Stable generalized complex structures

Gil Cavalcanti
joint work with Gualtieri

Utrecht University

October 2015
Bedlewo

Generalized complex geometry — 3 definitions

Definition (Definition 1)

Given (M, H), $H \in \Omega^3_{cl}(M)$ a GCS is

- a complex structure $\mathcal{J} : T \oplus T^* \longrightarrow T \oplus T^*$, orthogonal with respect to the natural paring and integrable with respect to the Courant bracket.

Generalized complex geometry — 3 definitions

Definition (Definition 2)

Given (M, H), $H \in \Omega^3_{cl}(M)$ a GCS is

- A complex Lagrangian subbundle $L \subset T_{\mathbb{C}} \oplus T^*_{\mathbb{C}}$ s.t. $L \cap \overline{L} = \{0\}$ & L is involutive;

Generalized complex geometry — 3 definitions

Definition (Definition 3)

Given (M, H), $H \in \Omega^3_{cl}(M)$ a GCS is

- A complex line bundle $K \subset \Omega^\bullet(M;\mathbb{C})$ generated pointwise by a pure spinor
$$\rho = e^{B+i\omega} \wedge \Omega$$
for which
$$\Omega \wedge \overline{\Omega} \wedge \omega^{n-k} \neq 0$$
and
$$d^H \rho = v \cdot \rho,$$
for any local section $\rho \in \Gamma(K \backslash \{0\})$, for some $v \in \Gamma(T \oplus T^*)$.

First Examples

Example

A complex structure is a GCS on $(M, 0)$: take $K = \wedge^{n,0} T^* M$.

Example

A symplectic structure is a GCS on $(M, 0)$: take $K = \langle e^{i\omega} \rangle$.

First Examples

Example

A holomorphic Poisson structure $\pi \in \Gamma(\wedge^{2,0}TM)$ is a GCS on $(M, 0)$: take $K = e^{\pi} \wedge^{n,0} T^*M$.
If M is holomorphic symplectic, one can deform complex structures into symplectic structures.

Example

In $\mathbb{C}^2$ take the bivector $z\partial/\partial z \wedge \partial/\partial w$. This gives the canonical bundle

$$K = \langle z + dz \wedge dw \rangle.$$

Questions

Question

What do they look like?

Question

Are there relevant subtypes?

Question

What are their differential topological properties?

Outline of Topics

Generalized complex geometry

Definition (Definition 2)

Given (M,H), $H \in \Omega^3_{cl}(M)$ a GCS is

- A complex Lagrangian subbundle $L \subset T_{\mathbb{C}} \oplus T^*_{\mathbb{C}}$ s.t. $L \cap \overline{L} = \{0\}$ & L is involutive;

L is a Lie algebroid $\Rightarrow$ de Rham theory on $\Gamma(\wedge^\bullet L^*) = \Gamma(\wedge^\bullet \overline{L})$.

Generalized complex geometry

Definition (Definition 3)

Given (M, H), $H \in \Omega^3_{cl}(M)$ a GCS is

- A complex line bundle $K \subset \Omega^\bullet(M; \mathbb{C})$ generated pointwise by a pure spinor
$$\rho = e^{B+i\omega} \wedge \Omega$$
for which
$$\Omega \wedge \overline{\Omega} \wedge \omega^{n-k} \neq 0$$
and
$$d^H \rho = v \cdot \rho,$$
for any local section $\rho \in \Gamma(K \backslash \{0\})$, for some $v \in \Gamma(T \oplus T^*)$.

Generalized complex geometry

Definition (Definition 3)

Given (M, H), $H \in \Omega^3_{cl}(M)$ a GCS is

- A complex line bundle $K \subset \Omega^\bullet(M;\mathbb{C})$ generated pointwise by a pure spinor
$$\rho = e^{B+i\omega} \wedge \Omega$$
for which
$$\Omega \wedge \overline{\Omega} \wedge \omega^{n-k} \neq 0$$
and
$$d^H \rho = v \cdot \rho,$$
for any local section $\rho \in \Gamma(K \backslash \{0\})$, for some $v \in \Gamma(\overline{L})$.

Generalized complex geometry

- K is the *canonical bundle*;
- *type of* $\mathcal{J} = \deg(\Omega)$;
- $v \in \Gamma(\overline{L})$ is the *modular field*;
- $\mathcal{J}$ is *generalized Calabi–Yau* if K has a nowhere vanishing d^H-closed section.
- $\pi_T \circ \mathcal{J}(T^*)$ is an integrable singular distribution (with symplectic leaves).

Generalized complex geometry — Local structure

Theorem (Gualtieri 2003)

If the type is locally constant, the GCS is locally equivalent to the product of a complex and a symplectic structure.

Theorem (Bailey 2011)

Any GCS is locally equivalent to a product of a symplectic and a holomorphic Poisson structure.

What do GCS look like? $\rightsquigarrow$ what do holomorphic Poisson structures look like?

Generalized Calabi–Yau geometry

Question

Is $\mathcal{J}$ GCY?

1. K has a nowhere vanishing section $\rho \Leftrightarrow c_1(K) = 0$;
2. $\rho \rightsquigarrow v$ (modular field). There is a *closed* section $\Leftrightarrow$ $[v] = 0 \in H^1(L)$;

Generalized Calabi–Yau geometry

Question

Is $\mathcal{J}$ GCY?

3. If $\mathcal{J}$ has constant type,

$$\rho = e^{B+i\omega} \wedge \Omega$$

then $d\Omega = 0$ and can ask whether $d^H e^{B+i\omega} = 0$.

$$\{0\} \longrightarrow \ker(\Omega\wedge)^i \longrightarrow \Omega^i(M;\mathbb{C}) \xrightarrow{\Omega\wedge} \mathcal{I}_\Omega^{i+k} \longrightarrow \{0\}$$

Twitsting class

$$\delta\rho \quad \rightsquigarrow \quad [H + d(B+i\omega)] \in H^3(\ker(\Omega\wedge)).$$

Type 1 GCY structures

Example (Kodaira–Thurston manifold)

Compact manifold corresponding to the Lie algebra

$$\mathfrak{n} = \langle e_1, e_2, e_3, e_4 : [e_1, e_2] = e_3 \rangle.$$

Type 1 GCY given by

$$e^{e^3 \wedge e^4} \wedge (e^1 + ie^2).$$

$\Omega \wedge \overline{\Omega} = e^1 \wedge e^2 = de^3 \;\Rightarrow$ Twisting class does not vanish.

Type 1 GCY structures

Theorem (Cavalcanti–Gualtieri)

Let D^{2n} be a compact, connected type 1 GCY. Then the following hold:

1. *There is a surjective submersion $\pi : D \longrightarrow T^2$, hence $b_1(D) \geq 2$ and $\chi(D) = 0$.*
2. *If D has a compact leaf, then fibers of π are the symplectic leaves of $\mathcal{J}$.*
3. *If the twisting class vanishes,*
   - *the structure can be deformed into one with a compact leaf;*
   - *there are classes $a, b \in H^1(D)$ and $c \in H^2(D)$ such that $abc^{n-1} \neq 0$. In particular $b_i(D) \geq 2$ for $0 < i < 2n$.*

Type 1 GCY structures

Theorem (Cavalcanti–Gualtieri)

Let $\pi : D \longrightarrow T^2$ be a fibration of a compact, connected, oriented 4-manifold over the torus. Then D admits a type 1 GCY structure for which the fibers of π are the symplectic leaves of $\mathcal{J}$.

Type 1 GCY structures

Relevant steps.

1. Oriented surface fibration $\Rightarrow$ symplectic fibration;
2. If there is a class $c \in H^2(D)$ with $c|_F = [\omega_F]$, then it is groovy (Thurston);
3. If the genus of the fiber is not 1, there is such a class c;
4. Torus bundles over the torus have been classified (Sakamoto–Fukuhara, Ue, Geiges). Only 2 do not satisfy Thurston's condition:
   - Kodaira–Thurston;
   - "the other" nilmanifold.

$\square$

Stable structures

The anticanonical bundle of $\mathcal{J}$ has a natural section:

$$s(\rho) = \rho_0.$$

$\mathcal{J}$ is *stable* if s has only nondegenerate zeros.
i.e., if ρ is a nonvanishing section of K then $d\rho_0 \neq 0$ along the locus $[\rho_0 = 0]$.

Stable structures

Theorem (Cavalcanti–Gualtieri, Goto–Hayano)

Logarithmic transform on a symplectic 4-manifold along a symplectic torus with trivial normal bundle produces stable structures.

Stable structures

Example (Cavalcanti–Gualtieri)

$$n\#\mathbb{C}P^2\#m\overline{\mathbb{C}P^2}$$

has a stable gcs if and only if it has an almost cplx str.

Theorem (Torres)

If M^4 and N^4 have sympletic tori with trivial normal bundle,

$$M\#(S^2 \times S^2)\#N \qquad M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}\#N$$

have stable generalized complex structure.

Stable structures

Keep in mind

Theorem (Iwase, Baykur–Sunukjian)

Every compact, simply connected 4-manifold is obtained from connected sums of $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$ and $S^1 \times S^3$ by means of logarithmic transforms along disjoint tori.

Stable structures

Question

What does the singular locus look like?

Bailey's theorem $\Rightarrow$ there are coordinates in $\mathbb{R}^{2n-4} \times \mathbb{C}^2$ for which

$$\rho = e^{i\omega_0} \wedge (z + dz \wedge dw) \sim e^{i\omega_0 + d\log z \wedge dw}.$$

Stable structures

Induce data on the singular locus, D:

- D is the zero locus of s, the anticanonical section $\Rightarrow$ codimension 2 submanifold;
- The residue of ρ gives D a type 1 GCY structure:

$$e^{i\omega_0} \wedge (z + dz \wedge dw) \rightsquigarrow e^{i\omega} dw$$

- adjunction formula:

$$ds : \mathcal{N}^* \otimes K^*|_D \cong \mathbf{1},$$

  $\mathcal{N}^* \cong K|_D$ gives $\mathcal{N}$ a generalized holomorphic structure.
- twisting class of $D \sim$ Chern class of K:

$$c_1(K) \wedge \Omega|_D = \text{twisting class of } D.$$

Stable structures

Proposition (Cavalcanti–Gualtieri)

Let D be a type 1 GCY and let $\mathcal{N}$ be a generalized holomorphic vector bundle over D. Then the total space of $\mathcal{N}$ admits a stable structure whose singular locus is the zero section.

Stable structures

Theorem (Cavalcanti–Gualtieri)

Let M be a stable GC manifold and let D be a compact component of the singular locus. Then a neighbourhood of D determines and is determined by the induced GCY on D and the holomorphic structure of $\mathcal{N}^*$.

Stable structures — Properties of the singular locus

The vector field $\partial/\partial w$ preserves the structure

$$\Rightarrow$$

The sphere bundle of $\mathcal{N}$ inherits a co-symplectic structure:
$\sigma = \omega|_{S^1\mathcal{N}}$ and $\alpha = \partial/\partial r \cdot \omega$ such that

$$d\sigma = 0 = d\alpha \qquad \text{and} \qquad \sigma^{n-1} \wedge \alpha \neq 0.$$

In 4-d this the pair (α, σ) gives rise to a taut foliation.

Stable structures — 4 d

Theorem (Cavalcanti–Gualtieri)

Let M^4 be a compact stable generalized complex manifold whose anticanonical divisor has connected components $D_1, \ldots, D_n$. Then for each i there is a tubular neighbourhood U_i of D_i, a symplectic manifold with boundary (X_i, ω_i) and an orientation reversing diffeomorphism of coisotropic submanifolds $\varphi_i : \partial X_i \xrightarrow{\cong} \partial \overline{U}_i$, so that

$$\widetilde{M} = M \backslash \overline{U} \cup_\varphi X \tag{1}$$

is a symplectic manifold.
Further, X_i can be chosen so that $b^+(X_i) > 1$ and the restriction map $H^2(X_i) \longrightarrow H^2(\partial X_i)$ is surjective.

Stable structures — 4 d

Proof.

Since the boundary of a tubular nhood of D has a taut foliation, the result follows from work of Elisashberg ad Etnyre on symplectic fillings of taut foliations. □

Theorem (Kronheimer–Mrowka)

Let Y be a closed 3-manifold with a smooth taut foliation $(\mathcal{F}, \sigma)$, then there exists a closed symplectic 4-manifold (X, ω) containing Y as a separating manifold for which $\omega|_Y = \sigma$.
Furthermore if $Y \neq S^1 \times S^2$, we can arrange so that the map $H^2(X;\mathbb{R}) \longrightarrow H^2(Y;\mathbb{R})$ is surjective and each component of $X \backslash Y$ has $b_2^+ > 1$.

Stable structures — 4 d

Corollary

An embedded surface $\Sigma \hookrightarrow M^4$ of a stable GCM disjoint from the singular locus must satisfy the adjunction inequality: if $\Sigma \cdot \Sigma \geq 0$, then

$$2g - 2 \geq |K \cdot \Sigma| + \Sigma \cdot \Sigma.$$

In particular, there are no spheres with nonnegative self-intersection that do not touch the singular locus.
N.B.: **There are** many (Lagrangian) spheres of zero self-intersection which cross the singular locus.
N.B. The singular locus is often a torus of positive self intersection.