{9(n-1)} \mathrm{Scal}^g . \quad (\dagger)$$

If this is the case, then the Killing number is given by $\kappa := 3\gamma/4n$ and

$$\boxed{(X \lrcorner T) \cdot \varphi_0 + \tfrac{3\gamma}{n} X \cdot \varphi_0 = 0, \quad \forall\, X \in \Gamma(TM).}$$

- For $n \leqslant 8$, the condition $(\dagger)$ is equivalent to

$$\gamma^2 = \frac{2n}{9-n}\|T\|^2, \quad \text{or} \quad \mathrm{Scal}^g = \frac{9(n-1)}{2(9-n)}\|T\|^2.$$

If this is the case, then $dT \cdot \varphi_0 = -\frac{3\gamma^2(n-3)}{2n}\varphi_0$.

Hints: (1) Since

$$D^c \equiv D^{1/4} = D^g + \frac{3}{4}T$$

and $\varphi_0 \in \Sigma_\gamma$ is ∇^c-parallel, it follows that φ_0 is an eigenspinor of the Riemannian Dirac operator D^g

$$D^g(\varphi_0) = -\frac{3\gamma}{4}\varphi_0.$$

(2) [Friedrich '80] Given a compact Riemannian spin manifold with positive scalar curvature, if $\varphi_0 \in \Gamma(\Sigma)$ is an eigenspinor of D^g with one of the eigenvalues

$$\pm\frac{1}{2}\sqrt{\frac{n\,\mathrm{Scal}^g_{\min}}{n-1}},$$

then φ_0 is a real Killing spinor with corresponding Killing number

$$\kappa := \mp\frac{1}{2}\sqrt{\frac{\mathrm{Scal}^g_{\min}}{n(n-1)}}.$$

(3) Since $\varphi \in \mathrm{Ker}(\nabla^c) := \{\varphi \in \Gamma(\Sigma_\gamma) \subset \Gamma(\Sigma) : \nabla^c\varphi = 0\}$, Scal^g is **constant**

Cor. Let $\varphi \in \mathrm{Ker}(\nabla^c)$. If $\boxed{\beta_{\mathrm{univ}}(\gamma)=\beta_{\mathrm{tw}}(\gamma)}$, then φ is a **real Killing spinor**, in particular $\varphi \in \mathcal{K}(M^n, g)_{\frac{3\gamma}{4n}} \cap \Gamma(\Sigma_\gamma)$.

Question: What about the converse? **YES**

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Observe that on a compact connected Riemannian spin manifold (M^n, g, T) with positive scalar curvature

- real Killing spinors with Killing number $\kappa = \frac{3\gamma}{4n}$,
- ∇^c-parallel spinors lying in Σ_γ, or
- characteristic spinors lying in Σ_γ,

if existent, they are sharing a common property:

They are all eigenspinors of the Riemannian Dirac operator with the same eigenvalue $-\frac{3\gamma}{4} \neq 0$, where $0 \neq \gamma \in \mathrm{Spec}(T)$ is a T-eigenvalue.

Examples.

(1) 6-dimensional nearly Kähler manifolds

[Gray'70], [Friedr.-Grunewald '86], [Friedr.-Ivanov.'01]

Consider a 6-dimensional NEARLY KÄHLER MANIFOLD, i.e. an almost Hermitan manifold (M^6, g, J) such that

$$(\nabla^g_X J)X = 0.$$

- There exists unique characteristic connection ∇^c with parallel skew-torsion, given by

$$T(X, Y) := (\nabla^g_X J)JY.$$

$\longrightarrow$ There exist two ∇^c-parallel spinors $\varphi^\pm$ lying in Σ_γ with $\gamma = \pm 2\|T\|$.

- Both are real Killing spinors with Killing number given by $\kappa := \pm\|T\|/4$.

Indeed,

$$\mathrm{Ric}^g = \frac{5}{4}\|T\|^2, \quad \mathrm{Scal}^g = \frac{15}{2}\|T\|^2$$

Notice that Scal^g coincides with $\frac{9(n-1)}{4n}\gamma^2 = \frac{9(n-1)}{2(9-n)}\|T\|^2$. Thus, the spinors $\varphi^\pm$ must be real Killing spinors with $\kappa = 3\gamma/4n = \pm\|T\|/4$. In particular, $dT \cdot \varphi^\pm = -\frac{1}{2}\,\mathrm{Scal}^c \cdot \varphi^\pm = -\frac{3\gamma^2(n-3)}{2n}\varphi^\pm$, i.e. $dT \cdot \varphi^\pm = -3\|T\|^2\varphi^\pm$.

(2) Nearly parallel G_2-manifolds

[Fr.-Kath.-Mor.-Sem.'97], [Friedr.-Iv.'01]

- G_2 14-dimensional compact simple Lie group can be viewed as;

$\Longrightarrow$ the stabiliser of a generic 3-form $\omega \in \wedge^3(\mathbb{R}^7)$

$$\omega := e_{127} + e_{135} - e_{146} - e_{236} - e_{245} - e_{347} + e_{567}.$$

$\Longrightarrow$ or the subgroup of $\mathrm{Spin}(7)$ preserving the unit spinor $\psi_0 = (1, 0, \ldots, 0)^t \in \mathrm{S}(\Delta_7) \subset \Delta_7 := \mathbb{R}^8$

$$\mathrm{G}_2 = \{A \in \mathrm{GL}_n\mathbb{R} : \omega = A^*\omega\} = \{g \in \mathrm{Spin}(7) : g\psi_0 = \psi_0\}.$$

- Under the action of G_2,

$$\underline{\Delta_7 = \mathbb{R}\psi_0 \oplus \{X \cdot \psi_0 : X \in \mathbb{R}^7\}.}$$

In particular ω and ψ_0 induce equivalent data:

$$\omega(X, Y, Z) = \langle X \cdot Y \cdot Z \cdot \psi_0, \psi_0\rangle.$$

Def. A G_2-structure on M^7 is a G_2-reduction of the frame bundle $\mathrm{SO}(TM^7)$.

Def. A G_2-structure on M^7 is said to be NEARLY PARALLEL if the associated spinor φ is a real Killing spinor. This is equivalent to say that $d\omega = -\tau_0 * \omega$ for some real constant $\tau_0 \neq 0$.

- $\exists!$ characteristic connection ∇^c with parallel skew-torsion T given by

$$T := \frac{1}{6}(d\omega, *\omega) \cdot \omega = -\frac{\tau_0}{6}\omega, \quad \|T\|^2 = \frac{7}{36}\tau_0^2,$$

$\longrightarrow$ There is a unique ∇^c-parallel spinor φ_0 with $\gamma = -\sqrt{7}\|T\|$. [Friedrich-Ivanov '01]

- φ_0 is a real Killing spinor with $\kappa = -\frac{3}{4\sqrt{7}}\|T\|$.

$$\mathrm{Ric}^g = \frac{27}{14}\|T\|^2\,\mathrm{Id}, \quad \mathrm{Scal}^g = \frac{27}{2}\|T\|^2$$

Indeed, Scal^g coincides with $\frac{9(n-1)}{4n}\gamma^2 = \frac{9(n-1)}{2(9-n)}\|T\|^2$. Thus, φ_0 must be a Killing spinor with $\kappa = 3\gamma/4n = -\frac{3}{4\sqrt{7}}\|T\|$. In particular, $dT \cdot \varphi_0 = -\frac{3\gamma^2(n-3)}{2n}\varphi_0$, i.e. $dT \cdot \varphi_0 = -6\|T\|^2\varphi_0$.

Lem.3 Let $0 \neq \gamma \in \mathrm{Spec}(T)$ be a real non-zero T-eigenvalue. Then, a real Killing spinor $\varphi \in \mathcal{K}(M^n, g)_{\frac{3\gamma}{4n}}$ is characteristic i.e. $D^c(\varphi) = 0$, iff $\varphi \in \Sigma_\gamma$.

Hints: $D^c(\varphi) = D^g(\varphi) + \frac{3}{4}T \cdot \varphi = -\frac{3\gamma}{4}\varphi + \frac{3}{4}T \cdot \varphi$.

- One **cannot** claim that a Killing spinor with $\kappa = 3\gamma/4n$ is ∇^c-parallel, **even if** it is characteristic.

- With the aim to construct the desired *one-to-one correspondence* between spinor fields in $\mathrm{Ker}(\nabla^c)$ and

$$\mathrm{Ker}(D^c) := \{\varphi \in \Gamma(\Sigma_\gamma) \subset \Gamma(\Sigma) : D^c(\varphi) = 0\},$$

the eigenvalues of the endomorphism $\boxed{dT + \frac{1}{2}\mathrm{Scal}^c}$ on Σ must be **non-negative** [Friedr.-Iv.'01].

Thm.2 [Chr.'15] Let (M^n, g, T) be compact connected Riemannian spin manifold (M^n, g, T), with $\nabla^c T = 0$ and positive scalar curvature given by $\mathrm{Scal}^g = \frac{9(n-1)\gamma^2}{4n}$ for some constant $0 \neq \gamma \in \mathrm{Spec}(T)$. If the symmetric endomorphism $\boxed{dT + \frac{1}{2}\Big[\frac{9(n-1)}{4n}\gamma^2 - \frac{3}{2}\|T\|^2\Big]}$ acts on Σ with **non-negative eigenvalues**, then the following classes of spinors, if existent, **coincide**

$$\mathrm{Ker}(\nabla^c) \cong \mathrm{Ker}(D^c) \cong \bigoplus_{\gamma \in \mathrm{Spec}(T)} \Big[\Gamma(\Sigma_\gamma) \cap \mathcal{K}(M^n, g)_{\frac{3\gamma}{4n}}\Big].$$

Extension to ∇^c-parallel TsT and KsT

Thm.3 [Chr.'15] Let (M^n, g, T) be a compact connected Riemannian spin manifold with $\nabla^c T = 0$ and assume that $\varphi \in \Gamma(\Sigma)$ is a spinor field such that $\nabla^c \varphi = 0$, where $\nabla^c = \nabla^g + \frac{1}{2}T$ is the characteristic connection. Let $\gamma \in \mathbb{R}\backslash\{0\}$ be a non-zero real number. Then, the following conditions are equivalent:

(a) $\varphi \in \Gamma(\Sigma_\gamma) \cap \mathrm{Ker}(P^s) := \mathrm{Ker}(P^s|_{\Sigma_\gamma})$ w.r.t. the family $\{\nabla^s : s \in \mathbb{R}\backslash\{1/4\}\}$,

(b) $\varphi \in \mathcal{K}^s(M, g)_\zeta$ w.r.t. the family $\{\nabla^s : s \in \mathbb{R}\backslash\{0, 1/4\}\}$ with $\zeta = 3(1-4s)\gamma/4n$,

(c) $\varphi \in \mathcal{K}(M, g)_\kappa$ with $\kappa = 3\gamma/4n$.

Hints: $\nabla^s_X \varphi = \nabla^c_X \varphi + \frac{4s-1}{4}(X \lrcorner T) \cdot \varphi$ and $D^s(\varphi) = D^c(\varphi) + \frac{3(4s-1)}{4} T \cdot \varphi$. Thus

$$\boxed{\nabla^c_X \varphi + \tfrac{4s-1}{4}(X \lrcorner T) \cdot \varphi + \tfrac{1}{n} X \cdot \{D^c(\varphi) + \tfrac{3(4s-1)}{4} T \cdot \varphi\} = 0.}$$

If $\varphi \in \mathrm{Ker}(\nabla^c) \cap \mathrm{Ker}(P^s|_{\Sigma_\gamma})$, then

$$(X \lrcorner T) \cdot \varphi + \frac{3\gamma}{n} X \cdot \varphi = 0.$$

Cor. If $\varphi \in \mathrm{Ker}(\nabla^c)$ is a real Killing spinor with $\kappa = 3\gamma/4n$ then $\boxed{\beta_{\mathrm{univ}}(\gamma) = \beta_{\mathrm{tw}}(\gamma)}$ identically.

Hints: If $\varphi \in \mathcal{K}(M^n, g)_{\frac{3\gamma}{4n}}$, then $\mathrm{Scal}^g = \frac{9(n-1)}{4n}\gamma^2$ is constant; because $\varphi \in \Sigma_\gamma$ is ∇^c-parallel and the scalar curvature satisfies the desired formula, for $n \leqslant 8$ we also have $\|T\|^2 = \frac{2(9-n)}{9(n-1)}\,\mathrm{Scal}^g$. Moreover, Thm.3 tell us that this is also a TsT for some $s \neq 1/4$; thus one may assume without loss of generality that $s = (n-1)/4(n-3) \neq 1/4$. Hence,

$$\beta_{\mathrm{tw}}(\gamma) = \frac{n\Big[9(n-3)^2 + (n-5)(9-n) + 4n(4-n)\Big]}{36(n-1)(n-3)^2}\,\mathrm{Scal}^g = \frac{n}{9(n-1)}\,\mathrm{Scal}^g = \frac{\gamma^2}{4} = \beta_{\mathrm{univ}}(\gamma)$$

Lem.4 Let $0 \neq \gamma \in \mathrm{Spec}(T)$ be a non-zero real T-eigenvalue. Then the following hold:

(a) A Killing spinor with torsion $\varphi \in \mathcal{K}^s(M, g)_{\frac{3\gamma(1-4s)}{4n}}$ for some $s \neq 0, 1/4$ is characteristic, if and only $\varphi \in \Sigma_\gamma$.

(b) A twistor spinor with torsion $\varphi \in \mathrm{Ker}(P^s|_{\Sigma_\gamma})$ for some $s \neq 0, 1/4$ is characteristic, if and only if φ is a D^s-eigenspinor with eigenvalue $-\frac{3\gamma(1-4s)}{4}$ i.e. $\varphi \in \mathcal{K}^s(M, g)_{\frac{3\gamma(1-4s)}{4n}} \cap \Gamma(\Sigma_\gamma)$. In particular, for $s = 0$, a twistor spinor $\varphi \in \mathrm{Ker}(P^g|_{\Sigma_\gamma})$ is characteristic, if and only if $D^g(\varphi) = -\frac{3\gamma}{4}\varphi$, i.e. $\varphi \in \mathcal{K}(M, g)_{\frac{3\gamma}{4n}} \cap \Gamma(\Sigma_\gamma)$.

Thm.4 [Chr.'15] Let (M^n, g, T) be compact connected Riemannian spin manifold (M^n, g, T), with $\nabla^c T = 0$ and positive scalar curvature given by $\mathrm{Scal}^g = \frac{9(n-1)\gamma^2}{4n}$ for some constant $0 \neq \gamma \in \mathrm{Spec}(T)$. If the symmetric endomorphism $\boxed{dT + \frac{1}{2}\Big[\frac{9(n-1)}{4n}\gamma^2 - \frac{3}{2}\|T\|^2\Big]}$ acts on Σ with non-negative eigenvalues, then the following classes of spinors, if existent, coincide

$$\begin{aligned}
\mathrm{Ker}(\nabla^c) &\cong \bigoplus_{\gamma\in\mathrm{Spec}\,(T)} \Big[\Gamma(\Sigma_\gamma) \cap \mathcal{K}(M,g)_{\frac{3\gamma}{4n}}\Big] \\
&\cong \bigoplus_{\gamma\in\mathrm{Spec}\,(T)} \Big[\Gamma(\Sigma_\gamma) \cap \mathcal{K}^s(M,g)_{\frac{3(1-4s)\gamma}{4n}}\Big] \\
&\cong \bigoplus_{\gamma\in\mathrm{Spec}\,(T)} \Big[\mathrm{Ker}(P^s|_{\Sigma_\gamma}) \cap \mathrm{Ker}(D^c)\Big].
\end{aligned}$$

Here, the parameter s takes values in $\mathbb{R}\backslash\{0, 1/4\}$ for the third set, and for the final set it is $s \in \mathbb{R}\backslash\{1/4\}$.

Representative examples

Thm.5 [Chr.'15] On a 6-**dimensional nearly Kähler manifold** (M^6, g, J) endowed with its characteristic connection ∇^c, the following classes of spinor fields coincide:

(1) TsT w.r.t. the family $\{\nabla^s : s \in \mathbb{R}\backslash\{1/4\}\}$, lying in $\Sigma_{\pm 2\|T\|}$,

(2) KsT w.r.t the family $\{\nabla^s : s \in \mathbb{R}\backslash\{0, 1/4\}\}$, with $\zeta := \mp\frac{(4s-1)}{4}\|T\|$,

(3) Riemannian Killing spinors,

(4) ∇^c-parallel spinors.

- Known for $s = 5/12 = \frac{(n-1)}{4(n-3)}$ by [Agricola.-Becker-B.-Kim'12].

Thm.6 [Chr.'15] On a **nearly-parallel** G_2**-manifold** (M^7, g, ω) endowed with its characteristic connection ∇^c, the following classes of spinor fields coincide:

(1) TsT w.r.t. the family $\{\nabla^s : s \in \mathbb{R}\backslash\{1/4\}\}$, lying in $\Sigma_{-\frac{7\tau_0}{6}} \equiv \Sigma_{-\sqrt{7}\|T\|}$,

(2) KsT w.r.t. the family $\{\nabla^s : s \in \mathbb{R}\backslash\{0, 1/4\}\}$, with $\zeta := \frac{(4s-1)\tau_0}{8} = \frac{3(4s-1)\|T\|}{4\sqrt{7}}$,

(3) Riemannian Killing spinors,

(4) ∇^c-parallel spinors.

Strong geometric constraints

Prop.3 [Chr.'15] Assume that $\nabla^c T = 0$ and that (M^n, g, T) is complete and admits a ∇^c-parallel spinor $0 \neq \varphi \in \Sigma_\gamma$ $(\mathbb{R} \ni \gamma \neq 0)$ lying in the kernel $\mathrm{Ker}(P^s)$ for some $s \neq 1/4$. Then, for any $s \in \mathbb{R}$ the following hold

$$\begin{aligned}
\mathrm{Ric}^s(X)\cdot\varphi &= \frac{6\gamma^2}{n^2}\Big[\frac{6(n-1)(1-4s)^2 + 96s(1-4s) + 16s(3-4s)(n-3)}{16}\Big] X\cdot\varphi, \\
\mathrm{Scal}^s\,\varphi &= \frac{6\gamma^2}{n}\Big[\frac{6(n-1)(1-4s)^2 + 96s(1-4s) + 16s(3-4s)(n-3)}{16}\Big]\varphi.
\end{aligned}$$

(a) (M^n, g) is a compact Einstein manifold with constant positive scalar curvature $\mathrm{Scal}^g = \frac{9(n-1)\gamma^2}{4n}$.

(b) For any $n > 3$, (M^n, g, T) is a strict ∇^c-Einstein manifold with parallel torsion and constant scalar curvature $\mathrm{Scal}^c = \frac{3(n-3)\gamma^2}{n}$. For $n = 3$, (M^3, g, T) is Ric^c-flat.

(c) (M^n, g, T) is ∇^s-Einstein (with non-parallel torsion) for any $s \in \mathbb{R}\backslash\{0, 1/4\}$ i.e.

$$\boxed{\mathrm{Ric}^s = \frac{\mathrm{Scal}^s}{n} g, \quad \forall s \in \mathbb{R}\backslash\{0, 1/4\}.}$$

Remarks : (1) The conditions $\nabla^c\varphi = 0$ and $\varphi \in \mathrm{Ker}(P^s|_{\Sigma_\gamma})$ for some $s \neq 1/4$, can be replaced by either

- $\nabla^c\varphi = 0$ and $\varphi \in \mathcal{K}^s(M,g)_{\frac{3\gamma(1-4s)}{4n}}$ for some $s \neq 0, 1/4$, or

- $\nabla^c\varphi = 0$ and $\varphi \in \mathcal{K}(M,g)_{\frac{3\gamma}{4n}}$!!!

(2)

$$\begin{aligned}\mathrm{Ric}^g(X)\cdot\varphi &= 4\kappa^2(n-1)X\cdot\varphi = \frac{9(n-1)\gamma^2}{4n^2}X\cdot\varphi,\\ \mathrm{Scal}^g &= 4\kappa^2 n(n-1) = \frac{9(n-1)\gamma^2}{4n}.\end{aligned}$$

Thus (M^n, g) must be Einstein with positive scalar curvature (compactness by Myers's theorem).

(3) We can present a *different proof* for the original Einstein condition, without using the fact that such a spinor must be a real Killing spinor. For this we provide first the existence of a ∇^c-Einstein structure (and its explicit form), and then we use this fact to describe the original Einstein condition.

Hints: By assumption $\nabla^c\varphi = 0$ and $\varphi \in \Sigma_\gamma$. Thus

(1) $\mathrm{Ric}^c(X)\cdot\varphi = \frac{1}{2}(X\lrcorner dT)\cdot\varphi = (X\lrcorner\sigma_T)\cdot\varphi$.

(2) $-2(X\lrcorner\sigma_T) = \frac{1}{2}(T^2\cdot X - X\cdot T^2) = (X\lrcorner T)\cdot T - T\cdot(X\lrcorner T) \Rightarrow$

$$\boxed{\mathrm{Ric}^c(X)\cdot\varphi = -\tfrac{1}{2}\Big[(X\lrcorner T)\cdot T - T\cdot(X\lrcorner T)\Big]\cdot\varphi. \quad (\clubsuit)}$$

(3) $T\cdot\varphi = \gamma\varphi, \quad (X\lrcorner T)\cdot\varphi + \frac{3\gamma}{n}X\cdot\varphi = 0$. Altogether:

$$\mathrm{Ric}^c(X)\cdot\varphi = \frac{3(n-3)\gamma^2}{n^2}X\cdot\varphi.$$

$\longrightarrow$ The *original Einstein condition*; we use the formula: [A.-Becker-B.-Kim'12]

$$\sum_i e_i\cdot R^g(X,e_i)\varphi = \sum_i e_i\cdot R^c(X,e_i)\varphi - \frac{6}{16}(X\lrcorner\sigma_T)\cdot\varphi + \frac{1}{8}\sum_i T(X,e_i)\cdot(e_i\lrcorner T)\cdot\varphi.$$

Notice that $\sum_i e_i\cdot R^g(X,e_i)\varphi = -\frac{1}{2}\mathrm{Ric}^g(X)\cdot\varphi$.

(4) For $s \neq 0, 1/4$ we apply the formulas (induced by our **Lemma 1** – see also [Becker-Bender's Phd'12])

$$\begin{aligned} \mathrm{Ric}^s(X)\cdot\varphi &= 4(n-1)\zeta^2 X\cdot\varphi - 16s\zeta(X\lrcorner T)\cdot\varphi + 2s(3-4s)(X\lrcorner\sigma_T)\cdot\varphi, \\ \mathrm{Scal}^s\,\varphi &= 4n(n-1)\zeta^2\varphi + 48s\zeta T\cdot\varphi - 8s(3-4s)\sigma_T\cdot\varphi. \end{aligned}$$

Examples

• Consider a nearly Kähler manifold (M^6, g, J). Recall that there exist two ∇^c-parallel spinors $\varphi^\pm$ with $\gamma = \pm 2\|T\|$ which are both TsT for some $s \neq 1/4$. Hence,

$$\boxed{\mathrm{Ric}^s(X)\cdot\varphi^\pm = \tfrac{(5-16s^2)}{4}\|T\|^2 X\cdot\varphi^\pm = \tfrac{(5-16s^2)}{2}\tau_0 X\cdot\varphi^\pm, \quad \forall\ s\in\mathbb{R}}$$

$$\begin{aligned}
\mathrm{Ric}^c(X)\cdot\varphi^\pm &= \frac{3(n-3)\gamma^2}{n^2}X\cdot\varphi^\pm \Rightarrow \mathrm{Ric}^c(X)\cdot\varphi^\pm = \|T\|^2 X\cdot\varphi^\pm,\\
\mathrm{Ric}^g(X)\cdot\varphi^\pm &= \frac{9(n-1)\gamma^2}{4n^2}X\cdot\varphi^\pm \Rightarrow \mathrm{Ric}^g(X)\cdot\varphi^\pm = \frac{5}{4}\|T\|^2\cdot\varphi^\pm.
\end{aligned}$$

....by the twistor equation: $(X\lrcorner T)\cdot\varphi^\pm = \mp\|T\|X\cdot\varphi^\pm$.

• A direct computation shows that :

$$\Big[(X\lrcorner T)\cdot T - T\cdot(X\lrcorner T)\Big]\cdot\varphi^\pm = -2\|T\|^2 X\cdot\varphi^\pm.$$

and the results follows by (♣).

• Consider a proper nearly parallel G_2-manifold (M^7, g, ω). Recall that there is a unique ∇^c-parallel spinor field φ_0 with $\gamma = -\sqrt{7}\|T\|$. Thus

$$\boxed{\mathrm{Ric}^s(X) \cdot \varphi_0 = \tfrac{6(9-16s^2)}{28}\|T\|^2 X \cdot \varphi_0 = \tfrac{(9-16s^2)}{24}\tau_0^2 X \cdot \varphi_0, \quad \forall\ s \in \mathbb{R}},$$

in particular

$$\mathrm{Ric}^c(X) \cdot \varphi_0 = \frac{12}{7}\|T\|^2 X \cdot \varphi_0, \quad \mathrm{Ric}^g(X) \cdot \varphi_0 = \frac{27}{14}\|T\|^2 X \cdot \varphi_0.$$

• In a line with nearly Kähler manifolds in dimension 6, we can compute Ric^c in a direct way, since

$$(X \lrcorner T) \cdot \varphi_0 = \frac{\tau_0}{2} X \cdot \varphi_0 = \frac{3\|T\|}{\sqrt{7}} X \cdot \varphi_0.$$

Thus

$$\Big[(X \lrcorner T) \cdot T - T \cdot (X \lrcorner T)\Big] \cdot \varphi_0 = -\frac{24}{7}\|T\|^2 X \cdot \varphi_0.$$

and the result follows by ($\clubsuit$).

Conclusions

• We deduce that on a triple (M^n, g, T) with $\nabla^c T = 0$, the existence of a spinor field $\varphi \in \Gamma(\Sigma)$ satisfying simultaneously the equations

$$\boxed{\nabla^c_X \varphi = 0, \quad \nabla^s_X \varphi = \zeta X \cdot \varphi,}$$

for some real numbers $s \neq 0, 1/4$, $\zeta \neq 0$, where $\nabla^s = \nabla^g + 2sT$, imposes much harder geometric restrictions than the original Killing spinor equation, namely:

Type of Killing spinors	Geometric conclusions
Killing spinors with Killing number $\kappa \in \mathbb{R}\backslash\{0\}$	• $\mathrm{Ric}^g = 4\kappa^2(n-1)g$, $\mathrm{Scal}^g = 4\kappa^2 n(n-1)$
∇^c-parallel KsT w.r.t. $\nabla^s = \nabla^g + 2sT$ with Killing number $\zeta = \frac{3(1-4s)\gamma}{n} \neq 0$ for some $\mathbb{R} \ni \gamma \neq 0$, $\mathbb{R} \ni s \neq 0, 1/4$	• φ is a real Killing spinor: $T \cdot \varphi = \gamma \cdot \varphi \neq 0$ • $\mathrm{Ric}^s = \frac{\mathrm{Scal}^s}{n} g \quad \forall\, s \in \mathbb{R}$, in particular : – $\mathrm{Ric}^g = \frac{9(n-1)\gamma^2}{4n^2} g$, $\mathrm{Scal}^g = \frac{9(n-1)\gamma^2}{4n}$ – $\mathrm{Ric}^c = \frac{3(n-3)\gamma^2}{n^2} g$, $\mathrm{Scal}^c = \frac{3(n-3)\gamma^2}{n}$

Remark : One has to *stress* that this is not the case in general; there exist KsT which are not real Killing spinors, and thus manifolds which are *not* necessarily Einstein can be endowed with them, e.g. the Heisenberg group. [Becker-Bender's Phd'12]

• The Killing/twistor spinor equation with **torsion** behave very different than their Riemannian analogues, depending on the geometry!

...we need the classification of simply connected Riemannian manifolds admitting real KS

$\Longrightarrow$ dimensions $4 \leqslant n \leqslant 8$ **Th. Friedrich's school** (Berlin, end of 80s).

$\rightarrow$ Any Einstein-Sasakian manifold M^{2m+1} admits real KS [Friedr.-Kath'90]

- $n = 3, 4, 8 \;\Rightarrow\; M^n = \mathrm{S}^n$. [Friedrich'81], [Hijazi'81]
- $n = 5 \;\Rightarrow\; M^5$ Einstein-Sasakian manifold. [Friedrich-Kath'89]
- $n = 6 \;\Rightarrow\; M^6$ nearly Kähler manifold. [Friedr.-Grunewald '85-'90]
- $n = 7 \;\Rightarrow\; M^7$ nearly parallel G_2-manifold. [Friedr.-Kath'90], [F.K.M.S.'97]
- in odd dimensions $4m+1 \geqslant 9, 4m+3 \geqslant 11$ only spheres, Einstein-Sasakian manifolds and 3-Sasakian manifolds can admit real KS [Bär 93]

$\Longrightarrow$ Notice that: An Einstein-Sasaki manifold M^{2m+1} $(2m+1 \geqslant 5)$ is **never** ∇^c-Einstein. [Agricola-Ferreira'12]

Thm.7 [Chr.'15] Let (M^n, g, T) be a compact connected Riemannian spin manifold with $\nabla^c T = 0$, endowed with a spinor field satisfying

$$\boxed{\nabla^c_X \varphi = 0, \quad \nabla^s_X \varphi = \zeta X \cdot \varphi, \quad \text{for some real numbers } s \neq 0, 1/4, \text{ and } \zeta \neq 0,}$$

with respect to the same Riemannian metric g. Then,

- $n = 3 \Rightarrow M^3 \cong \mathrm{S}^3$ is isometric to the **3-sphere** $(\mathrm{S}^3, g_{\mathrm{can}})$
- $n = 6 \Rightarrow M^6$ is isometric to a **strict nearly Kähler manifold**
- $n = 7 \Rightarrow M^7$ is isometric to **a nearly parallel G_2-manifold**

...some references

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