Kähler surfaces with quasi constant holomorphic curvature

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Workshop

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The aim of the present talk is to describe connected Kähler surfaces (M, g, J) admitting a global, 2-dimensional, J-invariant distribution $\mathcal{D}$ having the following property: The holomorphic curvature $K(\pi) = R(X, JX, JX, X)$ of any J-invariant 2-plane $\pi \subset T_xM$, where $X \in \pi$ and $g(X, X) = 1$, depends only on the point x and the number $|X_{\mathcal{D}}| = \sqrt{g(X_{\mathcal{D}}, X_{\mathcal{D}})}$, where $X_{\mathcal{D}}$ is an orthogonal projection of X on $\mathcal{D}$. In this case we have

$$R(X, JX, JX, X) = \phi(x, |X_{\mathcal{D}}|)$$

where $\phi(x, t) = a(x) + b(x)t^2 + c(x)t^4$ and a, b, c are smooth functions on M. Also $R = a\Pi + b\Phi + c\Psi$ for certain curvature tensors $\Pi, \Phi, \Psi \in \bigotimes^4 \mathfrak{X}^*(M)$ of Kähler type. The investigation of such manifolds, called QCH Kähler manifolds, was started by G. Ganchev and V. Mihova in [G-M-1],[G-M-2]. In our paper [J-2] we used their local results to obtain a global classification of such manifolds under the assumption that $\dim M = 2n \geq 6$. By $\mathcal{E}$ we shall denote the 2-dimensional distribution which is the orthogonal complement of $\mathcal{D}$ in TM. In the present paper we show that a Kähler surface (M, g, J) is a QCH manifold with respect to a distribution $\mathcal{D}$ if and only if is a QCH manifold with respect to the distribution $\mathcal{E}$. We also prove that (M, g, J) is a QCH Kähler surface if and only if the antiselfdual Weyl tensor W^- is degenerate and there exist a negative almost complex structure $\overline{J}$ which preserves the Ricci tensor Ric of (M, g, J) i.e. $Ric(\overline{J}., \overline{J}.) = Ric(., .)$ and such that $\overline{\omega} = g(\overline{J}., .)$ is an eigenvector of W^- corresponding to simple eigenvalue of W^-. Equivalently (M, g, J) is a QCH Kähler surface iff it admits a negative almost complex structure $\overline{J}$ satisfying the Gray second condition $R(X, Y, Z, W) - R(\overline{J}X, \overline{J}Y, Z, W) = R(\overline{J}X, Y, \overline{J}Z, W) + R(\overline{J}X, Y, Z, \overline{J}W)$. In [A-C-G-1] Apostolov, Calderbank and Gauduchon have classified weakly selfdual Kähler surfaces, extending the result of Bryant who classified self-dual Kähler surfaces [B]. Weakly self-dual Kähler surfaces turned out to be of Calabi type and of orthotoric type or surfaces with parallel

Ricci tensor.

We show that any Calabi type Kähler surfce and every orthotoric Kähler surface is a QCH manifold. In both cases the opposite complex strucure $\overline{J}$ is conformally Kähler. We also classify locally homogeneous QCH Kähler surfaces.

Let (M, g, J) be a 4-dimensional Kähler manifold with a 2-dimensional J-invariant distribution $\mathcal{D}$. Let $\mathfrak{X}(M)$ denote the algebra of all differentiable vector fields on M and $\Gamma(\mathcal{D})$ denote the set of local sections of the distribution $\mathcal{D}$. If $X \in \mathfrak{X}(M)$ then by $X^\flat$ we shall denote the 1-form $\phi \in \mathfrak{X}^*(M)$ dual to X with respect to g, i.e. $\phi(Y) = X^\flat(Y) = g(X, Y)$. By ω we shall denote the Kähler form of (M, g, J) i.e. $\omega(X, Y) = g(JX, Y)$. Let (M, g, J) be a QCH Kähler surface with respect to $J - invariant$ 2-dimensional distribution $\mathcal{D}$. Let us denote by $\mathcal{E}$ the distribution $\mathcal{D}^\perp$, which is a 2-dimensional, J-invariant distribution. By h, m respectively we shall denote the tensors $h = g \circ (p_\mathcal{D} \times p_\mathcal{D}), m = g \circ (p_\mathcal{E} \times p_\mathcal{E})$, where $p_\mathcal{D}, p_\mathcal{E}$ are the orthogonal projections on $\mathcal{D}, \mathcal{E}$ respectively. It follows that $g = h + m$. Let us define almost complex structure $\overline{J}$ by $\overline{J}_{|\mathcal{E}} = -J_{|\mathcal{E}}$ and $\overline{J}_{|\mathcal{D}} = J_{|\mathcal{D}}$. Let $\theta(X) = g(\xi, X)$ and $J\theta = -\theta \circ J$ which means that $J\theta(X) = g(J\xi, X)$. For every almost Hermitian manifold (M, g, J) the self-dual Weyl tensor W^+ decomposes under the action of the unitary group $U(2)$. We have $\bigwedge^* M = \mathbb{R} \oplus LM$ where $LM = [[\bigwedge^{(0,2)} M]]$ and we can write W^+ as a matrix with respect to this block decomposition

$$W^+ = \begin{pmatrix} \frac{\kappa}{6} & W_2^+ \\ (W_2^+)^* & W_3^+ - \frac{\kappa}{12} Id_{|LM} \end{pmatrix}$$

where κ is the conformal scalar curvature of (M, g, J) (see [A-A-D]). The selfdual Weyl tensor W^+ of (M, g, J) is called degenerate if $W_2 = 0, W_3 = 0$. In general the self-dual Weyl tensor of 4-manifold (M, g) is called degenerate if it has at most two eigenvalues as an endomorphism $W^+ : \bigwedge^+ M \to \bigwedge^+ M$. We say that an almost Hermitian structure J satisfies the second Gray curvature condition if

$$R(X,Y,Z,W)-R(JX,JY,Z,W)=R(JX,Y,JZ,W)$$

$$+R(JX,Y,Z,JW)$$

which is equivalent to $Ric(J,J)=Ric$ and $W_2^+=W_3^+=0$. Hence (M,g,J) satisfies the second Gray condition if J preserves the Ricci tensor and W^+ is degenerate. We shall denote by Ric_0 and ρ_0 the trace free part of the Ricci tensor Ric and the Ricci form ρ respectively. An ambikähler structure on a real 4-manifold consists of a pair of Kähler metrics (g_+,J_+,ω_+) and (g_-,J_-,ω_-) such that g_+ and g_- are conformal metrics and J_+ gives an opposite orientation to that given by J_- (i.e the volume elements $\frac{1}{2}\omega_+\wedge\omega_+$ and $\frac{1}{2}\omega_-\wedge\omega_-$ have opposite signs).

We shall recall some results from [G-M-1]. Let

$$R(X,Y)Z=([\nabla_X,\nabla_Y]-\nabla_{[X,Y]})Z \tag{1}$$

and let us write

$$R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

If R is the curvature tensor of a QCH Kähler manifold (M,g,J), then there exist functions $a,b,c \in C^\infty(M)$ such that

$$R = a\Pi + b\Phi + c\Psi, \tag{2}$$

where Π is the standard Kähler tensor of constant holomorphic curvature i.e.

$$\Pi(X,Y,Z,U) = \frac{1}{4}(g(Y,Z)g(X,U) - g(X,Z)g(Y,U) \tag{3}$$

$$+g(JY,Z)g(JX,U) - g(JX,Z)g(JY,U) - 2g(JX,Y)g(JZ,U)),$$

the tensor Φ is defined by the following relation

$$\Phi(X,Y,Z,U) = \frac{1}{8}(g(Y,Z)h(X,U) - g(X,Z)h(Y,U) \tag{4}$$

$$+g(X,U)h(Y,Z) - g(Y,U)h(X,Z) + g(JY,Z)h(JX,U)$$

$$-g(JX,Z)h(JY,U) + g(JX,U)h(JY,Z) - g(JY,U)h(JX,Z)$$

$$-2g(JX,Y)h(JZ,U) - 2g(JZ,U)h(JX,Y))$$

and finally

$$\Psi(X,Y,Z,U) = -h(JX,Y)h(JZ,U) = -(h_J \otimes h_J)(X,Y,Z,U). \tag{5}$$

where $h_J(X,Y) = h(JX,Y)$. Let $V = (V,g,J)$ be a real $2n$ dimensional vector space with complex structure J which is skew-symmetric with respect to the scalar product g on V. Let assume further that $V = D \oplus E$ where D is a 2-dimensional, J-invariant subspace of V, E denotes its orthogonal complement in

V. Note that the tensors Π, Φ, Ψ given above are of Kähler type. It is easy to check that for a unit vector $X \in V$ $\Pi(X, JX, JX, X) = 1, \Phi(X, JX, JX, X) = |X_D|^2, \Psi(X, JX, JX, X) = |X_D|^4$, where X_D means an orthogonal projection of a vector X on the subspace D and $|X| = \sqrt{g(X, X)}$. It follows that for a tensor (2.2) defined on V we have

$$R(X, JX, JX, X) = \phi(|X_D|)$$

where $\phi(t) = a + bt^2 + ct^4$.

Let $J, \overline{J}$ be hermitian, opposite orthogonal structures on a Riemannian 4-manifold (M, g) such that J is a positive almost complex structure. Let $\mathcal{E} = ker(J\overline{J} - Id), \mathcal{D} = ker(J\overline{J} + Id)$ and let the tensors Π, Φ, Ψ be defined as above where $h = g(p_{\mathcal{D}}, p_{\mathcal{D}})$. Let us define a tensor $K = \frac{1}{6}\Pi - \Phi + \Psi$. Then K is a curvature tensor, $b(K) = 0, c(K) = 0$ where b is Bianchi operator and c is the Ricci contraction. Define the endomorphism $K : \bigwedge^2 M \to \bigwedge^2 M$ by the formula $g(K\phi, \psi) = -K(\phi, \psi)$ (see (2.1)). Then we have

Lemma

The tensor K satisfies $K(\bigwedge^+ M) = 0$. Let $\phi, \psi \in \bigwedge^- M$ be the local forms orthogonal to $\overline{\omega}$ such that $g(\phi, \phi) = g(\psi, \psi) = 2$ and $g(\phi, \psi) = 0$. Then $K(\overline{\omega}) = \frac{1}{3}\overline{\omega}, K(\phi) = -\frac{1}{6}\phi, K(\psi) = -\frac{1}{6}\psi$.

Proof.

A straightforward computation. □

In the special case of a Kähler surface (M, g, J) we get for a QCH manifold (M, g, J)

Proposition

Let (M, g, J) be a Kähler surface which is a QCH manifold with respect to the distribution $\mathcal{D}$. Then (M, g, J) is also QCH manifold with respect to the distribution $\mathcal{E} = \mathcal{D}^{\perp}$ and if Φ', Ψ' are the above tensors with respect to $\mathcal{E}$ then

$$R = (a + b + c)\Pi - (b + 2c)\Phi' + c\Psi'. \tag{6}$$

Proof.

Let us assume that

$$X \in TM, |X| = 1.$$

Then if $\alpha = |X_{\mathcal{D}}|, \beta = ||X_{\mathcal{E}}||$ then $1 = \alpha^2 + \beta^2$. Hence $R(X, JX, JX, X) = a + b\alpha^2 + c\alpha^4 = a + b(1 - \beta^2) + c(1 - \beta^2)^2 = a + b + c - (b + 2c)\beta^2 + c\beta^4$. $\square$

If (M, g, J) is a QCH Kähler surface then one can show that the

Ricci tensor ρ of (M, g, J) satisfies the equation

$$\rho(X, Y) = \lambda m(X, Y) + \mu h(X, Y) \tag{7}$$

where $\lambda = \frac{3}{2}a + \frac{b}{4}, \mu = \frac{3}{2}a + \frac{5}{4}b + c$ are eigenvalues of ρ (see [G-M-1], Corollary 2.1 and Remark 2.1.) In particular the distributions $\mathcal{E}, \mathcal{D}$ are eigendistributions of the tensor ρ corresponding to the eigenvalues λ, μ of ρ. The Kulkarni-Nomizu product of two symmetric $(2,0)$-tensors $h, k \in \bigotimes^2 TM^*$ we call a tensor $h \oslash k$ defined as follows:

$$h \oslash k(X, Y, Z, T) = h(X, Z)k(Y, T) + h(Y, T)k(X, Z)$$
$$-h(X, T)k(Y, Z) - h(Y, Z)k(X, T).$$

Similarly we define the Kulkarni-Nomizu product of two 2-forms ω, η

$$\omega \oslash \eta(X, Y, Z, T) = \omega(X, Z)\eta(Y, T) + \omega(Y, T)\eta(X, Z)$$

$$-\omega(X,T)\eta(Y,Z)-\omega(Y,Z)\eta(X,T).$$

Then $b(\omega \oslash \eta) = -\frac{2}{3}\omega \wedge \eta$ where b is the Bianchi operator. In fact

$$3b(\omega \oslash \eta)(X,Y,Z,T)=\omega(X,Z)\eta(Y,T)+\omega(Y,T)\eta(X,Z)$$

$$-\omega(X,T)\eta(Y,Z)$$

$$-\omega(Y,Z)\eta(X,T)+\omega(Y,X)\eta(Z,T)+\omega(Z,T)\eta(Y,X)$$

$$-\omega(Y,T)\eta(Z,X)-\omega(Z,X)\eta(Y,T)+\omega(Z,Y)\eta(X,T)$$

$$+\omega(X,T)\eta(Z,Y)-\omega(Z,T)\eta(X,Y)-\omega(X,Y)\eta(Z,T)$$

$$=-2\omega \wedge \eta(X,Y,Z,T).$$

Note that

$$\Pi = -\frac{1}{4}\left(\frac{1}{2}(g \oslash g + \omega \oslash \omega) + 2\omega \otimes \omega\right), \tag{8}$$

$$\Phi = -\frac{1}{8}(h \oslash g + h_J \oslash \omega + 2\omega \otimes h_J + 2h_J \otimes \omega), \tag{9}$$

$$\Psi = -h_J \otimes h_J, \tag{10}$$

where $\omega = g(J.,.)$ is the Kähler form. Note that $b(\Psi) = \frac{1}{3} h_J \wedge h_J = 0$ since $h_J = e_1 \wedge e_2$ is primitive, where e_1, e_2 is an orthonormal basis in $\mathcal{D}$.

Theorem

Let (M, g, J) be a Kähler surface. If (M, g, J) is a QCH manifold then $W^- = c(\frac{1}{6}\Pi - \Phi + \Psi)$ and W^- is degenerate. The 2-form $\overline{\omega}$ is an eigenvector of W^- corresponding to a simple eigenvalue of W^- and $\overline{J}$ preserves the Ricci tensor. On the other hand let us assume that (M, g, J) admits a negative almost complex structure $\overline{J}$ such that $Ric(\overline{J}, \overline{J}) = Ric$. Let $\mathcal{E} = ker(J\overline{J} - Id), \mathcal{D} = ker(J\overline{J} + Id)$. If $W^- = \frac{\kappa}{2}(\frac{1}{6}\Pi - \Phi + \Psi)$ or equivalently if the half-Weyl tensor W^- is degenerate and $\overline{\omega}$ is an eigenvector of W^- corresponding to a simple eigenvalue of W^- then (M, g, J) is a QCH manifold.

Note that for a Kähler surface (M, g, J) the Bochner tensor coincides with W^- and we have

$$R = -\frac{\tau}{12}(\frac{1}{4}(g \oslash g + \omega \oslash \omega) + \omega \otimes \omega)$$

$$-\frac{1}{4}(\frac{1}{2}(Ric_0 \oslash g + \rho_0 \oslash \omega) + \rho_0 \otimes \omega + \omega \otimes \rho_0) + W^- .$$

If (M, g, J) is a QCH Kähler surface then $Ric = \lambda m + \mu h$ where $\lambda = \frac{3}{2}a + \frac{b}{4}, \mu = \frac{3}{2}a + \frac{5}{4}b + c$. Consequently $Ric_0 = -\frac{b+c}{2}m + \frac{b+c}{2}h = \delta h - \delta m$ where $\delta = \frac{b+c}{2}$. Hence $Ric_0 = 2\delta h - \delta g$. Hence we have

$$R = -\frac{\tau}{12}(\frac{1}{4}(g \oslash g + \omega \oslash \omega) + \omega \otimes \omega)$$

$$-\frac{1}{4}(\frac{1}{2}((2\delta h - \delta g) \oslash g + (2\delta h_J - \delta\omega) \oslash \omega) + (2\delta h_J - \delta\omega) \otimes \omega +$$

$$\omega \otimes (2\delta h_J - \delta\omega)) + W^- .$$

Consequently

$$R = \frac{\tau}{6}\Pi + 2\delta\Phi - \delta\Pi + W^- = (a - \frac{c}{6})\Pi + (b + c)\Phi + W^-$$

and $a\Pi + b\Phi + c\Psi = (a - \frac{c}{6})\Pi + (b + c)\Phi + W^-$ hence $W^- = c(\frac{1}{6}\Pi - \Phi + \Psi)$. It follows that W^- is degenerate and $\overline{\omega}$ is an eigenvalue of W^- corresponding to the simple eigenvalue of W^-. It is also clear that $Ric(\overline{J}, \overline{J}) = Ric$.
On the other hand let us assume that a Kähler surface (M, g, J) admits a negative almost complex structure $\overline{J}$ preserving the Ricci tenor Ric and such that W^- is degenerate with eigenvector $\overline{\omega}$ corresponding to the simple eigenvalue of W^-. Equivalently it means that $\overline{J}$ satisfies the second Gray condition of the curvature i.e. $R(X, Y, Z, W) - R(\overline{J}X, \overline{J}Y, Z, W) = R(\overline{J}X, Y, \overline{J}Z, W) + R(\overline{J}X, Y, Z, \overline{J}W)$. Then $W^- = \frac{\kappa}{2}((\frac{1}{6}\Pi - \Phi + \Psi)$. If $Ric_0 = \delta(h - m)$ then as above $R = \frac{\tau}{6}\Pi + 2\delta\Phi - \delta\Pi + W^-$. Consequently $R = (\frac{\tau}{6} - \delta)\Pi + 2\delta\Phi + \frac{\kappa}{2}(\frac{1}{6}\Pi - \Phi + \Psi)$ and consequently

$$R = (\frac{\tau}{6} - \delta + \frac{\kappa}{12})\Pi + (2\delta - \frac{\kappa}{2})\Phi + \frac{\kappa}{2}\Psi. \qquad (11)$$

Remark

Note that κ is the conformal scalar curvature of $(M, g, \overline{J})$. The Bochner tensor of QCH manifold was first identified in [G-M-2].

Corollary

A Kähler surface (M, g, J) is a QCH manifold iff it admits a negative almost complex structure $\overline{J}$ satisfying the second Gray condition of the curvature i.e.

$$R(X, Y, Z, W) - R(\overline{J}X, \overline{J}Y, Z, W) =$$

$$R(\overline{J}X, Y, \overline{J}Z, W) + R(\overline{J}X, Y, Z, \overline{J}W)$$

The J-invariant distribution $\mathcal{D}$ with respect to which (M, g, J) is a QCH manifold is given by $\mathcal{D} = \ker(J\overline{J} - Id)$ or by $\mathcal{D} = \ker(J\overline{J} + Id)$.

Theorem

Let us assume that (M, g, J) is a Kähler surface admitting a negative Hermitian structure $\overline{J}$ such that $Ric(\overline{J}, \overline{J}) = Ric$. Then (M, g, J) is a QCH manifold.

Proof.

If a Hermitian manifold (M, g, J) has a J-invariant Ricci tensor *Ric* then the tensor W^+ is degenerate (see [A-G]). □

Remark

If a Kähler surface (M, g, J) is compact and admits a negative Hermitian structure $\overline{J}$ as above then $(M, g, \overline{J})$ is locally conformally Kähler and hence globally conformally Kähler if $b_1(M)$ is even. Thus (M, g, J) is ambiKähler since $b_1(M)$ is even.

Now we give examples of QCH Kähler surfaces. First we give (see

[A-C-G-1])

Definition

A Kähler surface (M, g, J) is said to be of Calabi type if it admits a non-vanishing Hamiltonian Killing vector field ξ such that the almost Hermitian pair (g, I) -with I equal to J on the distribution spanned by ξ and $J\xi$ and $-J$ on the orthogonal distribution - is conformally Kähler.

Every Kähler surface of Calabi type is given locally by

$$g = (az - b)g_\Sigma + w(z)dz^2 + w(z)^{-1}(dt + \alpha)^2, \qquad (12)$$
$$\omega = (az - b)\omega_\Sigma + dz \wedge (dt + \alpha), d\alpha = a\omega_\Sigma$$

where $\xi = \frac{\partial}{\partial t}$.
The Kähler form of Hermitian structure I is given by $\omega_I = (az - b)\omega_\Sigma - dz \wedge (dt + \alpha)$ and the Kähler metric corresponding to I is $g_- = (az - b)^2 g$.

If $a \neq 0$ then the metric (*) is a product metric. If $a \neq 0$ then we set $a = 1, b = 0$ and write $w(z) = \frac{z}{V(z)}$ hence

$$g = zg_\Sigma + \frac{z}{V(z)}dz^2 + \frac{V(z)}{z}(dt + \alpha)^2, \tag{13}$$

$$\omega = z\omega_\Sigma + dz \wedge (dt + \alpha), d\alpha = \omega_\Sigma$$

It is known that for a Kähler surface of Calabi type of non-product type we have $\rho_0 = \delta\omega_I$ where $\delta = -\frac{1}{4z}(\tau_\Sigma + (\frac{V_z}{z^2})_z z^2)$ (see [A-C-G-1]) and consequently $Ric(I, I) = Ric$. This last relation remains true in the product case metric. Hence we have

Theorem

Every Kähler surface of Calabi type is a QCH Kähler surface.

Definition

A Kähler surface (M, g, J) is ortho-toric if it admits two independent Hamiltonian Killing vector fields with Poisson commuting momentum maps $\xi\eta$ and $\xi + \eta$ such that $d\xi$ and $d\eta$ are orthogonal.

An explicit classification of ortho-toric Kähler metrics is given in [A-C-G]. We have (this Proposition is proved in [A-C-G], Prop.8)

Proposition

The almost Hermitian structure (g, J, ω) *defined by*

$$g = (\xi-\eta)(\frac{d\xi^2}{F(\xi)}-\frac{d\eta^2}{G(\eta)})+\frac{1}{\xi-\eta}(F(\xi)(dt+\eta dz)^2-G(\eta)(dt+\xi dz)^2 \tag{14}$$

$$Jd\xi = \frac{F(\xi)}{\xi-\eta}(dt+\eta dz), Jdt = -\frac{\xi d\xi}{F(\xi)} - \frac{\eta d\eta}{G(\eta)} \tag{15}$$

$$Jd\eta = -\frac{G(\eta)}{\xi-\eta}(dt+\xi dz), Jdz = \frac{d\xi}{F(\xi)} + \frac{d\eta}{G(\eta)},$$

$$\omega = d\xi \wedge (dt+\eta dz) + d\eta \wedge (dt+\xi dz) \tag{16}$$

is orthotoric where F, G *are any functions of one variable. Every orthotoric Kähler surface* (M, g, J) *is of this form.*

Any orthotoric surface has a negative Hermitian structure $\overline{J}$, whose

Kähler form $\overline{\omega}$ is given by

$$\overline{\omega} = d\xi \wedge (dt + \eta dz) - d\eta \wedge (dt + \xi dz)$$

and

$$\overline{J}d\xi = Jd\xi = \frac{F(\xi)}{\xi - \eta}(dt + \eta dz), \overline{J}dt = -\frac{\xi d\xi}{F(\xi)} + \frac{\eta d\eta}{G(\eta)} \qquad (17)$$

$$\overline{J}d\eta = Jd\eta = -\frac{G(\eta)}{\xi - \eta}(dt + \xi dz), \overline{J}dz = \frac{d\xi}{F(\xi)} - \frac{d\eta}{G(\eta)},$$

The structure $(g_- = (\xi - \eta)^2 g, \overline{J})$ is Kähler. We also have $\rho_0 = \delta\overline{\omega}$ where $\delta = \frac{F'(\xi) - G'(\eta)}{(2(\xi-\eta)^2} - \frac{F''(\xi) + G''(\eta)}{(4(\xi-\eta))}$.
In particular the Hermitian strucure $\overline{J}$ preserves Ricci tensor *Ric*.
Hence we get

Theorem

Every orthotoric Kähler surface is a QCH Kähler surface.

Note that both Calabi type and orthotoric Kähler surfaces are ambikähler. On the other hand we have

Theorem

Let (M, g, J) be ambi-Kähler surface which is a QCH manifold. Then locally (M, g, J) is orthotoric or of Calabi type or a product of two Riemannian surfaces or is an anti-selfdual Einstein-Kähler surface.

Proof.

(We follow [A-C-G-2]). Let us denote by g_- the second Kähler metric. Let us assume that $g_- \neq g$. Then $g = \phi^{-2} g_-$ and the field $X = grad_{\omega_-}\phi$ is a Killing vector field $L_X g = L_X g_- = 0$ and is holomorphic with respect to $\overline{J}$). We shall show that X is also holomorphic with respect to J. In fact $Ric_0 = \delta g(J\overline{J}, .)$ and $L_X Ric = 0, L_X\delta = 0$. Hence $0 = \delta g((L_X J)\overline{J}, .)$ and consequently $L_X J = 0$ in $U = \{x : Ric_0(x) \neq 0\}$. If (M, g) is Einstein then $W^+ \neq 0$ everywhere or (M, g, J) is anti-selfdual. In the first case X preserves the simple eigenspace of W^+ and hence ω, cosequently $L_X J = 0$.

Note that $X = \overline{J} grad_g \psi$ where $\psi = -\frac{1}{\phi}$. Since $L_X\omega = 0$ we have $dX \lrcorner \omega = 0$ and consequently the 1-form $J\overline{J}d\psi$ is closed and locally equals $\frac{1}{2}d\sigma$. Thus the two form $\Omega = \frac{3}{2}\sigma\omega + \psi^3\omega_-$, where ω_- is the Kähler form of $(M, g_-, \overline{J})$, is a Hamiltonian form in the sense of [A-C-G-1] and the result follows from the classification in [A-C-G-1]. This form is defined globally if $H^1(M) = 0$. $\square$

Remark

Note that in the compact case every Killing vector field on a Kähler surface is holomorphic. If (M, g, J) is an Einstein Kähler anti-selfdual then in the case where it is not conformally flat the manifold $(M, g, \overline{J})$ is a self-dual Einstein Hermitian conformal to self-dual Kähler metric. Such a metric must be either orthotoric or of Calabi type. Thus (M, g, J) is of Calabi type if $(M, g, \overline{J})$ is of Calabi type, however (M, g, J) can not be orthotoric if $(M, g, \overline{J})$ is orthotoric.

Now we shall investigate Einstein QCH Kähler surfaces.

Theorem

Let (M, g, J) be a Kähler-Einstein surface. Then (M, g, J) is a QCH Kähler surface if and only if it admits a negative Hermitian structure $\overline{J}$ or it has constant holomorphic curvature and admits any negative almost complex structure. If (M, g, J) is QCH and the second case does not hold then $\overline{J}$ is conformally Kähler hence (M, g, J) is ambiKähler.

Proof.

If an Einstein 4-manifold (M, g) admits a degenerate tensor W^- then $W^- = 0$ or $W^- \neq 0$ on the whole of M. In the second case by the result of Derdzinski it admits a Hermitian structure $\overline{J}$ which is conformally Kähler and the metric $(g(W^-, W^-))^{\frac{1}{3}} g$ is a Kähler metric with respect to $\overline{J}$. □

Remark

(Compare [A-C-G-2]). If (M, g, J) is a QCH Kähler Einstein surface which is not anti-selfdual then in the case $H^1(M) = 0$ on (M, g, J) there is defined global Hamiltonian two form and on the open and dense subset U of M the metric g is:
(a) a Kähler product metric of two Riemannian surfaces of the same Gauss curvature
(b) Kähler Einstein metric of Calabi type over a Riemannian surface (Σ, g_Σ) of constant Gauss curvature k of the form (2.13) where $V(z) = a_1 z^3 + kz^2 + a_2$
(c) Kähler Einstein ambitoric metric of parabolic type (see [A-C-G-2], section 5.4.

Theorem

Let (M, g, J) be a self-dual Kähler surface with $Ric_0 \neq 0$ everywhere on M. Then $(M, g, \overline{J})$ is a QCH Kähler surface with Hermitian complex structure $\overline{J}$.

Proof.

We show as in Th.1 that $R = \frac{\tau}{6}\Pi + 2\delta\Phi - \delta\Pi$ where $\rho_0 = \delta\overline{\omega}$. Note that in $U = \{x : Ric_0 \neq 0\}$ the negative structure $\overline{J}$ is uniquely determined and is Hermitian in U (see Prop.4 in [A-G]). □

Remark

Note that a selfdual Kähler surface (M, g, J) is QCH if admits any negative almost complex structure $\overline{J}$ preserving the Ricci tensor Ric. For example $\mathbb{CP}^2$ with standard Fubini-Studi metric is selfdual however is not QCH since it does not admit any negative almost complex structure. However the manifold $M = \mathbb{CP}^2 - \{p_0\}$ for any point $p_0 \in \mathbb{CP}^2$ is QCH and admits a negative Hermitian complex structure (see [J-3]). In [D-2] there are constructed many examples of self-dual Kähler surfaces with $Ric_0 \neq 0$ hence QCH Kähler self-dual surfaces. Every self-dual Kähler metric is weakly selfdual. These metrics were classified by Bryant in [B]. From [A-C-G-1] it follows that self dual Kähler metrics are orthotoric or of Calabi type and in fact are ambi-Kähler. They are
(a) Kähler self-dual metrics of Calabi type over a Riemannian surface (Σ, g_Σ) of constant scalar curvature k where
$V(z) = a_1 z^4 + a_2 z^3 + kz^2$

Remark

(b) Kähler self-dual metrics of orthotoric type where $F(x) = lx^3 + Ax^2 + Bx, G(x) = lx^3 + Ax^2 + Bx$
(c) complex space forms and a product $\Sigma_c \times \Sigma_{-c}$ of Riemann surfaces of constant scalar curvatures c and $-c$.

Lemma

Let M be a connected QCH Kähler surface which is not Einstein. Then the following conditions are equivalent:
(a) The scalar curvature τ of (M, g, J) is constant and $\overline{J}$ is almost Kähler
(b) The eigenvalues λ, μ of Ric are constant.

Proof.

(a)⇒(b) Note that $\rho = \lambda\omega_1 + \mu\omega_2$ where λ, μ are eigenvalues of *Ric* and $\omega_2 = h_J, \omega_1 = m_J$. Note that $d\omega_1 + d\omega_2 = 0$ and

$$(\mu - \lambda)d\omega_1 = d\lambda \wedge \omega_1 + d\mu \wedge \omega_2 \tag{18}$$

Note that $\overline{J}$ is almost Kähler if and only if $d\omega_1 = 0$. Hence from (2.7) we get $p_{\mathcal{D}}(\nabla\lambda) = 0, p_{\mathcal{E}}(\nabla\mu) = 0$. Since τ is constant we get $\nabla\lambda = -\nabla\mu$ in an open set $U = \{x : \lambda(x) \neq \mu(x)\}$. Thus $\nabla\lambda = \nabla\mu = 0$ in U and consequently $U = M$ and λ, μ are constant.

$(b) \Rightarrow (a)$ This implication is trivial. □

Now we give a classification of locally homogeneous QCH Kähler surfaces.

Proposition

Let (M, g, J) be a QCH locally homogeneous manifold. Then the following cases occur:
(a) (M, g, J) has constant holomorphic curvature (hence is locally symmetric and self-dual)
(b) (M, g, J) is locally a product of two Riemannian surfaces of constant scalar curvature
(c) (M, g, J) is locally isometric to a unique 4-dimensional proper 3-symmetric space.

Proof.

If (M, g) is Einstein locally homogeneous 4-manifold then is locally symmetric (see [Jen]). A locally irreducible locally symmetric Kähler surface is self-dual.(see [D-1]). If (M, g) is not Einstein then using Lemma we see that $(M, g, \overline{J})$ is an almost Kähler manifold satisfying the Gray condition G_2. Hence $||\nabla\overline{J}||$ is constant on M and in the case $||\nabla\overline{J}|| \neq 0$ it is strictly almost Kähler manifold satisfying G_2. Such manifolds are classified in [A-A-D] and are locally isometric to a proper 3-symmetric space. Note that they are Kähler in an opposite orientation. If $||\nabla\overline{J}|| = 0$ then the case (b) holds. □

Remark

A Riemannian 3-symmetric space is a manifold (M, g) such that for each $x \in M$ there exists an isometry $\theta_x \in Iso(M)$ such that $\theta_x^3 = Id$ and x is an isolated fixed point. On a such manifold there is a natural canonical g-ortogonal almost complex structure $\overline{J}$ such that all θ_x are holomorphic with respect to $\overline{J}$. Such structure in dimension 4 is almost Kähler and satisfies the Gray condition G_2. The example of 3-symmetric 4-dimensional Riemannian space with non-integrable structure $\overline{J}$ was constructed by O. Kowalski in [Ko],Th.VI.3. This is the only proper generalized symmetric space in dimension 4. This example is defined on $\mathbb{R}^4 = \{x, y, u, v\}$ by the metric

$$g = (-x+\sqrt{x^2+y^2+1})du^2+(x+\sqrt{x^2+y^2+1})dv^2-2ydu\odot dv \tag{19}$$
$$+[\frac{(1+y^2)dx^2+(1+x^2)dy^2-2xydx\odot dy}{1+x^2+y^2}]$$

Proposition

Let (M, g, J) be a QCH Kähler surface. If (M, g) is conformally Einstein then the almost Hermitian structure $\overline{J}$ is Hermitian or (M, g, J) is self-dual.

Proof.

Let us assume that (M, g_1) is an Einstein manifold where $g_1 = f^2 g$. Then (M, g_1) is an Einstein manifold with degenerate half-Weyl tensor W^-. Consequently $W^- = 0$ or $W^- \neq 0$ everywhere. In the second case the metric

$$(g_1(W^-, W^-))^{\frac{1}{3}} g_1$$

is a Kähler metric with respect to $\overline{J}$. Thus $\overline{J}$ is Hermitian and conformally Kähler. □

Remark

Every QCH Kähler surface is a holomorphically pseudosymmetric Kähler manifold. (see [O],[J-1]). In fact from [J-1] it follows that $R.R = (a + \frac{b}{2})\Pi.R$. Hence in the case of QCH Kähler surfaces we have

$$R.R = \frac{1}{6}(\tau - \kappa)\Pi.R \tag{20}$$

where τ is the scalar curvature of (M, g, J) and κ is the conformal scalar curvature of $(M, g, \overline{J})$. Note that (2.19) is the obstruction for a Kähler surface to have a negative almost complex $\overline{J}$ structure satisfying the Gray condition (G_2). In an extremal situation where $(M, g, \overline{J})$ satisfies the Gray (G_1) condition we have $R.R = 0$.

Now we classify QCH Kähler surfaces for which a, b, c are all constant. Then λ, μ are constant and if (M, g) is not Einstein the almost complex structure $\overline{J}$ is almost Kähler. Hence $(M, g, \overline{J})$ is a G_2 almost Kähler manifold. Consequently $|\nabla\overline{\omega}|$ is constant and

(M,g,J) is a product of two Riemannian surfaces of constant scalar curvature or is a proper 3-symmetric space. If (M,g) is Einstein then $\kappa = 2c$ is constant and $|W^-|^2 = \frac{1}{24}\kappa^2$ is constant. Thus $\kappa = 0$ and (M,g,J) has constant holomorphic curvature (is a real space form) or by [D-1] the manifold $(M,g,\overline{J})$ is Kähler hence (M,g,J) is a product of two Riemannian surfaces of constant scalar curvature. Note that for a proper 3-symmetric space we have $\delta = \frac{\kappa}{4}$ for the distribution $\mathcal{D}$ perpendicular to the Kähler nullity of $\overline{J}$(see [A-A-D]), thus $b = 2\delta - \frac{\kappa}{2} = 0$ and $a = \frac{1}{6}(\tau - \kappa) = -\frac{1}{2}|\nabla\overline{\omega}|^2$. Since $\mu = 0$ $c = -\frac{3}{2}a$ and $\tau = -\kappa$ where $\kappa = \frac{3}{2}|\nabla\overline{\omega}|^2$. Hence

$$R.R = -\frac{\kappa}{3}\Pi.R \tag{21}$$

where $\kappa = \frac{3}{2}|\nabla\overline{\omega}|^2$ is constant. Summarizing we have proved

Proposition

Let us assume that (M, g, J) is a QCH Kähler surface with constant a, b, c. Then the following cases occur:
(a) (M, g, J) has constant holomorphic curvature (hence is locally symmetric and self-dual)
(b) (M, g, J) is locally a product of two Riemannian surfaces of constant scalar curvature
(c) (M, g, J) is locally isometric to a unique 4-dimensional proper 3-symmetric space and $a = -\frac{1}{3}\kappa, b = 0, c = \frac{1}{2}\kappa$ where $\kappa = \frac{3}{2}|\nabla\overline{\omega}|^2$ is constant scalar curvature of $(M, g, \overline{J})$, consequently $R = -\frac{1}{3}\kappa\Pi + \frac{1}{2}\kappa\Psi$.

Remark

We consider above the proper 3-symmetric space as a QCH manifold with respect to the distribution $\mathcal{D}$ perpendicular to the Kähler nullity of $\overline{J}$. If we consider it as a QCH manifold with respect to the distribution $\mathcal{E} = \mathcal{D}^{\perp}$ then $R = \frac{1}{6}\kappa\Pi - \kappa\Phi' + \frac{1}{2}\kappa\Psi'$ (see Prop.1.).

Theorem

Let (M, g, J) be a Kähler surface admitting opposite Hermitian structure I satisfying the first Gray condition G_1 which is locally conformally Kähler. Then locally

$$g = zg_\Sigma + \frac{1}{Cz}dz^2 + Cz(dt + \alpha)^2$$

where (Σ, g_Σ) is a Riemannian surface with area form ω_Σ and $d\alpha = \omega_\Sigma$ or (M, g, J) is a product of Riemannian surfaces or a space form with zero holomorphic sectional curvature. The Kähler form of (M, g, J) is $\Omega = z\omega_\Sigma + dz \wedge (dt + \alpha)$.

Proof.

First assume that (M, g, J) is a Kähler semi-symmetric surface foliated by 2-dimensional Euclidean space. Hence it follows that $\mathcal{D}$ is totally geodesic homothetic foliation. Such foliations were classified locally in [Ch-N]. Thus (M, g, J) is a Kähler surface of Calabi type. From [A-C-G] it follows that $\kappa = \tau$ if $V(z) = Cz^2$ where $g = zg_\Sigma + \frac{z}{V(z)}dz^2 + \frac{V(z)}{z}(dt + \alpha)^2$ is a general Calabi type metric which is not a Kähler product. For the general case let us note that QCH Kähler surface for which the structure I is Hermitian and locally conformally Kähler are of Calabi type or are orthotoric surfaces or $W = 0$ (see [J-4]). Semi-symmetric surfaces with $W = 0$ are products of Riemannian surfuces of constant opposite scalar curvatures (see [B]). One can easily check that orthotoric surface can be semi-symmetric only if $W = 0$ which finishes the proof. $\square$

Theorem

Let (M, g, J) be a Kähler surface. Then the following conditions are equivalent:
(1) There exists a vector field ξ such that $\nabla\xi = cI, c \in \mathbb{R} - \{0\}$
(2)There exists a vector field η such that $\nabla\eta = cJ, c \in \mathbb{R} - \{0\}$
(3) There exists a function $\phi \in C^{\infty}(M)$ such that $H^{\phi} = cg$ where H^{ϕ} is the Hessian of ϕ and $c \in \mathbb{R} - \{0\}$,
(4) (U, g, J) is a semi-symmetric Kähler surface of Calabi type where U is an open dense subset of M ,
(5)There exists an open and dense subset U of M such that (U, g, J) is locally isometric to $g = zg_{\Sigma} + \frac{1}{Cz}dz^2 + Cz(dt + \alpha)^2$ where (Σ, g_{Σ}) is a Riemannian surface with area form ω_{Σ} and $d\alpha = \omega_{\Sigma}$. The Kähler form of (U, g, J) is $\Omega = z\omega_{\Sigma} + dz \wedge (dt + \alpha)$.

Proof.

Let us assume that (1) holds. It is easy to see that (1),(2) are equivalent and (3) implies (1).
Note that $\nabla_X J\xi = cJX$ so we can take $\eta = J\xi$. Let us define the distribution $\mathcal{D} = span\{\xi, J\xi\}$. We can assume that $c = \frac{1}{2}$. It is clear that ξ is a holomorphic vector field and $J\xi$ is a holomorphic Killing vector field. Note that $\xi, J\xi$ are different from zero on an open dense subset U of M. What is more if $T = \nabla J\xi = \frac{1}{2}J$ then $R(X,\xi)Y = \nabla T(X,Y) = 0$. Hence $R(X,Y)\xi = R(X,Y)J\xi = 0$ and

$$R(X,JX,JX,X) = ||X_{\mathcal{D}^\perp}||^4 K(\mathcal{D}^\perp)$$

where $K(\mathcal{D}^\perp)$ is a sectional curvature of the distribution $\mathcal{D}^\perp$. It follows that (M,g,J) is a QCH Kähler surface i $R = c\Psi$ and $R.R = 0$. The distribution $\mathcal{D}$ is totally geodesic in particular is integrable. Since $L_{J\xi}g = 0, L_\xi g = g$ it follows that $\mathcal{D}$ is a complex conformal foliation and the almost Hermitian structure I determined by $\mathcal{D}$ is Hermitian.

Proof.

Note that $L_\eta g = \theta(\eta)g$ on $\mathcal{D}^\perp$ for $\eta \in \Gamma(\mathcal{D})$ where $\theta^\sharp = \frac{1}{|\xi|^2}\xi$. Hence $|\theta| = \frac{1}{|\xi|}$. Vector field $\xi = \frac{1}{|\theta|^2}\theta^\sharp$ is holomorphic. One can easily verify that $d\theta = 0$ since $X|\theta|^2 = 0$ for $X \in \mathcal{D}^\perp$ since $\nabla_X \xi = \frac{1}{2}X - \frac{1}{2}X \ln|\theta|^2 \xi + \frac{1}{2}JX \ln|\theta|^2 J\xi$ for $X \in \mathcal{D}^\perp$. Hence (M, g, J) is a semi-symmetric Kähler surface of Calabi type. Note that since $\theta = -d\ln|\theta|^2$ on U it follows that if the distribution $\mathcal{D}$ extends over the whole of M and consequently θ is defined on M then $\neq 0$ on the whole of M and consequently $U = M$. Note that the function $\phi = \frac{1}{|\theta|^2}$ satisfies (3) and the field $\xi = \nabla\phi$ satisfies (1). On the other hand if (M, g, J) is a semi-symmetric Kähler surface of Calabi type then vector field $\xi = \frac{1}{|\theta|^2}\theta^\sharp$ satisfies $\nabla_X \xi = \frac{1}{2}X$. □

Remark

It is known that the only complete Kähler surface satisfying (3) is a Euclidean space $(\mathbb{C}^2, can)$ with standard metric can. Let $\{e_1, e_2, e_3, e_4\}$ be a standard orthonormal basis of $\mathbb{C}^2$, $Je_1 = e_2, Je_3 = e_4$. Then $\xi = x_1 e_1 + y_1 e_2 + x_2 e_3 + y_2 e_4, J\xi = x_1 e_2 - y_1 e_1 + x_2 e_4 - y_2 e_3$ where $z_1 = x_1 + iy_1, z_2 + iy_2$ are standard complex coordinates on $\mathbb{C}^2$ and $\phi = \frac{1}{2}(x_1^2 + y_1^2 + x_2^2 + y_2^2)$. In this case we have $U = \mathbb{C}^2 - \{0\}$ and the totally geodesic complex foliation $\mathcal{D} = span\{\xi, J\xi\}$ defines on U a Hermitian structure I which does not extend to the whole of $\mathbb{C}^2$ and $\xi(0) = 0$. Hence $(\mathbb{C}^2 - \{0\}, can)$ is a semisymmetric surface of Calabi type (clearly $R = 0$ in this case).

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