Canonical structures on generalized symmetric spaces

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Main goals:

- to present selected recent results and trends in the theory of *canonical structures* on homogeneous k-symmetric spaces;
- to give some applications of canonical structures to *generalized Hermitian geometry* and *Riemannian geometry*

Contents

1. Homogeneous k–symmetric spaces and canonical structures

Researchers who founded this theory: V.I.Vedernikov, N.A.Stepanov, A.Ledg
A.Gray, J.A.Wolf, A.S.Fedenko, O.Kowalski, L.V.Sabinin, V.Kac . . .

Definition 1. *Let G be a connected Lie group, Φ its (analytic) automorphism, G^Φ the subgroup of all fixed points of Φ, and G^Φ_o the identity component of G^Φ. Suppose a closed subgroup H of G satisfies the condition*

$$G^\Phi_o \subset H \subset G^\Phi.$$

Then G/H is called a homogeneous Φ-space.

Homogeneous Φ-spaces include homogeneous symmetric spaces $(\Phi^2 = id)$ and, more general, *homogeneous Φ-spaces of order k* $(\Phi^k = id)$ or, in the other terminology, *homogeneous k-symmetric spaces*

For any homogeneous Φ-space G/H one can define the mapping

$$S_o = D\colon\ G/H \to G/H,\ xH \to \Phi(x)H.$$

It is evident that in view of homogeneity the "symmetry" S_p can be defined at any point $p \in G/H$.

The class of homogeneous Φ–spaces is very large and contains even non-reductive homogeneous spaces. At this stage we dwell on homogeneous k-symmetric spaces G/H only.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the corresponding Lie algebras for G and H, $\varphi = d\Phi_e$ the automorphism of $\mathfrak{g}$, where $\varphi^k = id$. Consider the linear operator $A = \varphi - id$. It is known (N.A.Stepanov, 1967) that G/H is a reductive space for which the corresponding *canonical reductive decomposition* is of the form:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \ \ \mathfrak{m} = A\mathfrak{g}.$$

Besides, this decomposition is obviously φ–invariant. Denote by θ the restriction of φ to $\mathfrak{m}$. As usual, we identify $\mathfrak{m}$ with the tangent space $T_o(G/H)$ at the point $o = H$.

Definition 2 (VVB, N.A.Stepanov, 1991)**.** *An invariant affinor structure F (i.e. a tensor field of type $(1,1)$) on a homogeneous k-symmetric space G/H is called canonical if its value at the point $o = H$ is a polynomial in θ.*

Denote by $\mathcal{A}(\theta)$ the set of all canonical affinor structures on G/H. It is easy to see that $\mathcal{A}(\theta)$ is a commutative subalgebra of the algebra $\mathcal{A}$ of all invariant affinor structures on G/H. It should be mentioned that all canonical structures are, in addition, invariant with respect to the "symmetries" $\{S_p\}$ of G/H.

Note that the algebra $\mathcal{A}(\theta)$ for any symmetric Φ-space ($\Phi^2 = id$) is trivial, i.e. it is isomorphic to $\mathbb{R}$.

The most remarkable example of canonical structures is the canonical almost complex structure $J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$ on a homogeneous 3-symmetric space (N.A.Stepanov, J.Wolf, A.Gray, 1967-1968).

It turns out that for homogeneous k-symmetric spaces $(k \geq 3)$ the algebra $\mathcal{A}(\theta)$ contains a rich collection of classical structures. All these canonical structures on homogeneous k–symmetric spaces were completely described.

We will concentrate on the following affinor structures of classical types:

almost complex structures J $(J^2 = -1)$;

almost product structures P $(P^2 = 1)$;

f*-structures* $(f^3 + f = 0)$ (K.Yano, 1963);

f-structures of hyperbolic type or, briefly, *h-structures* $(h^3 - h = 0)$ (V.F.Kirichenko, 1983).

Clearly, f-structures and h-structures are generalizations of structures J and P respectively.

We use the notation: $s = [\frac{k-1}{2}]$ (integer part), $u = s$ (for odd k), and $u = s + 1$ (for even k).

Theorem 1 (VVB,N.A.Stepanov,1991, 1998). *Let G/H be a homogeneous k-symmetric space.*

(1) *All non-trivial canonical f-structures on G/H can be given by the operators*

$$f = \frac{2}{k} \sum_{m=1}^{u} \left(\sum_{j=1}^{u} \zeta_j \sin \frac{2\pi m j}{k} \right) \left(\theta^m - \theta^{k-m} \right),$$

where $\zeta_j \in \{-1; 0; 1\}$, $j = 1, 2, \ldots, u$, and not all coefficients ζ_j are zero. In particular, suppose that $-1 \notin \operatorname{spec} \theta$. Then the polynomials f define canonical almost complex structures J iff all $\zeta_j \in \{-1; 1\}$.

(2) *All canonical h-structures on* G/H *can be given by the polynomials* $h = \sum_{m=0}^{k-1} a_m \theta^m$, *where:*

(a) *if* $k = 2n+1$, *then*

$$a_m = a_{k-m} = \frac{2}{k} \sum_{j=1}^{u} \xi_j \cos \frac{2\pi m j}{k};$$

(b) *if* $k = 2n$, *then*

$$a_m = a_{k-m} = \frac{1}{k} \left(2 \sum_{j=1}^{u} \xi_j \cos \frac{2\pi m j}{k} + (-1)^m \xi_n \right)$$

Here the numbers ξ_j *take their values from the set* $\{-1; 0; 1\}$ *and the polynomials* h *define canonical structures* P *iff all* $\xi_j \in \{-1; 1\}$.

We now particularize the results above mentioned for homogeneous Φ-spaces of orders 3, 4, and 5 only.

Corollary 1. *Let G/H be a homogeneous 3-symmetric space. There are (up to sign) only the following canonical structures of classical type on G/H:*

$$J = \frac{1}{\sqrt{3}}(\theta - \theta^2), \ P = 1.$$

We noted that the existence of the structure J and its properties are well known (see N.A.Stepanov, J.Wolf, A.Gray, V.F.Kirichenko, ...).

Corollary 2. *On a homogeneous 4-symmetric space there are (up to sign) the following canonical classical structures:*

$$P = \theta^2, \ f = \frac{1}{2}(\theta - \theta^3), \ h_1 = \frac{1}{2}(1 - \theta^2), \ h_2 = \frac{1}{2}(1 + \theta^2).$$

Corollary 3. *There exist (up to sign) only the following canonical structures of classical type on any homogeneous 5-symmetric space:*

$$P = \frac{1}{\sqrt{5}}(\theta - \theta^2 - \theta^3 + \theta^4);$$

$$J_1 = \alpha(\theta - \theta^4) - \beta(\theta^2 - \theta^3); \quad J_2 = \beta(\theta - \theta^4) + \alpha(\theta^2 - \theta^3);$$

$$f_1 = \gamma(\theta - \theta^4) + \delta(\theta^2 - \theta^3); \quad f_2 = \delta(\theta - \theta^4) - \gamma(\theta^2 - \theta^3);$$

$$h_1 = \frac{1}{2}(1 + P); \quad h_2 = \frac{1}{2}(1 - P);$$

where $\alpha = \frac{\sqrt{5+2\sqrt{5}}}{5}$; $\beta = \frac{\sqrt{5-2\sqrt{5}}}{5}$; $\gamma = \frac{\sqrt{10+2\sqrt{5}}}{10}$; $\delta = \frac{\sqrt{10-2\sqrt{5}}}{10}$.

We give another explanation for canonical structures f and P.

Let us write the corresponding φ-invariant decomposition of the Lie algebra $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus ... \oplus \mathfrak{m}_u,$$

where the subspaces $\mathfrak{m}_1, \ldots, \mathfrak{m}_u$ correspond to the spectrum of the operator θ.

Denote by f_i, where $i = 1, 2, \ldots, s$, the base canonical f-structure whose image is the subspace $\mathfrak{m}_i$. All the other canonical f-structures are algebraic sums of some base canonical f-structures.

The base canonical almost product structure P_i has $\mathfrak{m}_i$ as a $(+1)$-subspace, the others $\mathfrak{m}_j, j \neq i$ form (-1)-subspace.

2. Canonical f-structures and the generalized Hermitian geometry.

The history of one motivation:

Invariant structures in Kähler, Hermitian and generalized Hermitian geometries:

1. Kähler manifolds (**K**) $\Longleftrightarrow$ *Hermitian symmetric spaces* (**HSS**) (A.Borel, A.Lichnerovich, . . .)

2. Almost Hermitian manifolds (**AH**) (1960 - . . .) $\Longleftrightarrow$ *Homogeneous* 3–*symmetric spaces* (1967 - . . .) (N.A.Stepanov, A.Gray, J.A.Wolf, V.F.Kirichenko, S.Salamon, . . .)

Many applications of the canonical structure $J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$:

homogeneous structures (F.Tricerri, L.Vanhecke, S.Garbiero, ...), Einstein metrics (K.Sekigawa, J.Watanabe, H.Yoshida), holomorphic and minimal submanifolds (S.Salamon), real Killing spinors (H.Baum, T.Friedrich, R.Grunewald, I.Kath).

3. Generalized almost Hermitian manifolds (**GAH)** (1983 - ...)
(V.F.Kirichenko, A.S.Gritsans, D.Blair, ...)

$\Longleftrightarrow$

Homogeneous k–symmetric spaces (1988 - ...)
(A.J.Ledger, L.Vanhecke, VVB, Yu.D.Churbanov, D.V.Vylegzhanin, A.Sakovich, A.Samsonov, N.Cohen, C.J.C.Negreiros, L.A.B.San Martin, M.Paredes, S.Pinzon, I.Khemar, ...)

2.1. Almost Hermitian structures

K	*Kähler structure*:	$\nabla J = 0;$
H	*Hermitian structure*:	$\nabla_X(J)Y - \nabla_{JX}(J)JY = 0;$
$\mathbf{G}_1$	AH-structure of class G_1, or G_1-*structure*:	$\nabla_X(J)X - \nabla_{JX}(J)JX = 0;$
QK	*quasi-Kähler structure*:	$\nabla_X(J)Y + \nabla_{JX}(J)JY = 0;$
AK	*almost Kähler structure*:	$d\,\Omega = 0;$
NK	*nearly Kähler structure*, or NK-*structure*:	$\nabla_X(J)X = 0.$

It is well known (see, for example, Gray-Hervella, 1980) that

$$\mathbf{K} \subset \mathbf{H} \subset \mathbf{G_1}; \quad \mathbf{K} \subset \mathbf{NK} \subset \mathbf{G_1}; \quad \mathbf{NK} = \mathbf{G_1} \cap \mathbf{QK}; \quad \mathbf{K} = \mathbf{H} \cap \mathbf{QK}.$$

As was already mentioned, the role of homogeneous almost Hermitian manifolds is particularly important "because they are the model spaces to which all other almost Hermitian manifolds can be compared" (A.Gray, 1983). In particular, after the detailed investigation of the 6-dimensional homogeneous nearly Kähler manifolds V.F.Kirichenko proved (1981) that naturally reductive strictly nearly Kähler manifolds $SO(5)/U(2)$ and $SU(3)/T_{max}$ are not isometric even locally to the 6-dimensional sphere S^6. These examples gave a negative answer to the conjecture of S.Sawaki and Y.Yamanoue (1976) claimed that any 6-dimensional strictly NK-manifold was a space of constant curvature.

We select here some known results closely related to the main subject of our future consideration.

Theorem 2. *(E.Abbena, S.Garbiero, 1993) Any invariant almost Hermitian structure on a naturally reductive space* $(G/H, g)$ *belongs to the class* $\mathbf{G}_1$*.*

Theorem 3. *(A.Gray, 1972) A homogeneous 3-symmetric space* G/H *with the canonical almost complex structure* J *and an invariant compatible metric* g *is a quasi-Kähler manifold. Moreover,* $(G/H, J, g)$ *belongs to the class* **NK** *if and only if* g *is naturally reductive.*

Theorem 4. *(M.Matsumoto, A.Gray, V.F.Kirichenko, 1976) A 6-dimensional strictly nearly Kähler manifold is Einstein.*

2.2. Metric f-structures

A fundamental role in the geometry of metric f-manifolds is played by the *composition tensor* T, which was explicitly evaluated (V.F.Kirichenko, 1986):

$$T(X,Y)=\frac{1}{4}f(\nabla_{fX}(f)fY-\nabla_{f^2X}(f)f^2Y), \tag{1}$$

where ∇ is the Levi-Civita connection of a (pseudo)Riemannian manifold (M,g), $X,Y\in\mathfrak{X}(M)$.Using this tensor T, the algebraic structure of a so-called *adjoint Q-algebra* in $\mathfrak{X}(M)$ can be defined by the formula: $X*Y=T(X,Y)$. It gives the opportunity to introduce some classes of metric f-structures in terms of natural properties of the adjoint Q-algebra. We enumerate below the main classes of metric f-structures together with their defining properties:

Kf	*Kähler f–structure*:	$\nabla f = 0$;
Hf	*Hermitian f–structure*:	$T(X,Y) = 0$, i.e. $\mathfrak{X}(M)$ is an abelian Q-algebra;
$\mathbf{G_1f}$	f-structure of class G_1, or *G_1f-structure*:	$T(X,X) = 0$, i.e. $\mathfrak{X}(M)$ is an anticommutative Q-algebra
QKf	*quasi-Kähler f–structure*:	$\nabla_X f + T_X f = 0$;
Kill f	*Killing f-structure*:	$\nabla_X(f)X = 0$;
NKf	*nearly Kähler f-structure*, or *NKf-structure*:	$\nabla_{fX}(f)fX = 0$.

The following relationships between the classes mentioned are evident:

$$\mathbf{Kf} = \mathbf{Hf} \cap \mathbf{QKf}; \quad \mathbf{Kf} \subset \mathbf{Hf} \subset \mathbf{G_1f}; \quad \mathbf{Kf} \subset \mathbf{Kill\ f} \subset \mathbf{NKf} \subset \mathbf{G_1f}.$$

It is important to note that in the special case $f = J$ we obtain the corresponding classes of almost Hermitian structures (16 Gray-Hervella classes). In particular, for $f = J$ the classes **Kill f** and **NKf** coincide with the well-known class **NK** of *nearly Kähler structures*.

2.3. Invariant metric f-structures on homogeneous manifolds

Recall that $(G/H, g)$ is *naturally reductive* with respect to a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ if

$$g([X,Y]_{\mathfrak{m}}, Z) = g(X, [Y,Z]_{\mathfrak{m}})$$

for all $X, Y, Z \in \mathfrak{m}$. Here the subscript $\mathfrak{m}$ denotes the projection of $\mathfrak{g}$ onto $\mathfrak{m}$ with respect to the reductive decomposition.

Any invariant metric f-structure on a reductive homogeneous space G/H determines the orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ such that $\mathfrak{m}_1 = Im\ f$, $\mathfrak{m}_2 = Ker\ f$.

Theorem 5. *(2001) Any invariant metric f-structure on a naturally reductive space $(G/H, g)$ is a G_1f-structure.*

As a special case ($Ker\ f = 0$), it follows Theorem 2 (Abbena-Garbiero). We stress the particular role of homogeneous 4- and 5-symmetric spaces.

Theorem 6. *The canonical f-structure $f = \frac{1}{2}(\theta - \theta^3)$ on any naturally reductive 4-symmetric space $(G/H, g)$ is both a Hermitian f-structure and a nearly Kähler f-structure. Moreover, the following conditions are equivalent:*

1) f is a Kähler f-structure; 2) f is a Killing f-structure; 3) f is a quasi-Kähler f-structure; 4) f is an integrable f-structure; 5) $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}$; 6) $[\mathfrak{m}_1, \mathfrak{m}_2] = 0$; 7) G/H is a locally symmetric space: $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$.

Theorem 7. *Let* $(G/H, g)$ *be a naturally reductive* 5*-symmetric space,* f_1 *and* f_2*,* J_1 *and* J_2 *the canonical structures on this space. Then* f_1 *and* f_2 *belong to both classes* $\mathbf{Hf}$ *and* $\mathbf{NKf}$*. Moreover, the following conditions are equivalent:*

1) f_1 *is a Kähler* f*-structure; 2)* f_2 *is a Kähler* f*-structure; 3)* f_1 *is a Killing* f*-structure; 4)* f_2 *is a Killing* f*-structure; 5)* f_1 *is a quasi-Kähler* f*-structure; 6)* f_2 *is a quasi-Kähler* f*-structure; 7)* f_1 *is an integrable* f*-structure; 8)* f_2 *is an integrable* f*-structure; 9)* J_1 *and* J_2 *are* NK*-structures; 10)* $[\mathfrak{m}_1, \mathfrak{m}_2] = 0$ *(here* $\mathfrak{m}_1 = Im\, f_1 = Ker\, f_2, \mathfrak{m}_2 = Im\, f_2 = Ker\, f_1$*); 11)* G/H *is a locally symmetric space:* $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$*.*

It should be mentioned that Riemannian homogeneous 4-symmetric spaces of classical compact Lie groups were classified and geometrically described (J.A.Jimenez, 1988).

The similar problem for homogeneous 5-symmetric spaces was considered by Gr.Tsagas-Ph.Xenos (1987).

By Theorem 6 and Theorem 7, it presents a collection of homogeneous f-manifolds in the classes **NKf** and **Hf**. Note that the canonical f-structures under consideration are generally non-integrable.

Remark. Note that homogeneous k–symmetric spaces with canonical f–structures admit *generalized almost Hermitian structures* of arbitrary rank r (VVB, D.V.Vylegzhanin).

Besides, there are invariant NKf-structures and Hf-structures on homogeneous spaces $(G/H, g)$, where the metric g is not naturally reductive. The example of such a kind can be realized on the 6-dimensional generalized Heisenberg group (N, g). These groups were introduced by A.Kaplan and studied by F.Tricerri, L.Vanhecke, J.Berndt and others.

Theorem 8. *The 6-dimensional generalized Heisenberg group* (N, g) *with respect to the canonical* f*-structure* $f = \frac{1}{2}(\theta - \theta^3)$ *of a homogeneous* Φ*-space of order* 4 *is both* Hf*- and* NKf*-manifold. This* f*-structure is neither Killing nor integrable on* (N, g)*.*

Remark 2. Theorems 6 and 8, in particular, illustrate simultaneously the analogy and the difference between the canonical almost complex structure J on homogeneous 3-symmetric spaces $(G/H, g, J)$ and the canonical f-structure on homogeneous 4-symmetric spaces $(G/H, g, f)$ (see Theorem 3).

Many particular examples of both semisimple and solvable types were investigated in detail. They are:

- the flag manifolds $SU(3)/T_{max}$,
- $SO(n)/SO(2) \times SO(n-3),\ n \geq 4$,
- the 6-dimensional generalized Heisenberg group

and some others.

Specifically, we present invariant Killing f–structures with non-naturally reductive metrics as well as construct invariant Kähler f–structures on some naturally reductive not locally symmetric homogeneous spaces.

We should mention other geometric structures on homogeneous k-symmetric spaces, which are of contemporary interest in geometry and topology:
- symplectic structures on k-symmetric spaces compatible with the corresponding "symmetries" of order k (A.Tralle, M.Bocheński);
- topology of homogeneous k-symmetric spaces, in particular, geometric formality (D. Kotschick, S. Terzić, Jelena Grbić);
- geometry of elliptic integrable systems (I.Khemar).

Let G be a semisimple compact Lie group, B the Killing form of the Lie algebra $\mathfrak{g}$, G/H a homogeneous k-symmetric space. Further, consider the canonical decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_u,$$

where some subspaces can be trivial. We define the collection of "diagonal" Riemannian metrics on G/H by the formula

$$\langle X, Y \rangle = \lambda_1 B(X_1, Y_1) + \ldots + \lambda_u B(X_u, Y_u),$$

where $X, Y \in \mathfrak{g}$, $i = \overline{1, u}$, $X_i, Y_i \in \mathfrak{m}_i$ from the above decomposition, $\lambda_i \in \mathbb{R}, \lambda_i < 0$.

The particular case $\lambda_1 = \cdots = \lambda_u$ gives exactly a naturally reductive Riemannian metric on G/H.

We continue this study in this more general aspect. Consider reductive homogeneous spaces G/H with invariant Riemannian metric $g = \langle \cdot, \cdot \rangle$. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

be the corresponding reductive decomposition. It is well known that the Nomizu function α for the Levi-Civita connection ∇ is of the form

$$\alpha(X, Y) = \frac{1}{2}[X, Y]_{\mathfrak{m}} + U(X, Y),$$

where $X, Y \in \mathfrak{m}$, and a bilinear symmetric mapping $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is defined from the equality:

$$2\langle U(X, Y), Z\rangle = \langle X, [Z, Y]_{\mathfrak{m}}\rangle + \langle [Z, X]_{\mathfrak{m}}, Y\rangle, \quad \forall Z \in \mathfrak{m}.$$

The Nomizu function α for the Levi-Civita connection ∇ in the case of "diagonal" metrics was calculated (2011, A.Samsonov), i.e. the corresponding mapping U is of the form:

$$U(X_i, Y_j)_{\mathfrak{m}_{i\pm j}}=\frac{\lambda_j-\lambda_i}{2\lambda_{i\pm j}}[X_i, Y_j]_{\mathfrak{m}_{i\pm j}}, \quad U(X_i, Y_i)=U(X_i, Y_j)_{\mathfrak{m}_n}=0,$$

where $\mathfrak{m}_{i+j}$ means $\mathfrak{m}_{k-(i+j)}$ for $i+j>u$, λ_{i+j} means $\lambda_{k-(i+j)}$ for $i+j>u$, $\mathfrak{m}_n$ is any of the subspaces $\mathfrak{m}_l$ excluding $\mathfrak{m}_{i-j}$ and $\mathfrak{m}_{i+j}$.

We also recall the important commutator inclusions for the subspaces from the canonical decomposition (2010, VVB, A.Samsonov). In the previous notations, they are

$$[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j} + \mathfrak{m}_{i-j}.$$

Note that for $k=2$ this formula gives the well-known classical inclusions for symmetric spaces, namely,

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \ [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \ [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}.$$

We formulate several recent general results:

Theorem 9 (A.Samsonov, 2011). *Let $(G/H, g)$ be a homogeneous k-symmetric space with any "diagonal" metric g. Then, any base canonical f-structure f_i, with $i = 1, 2, \ldots, s$ on G/H is a nearly Kähler f-structure.*

Theorem 10 (A.Samsonov, 2011). *Let $(G/H, g)$ be a homogeneous k-symmetric space with any "diagonal" metric g. Then, for any base canonical f-structure f_i on M, the following assertions hold:*

1) *if $3i \neq k$, then f_i belongs to the class* $\mathbf{Hf}$*;*
2) *if $3i = k$, then $f_i \in \mathbf{Hf} \Leftrightarrow [\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{h}$.*

Note that the above theorems generalize some known results obtained earlier for orders $k = 3, 4, 5$ (including the classical results of N.A.Stepanov and A.Gray for homogeneous 3-symmetric spaces).

3. Canonical distributions on Riemannian homogeneous k-symmetric spaces

Riemannian almost product manifold (M, g, P) naturally admits two complementary mutually orthogonal distributions **V** (*vertical*) and **H** (*horizontal*) corresponding to the eigenvalues 1 and -1 of P, respectively. In accordance with the Naveira classification there are 36 classes of Riemannian almost product structures (8 types for each of distributions). Here we consider the following types of distributions (in terms of vertical ones):

F (foliation): $\nabla_A(P)B = \nabla_B(P)A$;
AF (anti-foliation): $\nabla_A(P)A = 0$;
TGF (totally geodesic foliation): $\nabla_A P = 0$,
where A and B are vertical vector fields.

It is known (O. Gil-Medrano) that the system of conditions $\boldsymbol{AF}$ and $\boldsymbol{F}$ is equivalent to the condition $\boldsymbol{TGF}$.

Now we concentrate on invariant almost product structures on Riemannian homogeneous manifolds.

Let $(G/H, g = \langle \cdot, \cdot \rangle, P)$ be a naturally reductive homogeneous space. It was proved before (VVB, 1998) that both vertical and horizontal distributions of this structure P are always of type $\boldsymbol{AF}$. Besides, these distributions may be of type $\boldsymbol{F}$ (hence, $\boldsymbol{TGF}$) under simple algebraic criteria.

It means that, in accordance with the Naveira classification, there are exactly three classes of invariant naturally reductive almost product structures. They are $(\mathbf{TGF}, \mathbf{TGF})$, $(\mathbf{TGF}, \mathbf{AF})$, $(\mathbf{AF}, \mathbf{AF})$.

We continue this study in more general aspect. Consider reductive homogeneous spaces G/H with invariant almost product structure P and any compatible invariant Riemannian metric $g = \langle \cdot, \cdot \rangle$. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \;\; \mathfrak{m} = \mathfrak{m}_+ \oplus \mathfrak{m}_-$$

be the corresponding reductive decomposition generated by P. It is well known that the Nomizu function α for the Levi-Civita connection ∇ is of the form

$$\alpha(X,Y) = \frac{1}{2}[X,Y]_{\mathfrak{m}} + U(X,Y),$$

where $X, Y \in \mathfrak{m}$, and a bilinear symmetric mapping $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is defined from the equality:

$$2\langle U(X,Y), Z\rangle = \langle X, [Z,Y]_{\mathfrak{m}}\rangle + \langle [Z,X]_{\mathfrak{m}}, Y\rangle, \quad \forall Z \in \mathfrak{m}.$$

Theorem 11. *Let* $(G/H, g = \langle\cdot,\cdot\rangle, P)$ *be a Riemannian reductive almost product space. Then*

(1) *the vertical distribution* $\mathfrak{m}_+$ *belongs to type* $\boldsymbol{AF}$ *iff*

$$U(A,A) \in \mathfrak{m}_+\,, \qquad \forall\, A \in \mathfrak{m}_+.$$

(1) *the vertical distribution* $\mathfrak{m}_+$ *belongs to type* $\boldsymbol{F}$ *iff*

$$[\mathfrak{m}_+, \mathfrak{m}_+] \subset \mathfrak{m}_+ \oplus \mathfrak{h}.$$

It follows that the distribution $\mathfrak{m}_+$ *belongs to type* $\boldsymbol{TGF}$ *iff both above conditions are satisfied.*

The similar conditions can be written for the horizontal distribution $\mathfrak{m}_-$.

We apply these results for canonical structures P on homogeneous k-symmetric spaces with the "diagonal" metrics.

Let G be a semisimple compact Lie group, B the Killing form of the Lie algebra $\mathfrak{g}$, G/H a homogeneous k-symmetric space. As above, consider the canonical decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus ... \oplus \mathfrak{m}_u,$$

where some subspaces can be trivial. We define the collection of "diagonal" Riemannian metrics on G/H by the formula

$$\langle X, Y \rangle = \lambda_1 B(X_1, Y_1) + ... + \lambda_u B(X_u, Y_u),$$

where $X, Y \in \mathfrak{g}$, $i = \overline{1, u}$, $X_i, Y_i \in \mathfrak{m}_i$ from the above decomposition, $\lambda_i \in \mathbb{R}, \lambda_i < 0$.

Theorem 12. *Any the base canonical distribution* $\mathfrak{m}_i$, $\overline{1,u}$ *on Riemannian k-symmetric space* $(G/H, g = \langle\cdot,\cdot\rangle)$ *is of type* $\boldsymbol{AF}$ *for all "diagonal" metrics* g.

Further, the distribution $\mathfrak{m}_i$ *belongs to* $\boldsymbol{F}$ *(hence,* $\boldsymbol{TGF}$*) if and only if one of the following cases is realized:*

(1) *The subspace* $\mathfrak{m}_{2i}$ *is trivial.*
(2) *The index* i *satisfies the condition* $k = 3i$.
(3) $[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{h}$.
(4) *If* $k = 2n$, *then* $i = n$ *(i.e.* $\mathfrak{m}_n$ *belongs to* $\boldsymbol{F}$*).*

It follows that for base canonical distributions the result doesn't depend on the function U. Note that for 4- and 5-symmetric spaces we have the decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$, i.e. all canonical distributions are base.

However, for other canonical distributions (e.g., $\mathfrak{m}_i \oplus \mathfrak{m}_j$) the situation is more complicated.

Example (homogeneous 6-symmetric spaces).

Here the decomposition is the following: $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$.

Theorem 13. *Let G/H be a homogeneous 6-symmetric space, where G is a compact semisimple Lie group. Suppose g is any diagonal Riemannian metric on G/H represented by the collection $(\lambda_1, \lambda_2, \lambda_3)$. Then:*

(1) *$\mathfrak{m}_2$ and $\mathfrak{m}_3$ are of type **TGF**.*
(2) *$\mathfrak{m}_1$ belongs to type **TGF** if and only if $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}$.*
(3) *$\mathfrak{m}_1 \oplus \mathfrak{m}_2$ is of type **AF** if and only if any of the following two conditions is satisfied: (a)$\lambda_1 = \lambda_2$; (b) $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1$.*

(4) $\mathfrak{m}_1 \oplus \mathfrak{m}_2$ *is of type* $\boldsymbol{F}$ *if and only if* $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1$. *This is also a criterion for type* $\boldsymbol{TGF}$.

(5) $\mathfrak{m}_1 \oplus \mathfrak{m}_3$ *is of type* $\boldsymbol{AF}$ *if and only if any of the following two conditions is satisfied: (a)*$\lambda_1 = \lambda_3$*; (b)* $[\mathfrak{m}_1, \mathfrak{m}_3] = 0$.

(6) $\mathfrak{m}_1 \oplus \mathfrak{m}_3$ *is of type* $\boldsymbol{F}$ *if and only if both the following relations hold:* $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}$, $[\mathfrak{m}_1, \mathfrak{m}_3] = 0$. *This is also a criterion for type* $\boldsymbol{TGF}$.

(7) $\mathfrak{m}_2 \oplus \mathfrak{m}_3$ *is of type* $\boldsymbol{AF}$ *if and only if any of the following two conditions is satisfied: (a)*$\lambda_2 = \lambda_3$*; (b)* $[\mathfrak{m}_2, \mathfrak{m}_3] = 0$.

(8) $\mathfrak{m}_2 \oplus \mathfrak{m}_3$ *is of type* $\boldsymbol{F}$ *if and only if* $[\mathfrak{m}_2, \mathfrak{m}_3] = 0$. *This is also a criterion for type* $\boldsymbol{TGF}$.

This theorem gives the opportunity to characterize the Naveira classes for all combinations of the above canonical distributions. As an example, the canonical structure P_3 belongs to the class $(\boldsymbol{TGF},\ \boldsymbol{TGF})$ if and only if $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1$.

4. Left-invariant f-structures on nilpotent Lie groups.

Many results of this section were obtained jointly P.A.Dubovik.

3.1. Left-invariant f-structures on 2-step nilpotent Lie groups.

We start with several important examples.

Example 1.

The 6–dimensional generalized Heisenberg group (N, g) is a G_1–manifold with respect to the left-invariant canonical almost Hermitian structure $J = f_3$ of the Riemannian homogeneous 6–symmetric space (N, g, Φ). Besides, the structure J is neither nearly Kähler nor Hermitian structure on the manifold (N, g).

It should be mentioned that G_1–structures of such a kind have interesting applications in *heterotic strings* (P.Ivanov, S.Ivanov, 2005).

Example 2. We also consider the 5–dimensional Heisenberg group $H(2,1)$ as a Riemannian homogeneous 6–symmetric space. It is proved that all the canonical f-structures f_i, $i = 1, \dots, 4$ are Hermitian f-structures. Besides, the base f-structures f_1 and f_2 are integrable, but the other f-structures f_3 and f_4 are not integrable.

We notice that the group $H(2,1)$ is used in constructing the 6-dimensional nilmanifold connected with the *heterotic equations* of motion in *string theory* (M.Fernandez, S.Ivanov, L.Ugarte, R.Villacampa, 2009).

General approach. Let G be a 2-step nilpotent Lie group, $\mathfrak{g}$ its Lie algebra, $Z(\mathfrak{g})$ the center of $\mathfrak{g}$. Consider a left-invariant metric f-structure on G with respect to a left-invariant Riemannian metric g on G.

Theorem 14 (VVB, P.Dubovik, 2013)**.** (*i*) *If* $Z(\mathfrak{g}) \subset Ker\, f$ *then* f *is a Hermitian* f*-structure, but it is not a Kähler* f*-structure.*

(ii) If $Im\, f \subset Z(\mathfrak{g})$ *then* f *is both a Hermitian and a nearly Kähler* f*-structure, but it is not a Kähler* f*-structure.*

Example 3. Let $H(n,1)$ be a $(2n+1)$-dimensional matrix Heisenberg group. We can consider $H(n,1)$ as a Riemannian homogeneous k–symmetric space, where k is even.

Lemma 1. *Let* f *be any left-invariant canonical* f*-structure on a Riemannian homogeneous* k*–symmetric space* $H(n,1)$*. Then* $Z(\mathfrak{h}(n,1)) \subset Ker\, f$*.*

As an application of a previous theorem, we obtain

Theorem 15 (VVB, P.Dubovik, 2013)**.** *Any left-invariant canonical* f*-structure on a* $(2n+1)$*-dimensional matrix Heisenberg group* $H(n,1)$ *is a Hermitian* f*-structure, but it is not a Kähler* f*-structure.*

3.2. Left-invariant f-structures on Lie groups.

Let G be a connected Lie group, $\mathfrak{g}$ its Lie algebra. Denote by $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}]$ the first and the second ideal of the derived series. Consider a left-invariant Riemannian metric g on G determined by the Euclidean inner product on $\mathfrak{g}$.

Theorem 16 (P.Dubovik, 2013)**.** *Let f be a left-invariant metric f-structure on G satisfying any of the following conditions:*

(i) $\mathfrak{g}^{(1)} \subset Ker\, f$;

(ii) $Im\, f \subset \mathfrak{g}^{(1)}$, $\mathfrak{g}^{(2)} \subset Ker\, f$;

(iii) $Im\, f \subset Z(\mathfrak{g}) \subset \mathfrak{g}^{(1)}$.

Then f is a Hermitian f-structure. Moreover, the condition (*iii*) *implies that f is a nearly Kähler f-structure. In addition, under the* *condition* (i) f *is a nearly Kähler* f*-structure if and only if* $[fX, f^2X] = 0$ *for any* $X \in \mathfrak{g}$.

Note that, for example, the 6-dimensional generalized Heisenberg group and the 5-dimensional Heisenberg group $H(2,1)$ admit f-structures mentioned in the above theorem.

3.3. Filiform Lie groups.

Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension m. Let

$$C^0\mathfrak{g} \supset C^1\mathfrak{g} \supset \cdots \supset C^{m-2}\mathfrak{g} \supset C^{m-1}\mathfrak{g} = 0$$

be the descending central series of $\mathfrak{g}$, where

$$C^0\mathfrak{g} = \mathfrak{g}, C^i\mathfrak{g} = [\mathfrak{g}, C^{i-1}\mathfrak{g}], \quad 1 \leq i \leq m-1.$$

A Lie algebra $\mathfrak{g}$ is called *filiform* if $dim C^k\mathfrak{g} = m - k - 1$ for $k = 1, \ldots, m-1$. A Lie group G is called *filiform* if its Lie algebra is filiform.

Note that the filiform Lie algebras have the maximal possible nilindex, that is $m-1$.

Basic examples of $(n+1)$-dimensional filiform Lie algebras:

1. The Lie algebra L_n:

$[X_0, X_i] = X_{i+1},\ i = 1, \ldots, n-1.$

2. The Lie algebra $Q_n (n = 2k+1)$:

$[X_0, X_i] = X_{i+1},\ i = 1, \ldots, n-1,$

$[X_i, X_{n-i}] = (-1)^i X_n,\ i = 1, \ldots, k.$

3.4. Left-invariant f-structures on 6-dimensional filiform Lie groups.

The classification of 6-dimensional nilpotent Lie algebras was obtained by V.V.Morozov (1958), there exist 32 types of such algebras.

We select from this list 5 filiform Lie algebras:

(1) The Lie algebra $\mathfrak{g} = L_5$:

$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_1, e_5] = e_6.$

Proposition 1. If $e_1 \in Ker\, f$, then f is a Hermitian f-structure.

For example, the following f-structure satisfies the above condition:

$f(e_1) = f(e_2) = 0,\ f(e_3) = -e_4,\ f(e_4) = e_3,$

$f(e_5) = e_6,\ f(e_6) = -e_5.$

(2) The Lie algebra $\mathfrak{g} = Q_5$:

$[e_1, e_2] = e_3, [e_1, e_5] = e_6, [e_2, e_3] = e_4, [e_2, e_4] = e_5, [e_3, e_4] = e_6.$

Proposition 2. Suppose any of the following conditions is satisfied: $e_1, e_4 \in Ker\, f$, $\;e_3, e_5 \in Ker\, f$, $\;e_2, e_6 \in Ker\, f$. Then f is a Hermitian f-structure.

For example, the following f-structure satisfies the above condition:
$f(e_1) = f(e_4) = 0,\; f(e_2) = -e_3,\; f(e_3) = e_2,$
$f(e_5) = e_6,\; f(e_6) = -e_5.$

On analogy, the other three filiform Lie algebras were studied.

5. Canonical structures of "metallic family".

Recently a new type of affinor structures was introduced. It was initiated by the quadratic equation $x^2 - x - 1 = 0$ for the Golden ratio (Golden section, Golden proportion, Divine ratio, ...). The positive root $\frac{1+\sqrt{5}}{2} = \phi$ of this equation is the Golden ratio (the Phidias number).

Definition 1. (M.Crasmareanu, C.-E.Hretcanu, 2008). Affinor structure F on a manifold M is called a *Golden structure* if $F^2 = F + id$.

This notion is a particular case of a general concept of a *polynomial structure* on M (S.Goldberg, K.Yano, 1970).

It is easy to see that any Golden structure F induces an almost product structure $P = \frac{1}{\sqrt{5}}(2F - id)$. Conversely, an almost product structure P defines a Golden structure $F = \frac{1}{2}(id + \sqrt{5}P)$. Besides, in this correspondence $F \longleftrightarrow P$ we have: $\tilde{F} = id - F \longleftrightarrow \tilde{P} = -P$.

Very recently the same authors (C.-E.Hretcanu, M.Crasmareanu, 2013) generalized the above construction. It was based on the following classical equation.

Fix two positive integers p and q. The positive solution $\sigma_{p,q}$ of the equation $x^2 - px - q = 0$ is called a (p,q)-*metallic number*. These numbers

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}$$

of the *metallic means family* were considered by Vera W. de Spinadel (1997 and later).

Some particular cases of the numbers from the metallic means family:

the golden mean $\phi = \frac{1+\sqrt{5}}{2}$ if $p = q = 1$;

the silver mean $\sigma_{2,1} = 1 + \sqrt{2}$ for $p = 2, q = 1$;

the bronze mean $\sigma_{3,1} = \frac{3+\sqrt{13}}{2}$ for $p = 3, q = 1$;
the copper mean $\sigma_{1,2} = 2$ for $p = 1, q = 2$ and so on.

It should be mentioned that many authors wrote about close relation of some metallic numbers to classical Fibonacci numbers, Pell numbers, design, fractal geometry, dynamical systems, quasicrystals etc.

Definition 2. (M.Crasmareanu, C.-E.Hretcanu, 2013). Affinor structure F on a manifold M is called a *metallic structure* if $F^2 = pF + qI$. Further, for a Riemannian manifold (M, g) the structure F is called a *metallic Riemannian structure* if $g(FX, Y) = g(X, FY)$ for any vector fields X, Y.

Any almost product structure P induces two metallic structures on M:

$$F_1 = \frac{p}{2}I + (\frac{2\sigma_{p,q} - p}{2})P, \quad F_2 = \frac{p}{2}I - (\frac{2\sigma_{p,q} - p}{2})P.$$

Conversely, any metallic structure F on M determines two almost product structures:

$$P = \pm\left(\frac{2}{2\sigma_{p,q} - p}F - \frac{p}{2\sigma_{p,q} - p}I\right).$$

Moreover, P is a Riemannian almost product structure on (M, g) if and only if F_1, F_2 are metallic Riemannian structures.

An important observation is that the structures F and P define the same distributions on a manifold M. It means that the properties of the structures F and P practically coincide.

As usual, we are interested in invariant structures on homogeneous manifolds.

Question: Are there *homogeneous* manifolds with *invariant* metallic structures F?

Certainly, the answer is positive. For this purpose we can use a rich collection of canonical almost product structures P on homogeneous k-symmetric spaces. Moreover, we are able to completely describe *all* canonical structures F using the corresponding above formulae. For simplicity, we illustrate some particular cases.

Example 1. Homogeneous 4-symmetric spaces.

Here $P = \theta^2$. It follows that all canonical metallic structures are represented by the formula:

$$F = \frac{p}{2}I \pm (\frac{2\sigma_{p,q} - p}{2})\theta^2.$$

Example 2. Homogeneous 5-symmetric spaces.

We have: $P = \frac{1}{\sqrt{5}}(\theta - \theta^2 - \theta^3 + \theta^4)$.

Then, for instance, canonical silver structures can be written in the form

$$F = I \pm \sqrt{\frac{2}{5}}(\theta - \theta^2 - \theta^3 + \theta^4).$$

Main conclusion: The properties of the metallic structures F can be obtained from those of the corresponding almost product structures P. It follows that many previous results about invariant distributions and structures on homogeneous k-symmetric spaces and nilpotent Lie groups can be adapted and reformulated in terms of metallic structures.

6. Regular Φ-spaces (the most general case).

Definition 3 (N.A.Stepanov, 1967)**.** *Let G/H be a homogeneous Φ-space, $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras for G and H, $\varphi = d\Phi_e$ the automorphism of $\mathfrak{g}$. Consider the linear operator $A = \varphi - id$ and the Fitting decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with respect to A, where $\mathfrak{g}_0$ and $\mathfrak{g}_1$ denote 0- and 1-component of the decomposition respectively. It is clear that $\mathfrak{h} = Ker\,A$, $\mathfrak{h} \subset \mathfrak{g}_0$. A homogeneous Φ-space G/H is called a regular Φ-space if $\mathfrak{h} = \mathfrak{g}_0$.*

Two basic facts (N.A.Stepanov, 1967):

- *Any homogeneous Φ-space of order k ($\Phi^k = id$) is a regular Φ-space.*
- *Any regular Φ-space is reductive. More exactly, the Fitting decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $\mathfrak{m} = A\mathfrak{g}$ is a reductive one.*

This decomposition is obviously φ-invariant. As before, we denote by θ the restriction of φ to $\mathfrak{m}$. Now we also recall the construction of the algebra $\mathcal{A}(\theta)$ of canonical affinor structures on regular Φ-spaces.

Consider the commutative algebra $A_n(\mathbb{P})$ consisting of the matrices having the form

$$\begin{pmatrix} z_1 & z_2 & \dots & z_{n-1} & z_n \\ 0 & z_1 & \ddots & & z_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & z_2 \\ 0 & 0 & \dots & 0 & z_1 \end{pmatrix}$$

where all elements z_j, $j = 1, 2, \dots, n$ belong to the field $\mathbb{P}$.

The following theorem gives the description of the algebraic structure for the algebra $\mathcal{A}(\theta)$.

Theorem 17 (VVB, 2002). *Let G/H be a regular Φ-space, ν the unitary minimal polynomial of the operator θ, $\nu = \nu_1^{n_1}\nu_2^{n_2}\dots\nu_m^{n_m}\nu_{m+1}^{n_{m+1}}\dots\nu_s^{n_s}$ its decomposition into unitary irreducible factors over the field $\mathbb{R}$, where deg $\nu_j = 2$ for $j = 1, 2, \dots, m$, deg $\nu_j = 1$ for $j = m+1, \dots, s$, and all polynomials $\nu_1, \nu_2, \dots, \nu_s$ are pairwise mutually disjoint. Then, the algebra $\mathcal{A}(\theta)$ of canonical affinor structures of the space G/H is isomorphic to the direct sum*

$$A_{n_1}(\mathbb{C}) \oplus \dots \oplus A_{n_m}(\mathbb{C}) \oplus A_{n_{m+1}}(\mathbb{R}) \oplus \dots \oplus A_{n_s}(\mathbb{R})$$

of real commutative algebras.

Corollary 4. *If G/H is a regular Φ-space, where Φ is a semisimple automorphism of the Lie group G, then*

$$\mathcal{A}(\theta) \cong \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{m} \oplus \underbrace{\mathbb{R} \oplus \cdots \oplus \mathbb{R}}_{s-m}$$

Corollary 5. *Suppose that G/H is a homogeneous k-symmetric space. Let us denote the number of pairs of conjugate kth roots of unity in the spectrum of the operator θ by m. Then,*

$$\mathcal{A}(\theta) \cong \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{m} \oplus \mathbb{R}, \qquad \textit{if} \;\; -1 \in \textit{spec}\, \theta,$$
$$\mathcal{A}(\theta) \cong \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{m}, \qquad \textit{if} \;\; -1 \notin \textit{spec}\, \theta.$$

In accordance with the structure of the minimal polynomial ν of the operator θ, we have the following decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{m}_1 \oplus \sum_{\alpha \in spec\, \theta} \mathfrak{m}_\alpha.$$

Lemma 2. *For this decomposition we have*

$$[\mathfrak{m}_\alpha, \mathfrak{m}_\beta] \subset \mathfrak{m}_{\alpha\beta} + \mathfrak{m}_{\overline{\alpha}\beta},$$

where the subspace $\mathfrak{m}_{\alpha\beta}$ (respectively, $\mathfrak{m}_{\overline{\alpha}\beta}$) is trivial if $\alpha\beta$ (respectively, $\overline{\alpha}\beta$) doesn't belong to spec θ (over $\mathbb{C}$).

Using this lemma and other facts, we obtain

Theorem 18. *Let G/H be a regular Φ-space with naturally reductive metric g, f_α the base canonical f-structure (i.e. $\mathfrak{m}_\alpha$ is an image of*

f_α). Suppose α satisfies any of two conditions: (1) mod $\alpha = 1$, (2) $\overline{\alpha}\alpha \notin spec\,\theta$. Then f_α is a nearly Kähler f-structure.

As a particular case, it immediately follows

Corollary 6. *Any base canonical f-structure f_i on naturally reductive homogeneous k-symmetric space is a nearly Kähler f-structure.*

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