Conformal Killing 2-forms on low dimensional Lie groups

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CIEM - CONICET

Workshop on almost hermitian and contact geometry
MRCC-Bedlewo, Poland
October 18 – 24, 2015

Outline

- The conformal Killing equation.
- The invariant setting.
- Some general obstructions.
- Nilpotent Lie groups.
- Compact Lie groups.
- The classification in dimension 3.

The conformal Killing equation

Let (M, g) be a Riemannian manifold with $\dim M = n$. A p-form ω on M is called *conformal Killing* (CK) if it satisfies the following equation:

$$\nabla_X \omega = \frac{1}{p+1} \iota(X) d\omega - \frac{1}{n-p+1} X^* \wedge d^* \omega,$$

where ∇ is the Levi-Civita connection, X^* is the 1-form dual to X, d^* is the co-differential and ι is the interior product.

If moreover ω is co-closed, that is $d^*\omega = 0$, then ω is called *Killing*. Killing forms are forms whose covariant derivative is totally skew-symmetric. Indeed,

$$\nabla_X \omega = \frac{1}{p+1} \iota(X) d\omega.$$

We recall that $d^*\omega$ can be computed as follows: if $\{E_i : i = 1, \ldots, n\}$ is a local orthonormal basis of vector fields, then

$$d^*\omega(U_1, ..., U_{p-1}) = -\sum_{i=1}^{n} (\nabla_{E_i}\omega)(E_i, U_1, ..., U_{p-1})$$

for any vector fields $U_1, ..., U_{p-1}$ on M.

In particular, if ω is parallel, that is, $\nabla_X \omega = 0$ for all X, then ω is a Killing form.

Some properties:

1. If ω is a 1-form and $U = \omega^*$ then:
   - ω is Killing $\Rightarrow$ U is Killing vector field ($\mathcal{L}_U g = 0$);
   - ω is conformal Killing $\Rightarrow$ U is a conformal vector field ($\mathcal{L}_U g = \varphi g$, for some function φ).

2. If ω is a CK p-form on (M^n, g), then $*\omega$ is a CK $(n-p)$-form.

3. Conformal invariance: If ω is a CK p-form on (M, g) and $\tilde{g} := e^{2\lambda} g$ is a conformally equivalent metric, then the form $\tilde{\omega} := e^{(p+1)\lambda}\omega$ is a CK p-form on $(M, \tilde{g})$.

4. The space of CK p-forms has finite dimension $\leq \binom{n+2}{p+1}$.

Comments

- Killing p-forms were first introduced by Kentaro Yano in 1952 as a natural generalization of Killing vector fields to forms.
- They have been intensively studied by physicists since the work of Penrose and Walker (1970). Killing tensors give rise to quadratic first integrals of the geodesic equation.
- Conformal Killing tensors as a generalization of conformal Killing vector fields appear in different papers by Tachibana, Kashiwada in 1968, 1969 respectively. They are also called conformal Killing-Yano forms or twistor forms.

Some examples of the general results obtained are:

Theorem (Moroianu-Semmelmann, 2005)

Every Killing p-form on a compact quaternionic Kähler manifold is automatically parallel ($p \geq 2$).

Theorem (Belgun-Moroianu-Semmelmann, 2006)

A compact simply connected symmetric space carries a non-parallel Killing p-form if and only if it is isometric to a Riemannian product $S^k \times N$, *where* S^k *is a round sphere and* $k > p$.

Theorem (Moroianu-Semmelmann, 2008)

A CK form on a compact Riemannian product is a sum of parallel forms, Killing forms on one of the factors, and their Hodge duals.

Examples

(i) The round sphere S^n. CK forms are sums of closed and co-closed forms corresponding to the least eigenvalue of the Laplace operator.

(ii) A Riemannian manifold (M, g) is called Sasakian if there exists a unit length Killing vector field ξ such that for all X vector field on M,

$$\nabla_X d\eta = -2X^* \wedge \eta$$

where $\eta(Y) = g(\xi, Y)$ for any vector field Y on M.

In a Sasakian manifold, $d\eta^k$ and $\eta \wedge d\eta^k$ are closed (resp., co-closed) conformal Killing forms, for $k \geq 1$.

(iii) In the case of a 2-form ω on a Riemannian manifold (M, g) one can consider the associated skew-symmetric tensor $T : TM \to TM$ defined by $\omega(X, Y) = g(TX, Y)$. In this case the 2-form ω is Killing if and only if

$$(\nabla_X T)Y + (\nabla_Y T)X = 0,$$

or equivalently,

$$(\nabla_X T)X = 0, \quad \text{for all } X \in \mathfrak{X}(M).$$

As a consequence one has:

Let (M, g, T) be a Riemannian manifold with a skew-adjoint endomorphism T of TM. The associated 2-form $\omega(X, Y) = g(TX, Y)$ is Killing if and only if $(\nabla_X T)X = 0$ for all $X \in \mathfrak{X}(M)$.

Particular case:

(M, g, J) an almost Hermitian manifold with Kähler form $\omega(X, Y) = g(JX, Y)$.

Then:

ω is a Killing 2-form if and only if $(\nabla_X J)X = 0$, equivalently (M, g, J) is nearly Kähler.

Nearly Kähler manifolds were introduced by Alfred Gray in 1976 and have been much studied since then.

It is known that a 4-dimensional nearly Kähler manifold is Kähler. In this more general context we can prove

Proposition (Andrada-Dotti)

If (M, g) is a 4-dimensional Riemannian manifold and T is an invertible skew-symmetric endomorphism of TM such that $(\nabla_X T)X = 0$, then T is parallel.

According to [U. Semmelmann, Math. Zeit. 2003] a 2-form ω is CK if and only if there exists a 1-form θ on M such that

$$\nabla_X\omega(Y,Z)+\nabla_Y\omega(X,Z) = 2g(X,Y)\theta(Z)-g(X,Z)\theta(Y)-g(Y,Z)\theta(X),$$

for any vector fields X, Y, Z on M. Furthermore, the 1-form θ is given by

$$\theta = -\frac{1}{n-1}d^*\omega.$$

In particular, a 2-form ω is Killing if

$$\nabla_X\omega(Y,Z)+\nabla_Y\omega(X,Z) = 0,$$

for any vector fields X, Y, Z on M.

Invariant setting on Lie groups

In [B -Dotti-Santillán, Class. Quantum Grav. 2012] we begin the study of *left-invariant* Killing 2-forms on Lie groups equipped with left invariant metrics. Equivalently, we consider a Lie algebra $\mathfrak{g}$ with an inner product $\langle\,,\rangle$ and a 2-form $\omega \in \Lambda^2\mathfrak{g}^*$ satisfying the Killing equation.

Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra, that is, $[\mathfrak{g}', \mathfrak{g}] = 0$, where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is the commutator ideal. Fix an inner product $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{g}$ and let $\mathfrak{z}$ be its center, and $\mathfrak{v} = \mathfrak{z}^\perp$. For $z \in \mathfrak{z}$ define $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ defined by

$$\langle j_z x, y\rangle = \langle z, [x, y]\rangle \qquad \text{for } z \in \mathfrak{z}, \quad x, y \in \mathfrak{v}.$$

Theorem (BDS, 2012)

Let T be a skew-symmetric endomorphism of $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{v}$. Then T is a Killing tensor on $\mathfrak{g}$ if and only if T preserves the center $\mathfrak{z}$ and for all $z \in \mathfrak{z}$ the following equation holds:

$$T \circ j_z = -j_z \circ T|_{\mathfrak{v}} = \frac{1}{3} j_{Tz}.$$

Corollary

If T is a left-invariant Killing tensor on the $(2n+1)$-dimensional Heisenberg group H_{2n+1} then $T = 0$.

Proposition

The Iwasawa manifold with its standard metric carries a non-degenerate Killing tensor.

We recall that a left invariant metric on a Lie group G is called *bi-invariant* if right translations are also isometries.

Theorem

Let G be a Lie group with a bi-invariant metric. If T is a skew-symmetric endomorphism of $\mathfrak{g}$ then T is a Killing tensor if and only if $T|_{[\mathfrak{g},\mathfrak{g}]} = 0$. In particular, a compact semisimple Lie group has no non trivial Killing 2-forms.

Theorem

For any left invariant metric on $SU(2)$, there are no non-trivial left invariant Killing 2-forms.

Left-invariant CK 2-forms on Lie groups

We will consider CK 2-forms $\omega \in \Lambda^2\mathfrak{g}^*$, where $\mathfrak{g}$ is equipped with an inner product $\langle \cdot\, , \cdot \rangle$. Since ω is uniquely determined by a skew-symmetric endomorphism T of $\mathfrak{g}$ by $\omega(x, y) = \langle Tx, y\rangle$, we will often refer to T as a CK tensor.

As stated before, ω is conformal Killing if and only if there exists a 1-form $\theta \in \mathfrak{g}^*$ such that

$$(\nabla_x\omega)(y, z) + (\nabla_y\omega)(x, z) = 2\langle x, y\rangle\theta(z) - \langle x, z\rangle\theta(y) - \langle y, z\rangle\theta(x),$$

for any $x, y, z \in \mathfrak{g}$. Furthermore, the 1-form θ is given by

$$\theta = -\frac{1}{n-1}d^*\omega.$$

We will consider CK tensors T which are not Killing. In this case we say that the CK tensor is *strict*.

CK 2-forms on nilpotent Lie groups

Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra, that is, $[\mathfrak{g}', \mathfrak{g}] = 0$, where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is the commutator ideal. Fix an inner product $\langle \cdot , \cdot \rangle$ on $\mathfrak{g}$ and let $\mathfrak{z}$ be its center, and $\mathfrak{v} = \mathfrak{z}^{\perp}$. For $z \in \mathfrak{z}$ let $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ be defined by

$$\langle j_z x, y \rangle = \langle z, [x, y] \rangle \qquad \text{for } z \in \mathfrak{z}, \quad x, y \in \mathfrak{v}.$$

Theorem ([Andrada-B -Dotti, J. Geom. Phys. 2015])

If $\mathfrak{g}$ is a 2-step nilpotent Lie algebra admitting an inner product with a strict CK tensor T, then $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_{2m+1}$ and if z_0 is a unit generator of the center of $\mathfrak{h}_{2m+1}$, then $Tz_0 = 0$ and $T|_{\mathfrak{v}} = \lambda j_{z_0}^{-1}$, for some $\lambda \neq 0$, where $\mathfrak{v} = z_0^{\perp}$.

CK 2-forms on compact Lie groups

Theorem

Let G be a compact n-dimensional Lie group equipped with a bi-invariant metric g. If there exists a left-invariant strict conformal Killing tensor T, then $n = 3$ and the Lie algebra of G is isomorphic to $\mathfrak{su}(2)$.

Later we will exhibit all left invariant metrics on $SU(2)$ that admit CK 2-forms. The existence of left invariant CK 2-forms on compact Lie groups equipped with left invariant metrics has not yet been settled.

CK tensors in dimension 3.

In [Andrada-B -Dotti, J. Geom. Phys. 2015] we classified the CK 2-forms on 3-dimensional Lie groups, extending in this way the work of Perrone (1998) on Sasakian structures on such groups.

$$\begin{array}{rllll}
\mathfrak{su}(2): & [f_1, f_2] = f_3, & [f_2, f_3] = f_1, & [f_3, f_1] = f_2, \\
\mathfrak{sl}(2,\mathbb{R}): & [f_1, f_2] = -f_3, & [f_2, f_3] = f_1, & [f_3, f_1] = f_2, \\
\mathfrak{e}(2): & [f_3, f_1] = f_2, & [f_3, f_2] = -f_1, & \\
\mathfrak{h}_3: & [f_1, f_2] = f_3, & & \\
\mathfrak{aff}(\mathbb{R}) \times \mathbb{R}: & [f_1, f_2] = f_2. & &
\end{array}$$

Note that $\mathfrak{e}(2)$ is the Lie algebra of the isometry group of the Euclidean plane and $\mathfrak{h}_3$ is the Heisenberg Lie algebra.

Theorem ([ABD, 2015])

The 3-dimensional non-abelian Lie algebras admitting an inner product with non-zero CK 2-forms are $\mathfrak{e}(2)$, $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{su}(2)$, $\mathfrak{h}_3$ *and* $\mathfrak{aff}(\mathbb{R})\times\mathbb{R}$. *Furthermore,*

1. *Any left invariant metric on* $E(2)$ *admitting CK 2-forms is flat and any of these 2-forms is parallel.*
2. *On* $SU(2)$ *and* $SL(2,\mathbb{R})$ *there is a one-parameter family of left invariant metrics, pairwise non-isometric up to scaling, that admit CK 2-forms and any of these 2-forms is strict.*
3. *Any left invariant metric on the Lie group* H_3 *admits CK 2-forms and any of these 2-forms is strict.*
4. *Any left invariant metric on the Lie group* $\mathrm{Aff}(\mathbb{R})\times\mathbb{R}$ *admits CK 2-forms. Up to scaling, each of these metrics is isometric to* g_t *for one and only one* $t\geq 0$. *For* $t=0$ *the CK 2-forms are all parallel, while for* $t>0$, *the CK 2-forms are all strict.*

Remark. We point out that all the CK 2-forms from the Theorem are closed. Furthermore, when these CK 2-forms are strict, their Hodge duals are contact forms.

Corollary

A left-invariant metric g on $SU(2)$ admits non-trivial left invariant CK tensors if and only if g is homothetic to the Berger metric g_t for some $t > 0$.

• $\mathfrak{su}(2)$:

$$g_t = \begin{pmatrix} 1 & & \\ & 1 & \\ & & t \end{pmatrix}, \qquad t > 0.$$

• $\mathfrak{sl}(2,\mathbb{R})$:

$$g_t = \begin{pmatrix} 1 & & \\ & 1 & \\ & & t \end{pmatrix}, \qquad t > 0.$$

• $\mathfrak{aff}(\mathbb{R}) \times \mathbb{R}$:

$$g_t = \begin{pmatrix} 1 & & \\ & 1+t^2 & t \\ & t & 1 \end{pmatrix}, \qquad t > 0.$$

Coordinate expression of the metrics

In order to obtain the coordinate expression of the left invariant metrics we give a basis of left invariant 1-forms in local coordinates on each Lie group.

• $\mathfrak{su}(2)$: using the Euler angles (ψ, θ, ϕ) as a local coordinate system on $SU(2)$, a basis of left invariant 1-forms is given by

$$\sigma_1 = \sin\theta \sin\psi \, d\phi + \cos\psi \, d\theta, \quad \sigma_2 = \sin\theta \cos\psi \, d\phi - \sin\psi \, d\theta,$$

$$\sigma_3 = d\psi + \cos\theta \, d\phi.$$

There are two cases:

(i) $t > 0,\ t \neq 1$. The left invariant Berger metric g_t can be written as

$$g_t = \sigma_1^2 + \sigma_2^2 + t\sigma_3^2,$$

and any left invariant CK 2-form on $SU(2)$ with respect to this metric is a constant multiple of $\omega = \sigma_1 \wedge \sigma_2$.

(ii) $t = 1$. The left invariant metric g_1 can be written as

$$g_1 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2,$$

and any left invariant 2-form on $SU(2)$ is CK with respect to this bi-invariant metric.

• $\mathfrak{sl}(2,\mathbb{R})$: using the diffeomorphism $SL(2,\mathbb{R}) \simeq S^1 \times \mathbb{R}^2$ corresponding to the Iwasawa decomposition, we may consider a local coordinate system

$$(\theta, r, s) \longrightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^r & 0 \\ 0 & e^{-r} \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbb{R}).$$

A basis of left invariant 1-forms is given by

$$\sigma_1 = dr - se^{2r}\, d\theta, \quad \sigma_2 = 2s\, dr + ds - (s^2 e^{2r} + e^{-2r})\, d\theta,$$

$$\sigma_3 = e^{2r}\, d\theta,$$

and the left invariant metric g_t can be written as

$$g_t = 4\sigma_1^2 + (\sigma_2 + \sigma_3)^2 + t(-\sigma_2 + \sigma_3)^2 \quad t > 0.$$

Any left invariant CK 2-form on $SL(2,\mathbb{R})$ with respect to this metric is a constant multiple of $\omega = \sigma_1 \wedge (\sigma_2 + \sigma_3)$.

• $\mathfrak{h}_3$: Let $H_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$ be the simply connected Lie group with Lie algebra $\mathfrak{h}_3$.

A basis of left invariant 1-forms on H_3 is

$$\sigma_1 = dx, \quad \sigma_2 = dy, \quad \sigma_3 = dz - x\,dy,$$

and the left invariant metric g can be written as

$$g = \sigma_1^2 + \sigma_2^2 + \sigma_3^2.$$

Moreover, any left invariant CK 2-form on H_3 is a constant multiple of $\omega = \sigma_1 \wedge \sigma_2 = dx \wedge dy$.

• $\mathfrak{aff}(\mathbb{R}) \times \mathbb{R}$: Let $\mathrm{Aff}(\mathbb{R}) \times \mathbb{R} = \left\{ \begin{pmatrix} e^x & y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^z \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$. Then (x, y, z) define global coordinates on $\mathrm{Aff}(\mathbb{R}) \times \mathbb{R}$, and a basis of left invariant 1-forms is given by

$$\sigma_1 = dx, \quad \sigma_2 = e^{-x}\, dy, \quad \sigma_3 = dz.$$

The left invariant metric g_t is

$$g_t = \sigma_1^2 + \sigma_2^2 + (t\sigma_2 + \sigma_3)^2\,, \quad t > 0$$

and any left invariant CK 2-form is a constant multiple of $\omega = e^{-x}\, dx \wedge dy$.

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