LCS structures on nilpotent Lie algebras and almost abelian Lie algebras

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Outline

A bit of Hermitian geometry:
Let (M, J, g) be a Hermitian manifold, where J is a complex structure and g is a Hermitian metric. (M, J, g) is a locally conformally Kähler (LCK) manifold if there exists an open cover $\{U_i\}$ and smooth functions f_i on U_i such that each local metric

$$g_i = exp(-f_i)g$$

is Kähler on U_i.
This condition is equivalent to requiring that

$$d\omega = \theta \wedge \omega$$

for some closed 1-form θ, called the Lee form.
The Lee form θ is determined by

$$\theta = -\frac{1}{n-1}(\delta\omega) \circ J,$$

where $2n$ is the dimension of M.

Given a simply connected Lie group G, a lattice on G is a discrete subgroup Γ such that $\Gamma\backslash G$ is a compact manifold. If G is solvable (nilpotent), we have a solvmanifold (nilmanifold).

A left-invariant LCK structure on a $G \rightsquigarrow$ an LCK structure on $\Gamma\backslash G$.

There are many recent results on LCK structures on solvmanifold.

- Sawai (2007): if a non-toral nilmanifold admits an invariant LCK structure, then it is a quotient of $\mathbb{R} \times H_{2n+1}$, where H_{2n+1} is the $(2n+1)$-dimensional Heisenberg Lie group.
- What happens in the non-invariant case on nilmanifolds? Not known yet.
- Kasuya (2013) proved the non-existence of Vaisman metrics is some solvmanifolds.
- Andrada, O. (2014): if $\mathfrak{g}$ is a unimodular Lie algebra with an LCK structure where the complex structure is abelian, that is $[X, Y] = [JX, JY]$ then $\mathfrak{g} \simeq \mathbb{R} \times \mathfrak{h}_{2n+1}$.

LCS manifolds

A locally conformally symplectic (LCS) form on a manifold M is a non-degenerate 2-form ω such that there exists an open cover $\{U_i\}$ and smooth functions f_i on U_i such that

$$\omega_i = exp(-f_i)\omega$$

is a symplectic form on U_i.

This condition is equivalent to requiring that

$$d\omega = \theta \wedge \omega$$

for some closed 1-form θ, called the Lee form.

When the manifold is a Lie group and θ and ω are left invariant we obtain $\rightsquigarrow$ an LCS structure on the Lie algebra.

The p-adapted cohomology:

For a Lie algebra $\mathfrak{g}$ and a closed 1-form α on $\mathfrak{g}$, we have the adapted differential operator d_α:

$$d_\alpha \beta = \alpha \wedge \beta + d\beta,$$

for $\beta \in \bigwedge^p \mathfrak{g}$.
A p-form β is called α-closed if $d_\alpha \beta = 0$. Since $d_\alpha^2 = 0$, it defines the p-adapted cohomology group $H^p_\alpha(\mathfrak{g})$.

If $(\mathfrak{g}, \omega, \theta)$ is a LCS Lie algebra, then

$$d_{-\theta}\omega = -\theta \wedge \omega + d\omega = 0,$$

so ω is $-\theta$-closed.

Nilpotent case

Proposition (Dixmier '55)

Let $\mathfrak{g}$ be a nilpotent Lie algebra. For any non-trivial closed 1-form α on $\mathfrak{g}$, $H^p_\alpha(\mathfrak{g}) = 0$, $p \geq 2$.

If a nilpotent Lie algebra $\mathfrak{g}$ admits an LCS structure, then there exists a 1-form β such that

$$d_{-\theta}\beta = \omega.$$

Relation with Contact structures:

Let $A \in \mathfrak{g}$ such that $\theta(A) = 1$.
We can write

$$\mathfrak{g} = \mathbb{R}A \oplus \ker\theta.$$

Since ω is non-degenerate we get that

$$\beta \wedge (d\beta)^n \neq 0$$

on $\ker \theta$. The pair (θ, β) is a contact pair.

Proposition

If $\mathfrak{g}$ is a nilpotent Lie algebra with an LCS structure, then

- $\dim([\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{z}) = 1$.
- *if $\theta \neq 0$, then $1 \leq \dim \mathfrak{z} \leq 2$.*
- *if $\mathfrak{g}$ is a 2-step nilpotent, then $\mathfrak{g} = \mathbb{R} \times h_{2n+1}$.*

Proposition

If $\mathfrak{g}$ is a nilpotent Lie algebra with an LCS structure, then $\ker \theta$ has a contact form $\beta|_{\ker \theta}$.

6-dimensional case

Proposition

If $\mathfrak{g}$ is a 6-dimensional real nilpotent Lie algebras with an LCS structure, the $\mathfrak{g}$ is isomorphic to one and only one of the following Lie algebras.

- *(0,0,12,13,14+23,34+52)*
- *(0,0,12,13,14,34+52)*
- *(0,0,0,12,14-23,15+34)*
- *(0,0,0,12,14,15+23+24)*
- *(0,0,0,12,14,15+24)*
- *(0,0,0,12,13,14+35)*
- *(0,0,0,12,23,14+35)*
- *(0,0,0,12,23,14-35)*
- *(0,0,0,0,12,15+34)*
- *(0,0,0,0,12,14+25)*
- *(0,0,0,0,0,12+34)*

For example $(0,0,0,0,0,12+34)$ is the Lie algebra generated by $\{e_1,\ldots,e_6\}$ where $de^6 = e^1 \wedge e^2 + e^3 \wedge e^4$, that is, $\mathbb{R} \times \mathfrak{h}_5$. [Bazzoni-Marrero] also obtained this classification recently.

Almost abelian

A Lie algebra is called almost abelian if it has an abelian ideal of codimension one:

$$\mathfrak{g} = \mathbb{R} \ltimes \mathbb{R}^n.$$

We have two problems:

$\rightsquigarrow$ Do they admit LCS structures?

$\rightsquigarrow$ Let G be the associated simply connected Lie group, does G admit any lattice?

In order to answer the first question we will consider two cases according to the dimension of $\mathfrak{g}$:

Theorem (Andrada, -)

Let $\mathfrak{g}$ be a unimodular almost abelian Lie algebra with $\dim \mathfrak{g} \geq 6$. Then $\mathfrak{g}$ admits an LCS structure if and only if $\mathfrak{g} = \mathbb{R} \ltimes_M \mathbb{R}^{2n+1}$ where the action is given by

$$M = \left(\begin{array}{c|c} \mu & w^t \\ \hline 0 & -\frac{\mu}{2n} I + B \end{array}\right),$$

with $B \in \mathfrak{sp}(n, \mathbb{R})$.
Moreover, $\mathfrak{g}$ admits an LCK form if and only if $w = 0$ and $B \in \mathfrak{u}(n)$.

Remark

The LCS forms are symplectic if and only if $\mu = 0$.

4-dimensional case:

$\mathfrak{g} = \mathbb{R} \ltimes_M \mathbb{R}^3$ where the action is given by

$$M = \left(\begin{array}{c|cc} \mu & a & b \\ \hline m & & \\ & -\frac{\mu}{2}I & + \quad B \\ n & & \end{array}\right),$$

with $B \in \mathfrak{sl}(2, \mathbb{R})$ and $a, b, m, n \in \mathbb{R}$.

We classify up to Lie algebra isomorphism the unimodular Lie algebras of type $\mathbb{R} \ltimes_M \mathbb{R}^3$ with an LCS structure.

Theorem

Let $\mathfrak{g}$ be a unimodular almost abelian 4-dimensional Lie algebra with LCS structure, thus $\mathfrak{g}$ is isomorphic to one and only one of the following Lie algebras.

Lie algebra	Lie brackets	LCS form	Lee form
$\mathbb{R}^4$	$[\cdot,\cdot]=0$	$\omega=e^1\wedge e^2+e^3\wedge e^4$	$\theta=0$
$\mathfrak{h}_3\times\mathbb{R}$	$[e_1,e_2]=e_3$	$\omega=e^1\wedge e^2+e^3\wedge e^4$	$\theta=-e^4$
$\mathfrak{n}_4$	$[e_1,e_2]=e_3$, $[e_1,e_3]=e_4$	$\omega=e^1\wedge e^3+e^2\wedge e^4$	$\theta=e^2$
$\mathfrak{r}_{3,-1}\times\mathbb{R}$	$[e_1,e_2]=e_2$, $[e_1,e_3]=-e_3$	$\omega=e^1\wedge e^2+e^3\wedge e^4$	$\theta=e^1$
$\mathfrak{r}_{4,\lambda,-(1+\lambda)}$ $(\lambda\geq 1)$	$[e_1,e_2]=e_2$, $[e_1,e_3]=\lambda e_3$, $[e_1,e_4]=-(1+\lambda)e_4$	$\omega=e^1\wedge e^2+e^3\wedge e^4$	$\theta=e^1$
$\mathfrak{r}_{4,-1/2}$	$[e_1,e_2]=e_2$, $[e_1,e_3]=e_2+e_3$, $[e_1,e_4]=-2e_4$	$\omega=e^1\wedge e^2+e^3\wedge e^4$	$\theta=e^1$
$\mathfrak{r}'_{3,0}\times\mathbb{R}$	$[e_1,e_3]=e_4$, $[e_1,e_4]=-e_3$	$\omega=e^1\wedge e^3+e^2\wedge e^4$	$\theta=e^2$
$\mathfrak{r}'_{4,\lambda,-\lambda/2}$ $(\lambda>0)$	$[e_1,e_2]=\lambda e_2$, $[e_1,e_3]=-\frac{\lambda}{2}e_3-e_4$, $[e_1,e_4]=e_3-\frac{\lambda}{2}e_4$	$\omega=e^1\wedge e^2+e^3\wedge e^4$	$\theta=\lambda e^1$

Only $\mathbb{R}^4$, $\mathfrak{h}_3\times\mathbb{R}$ and $\mathfrak{r}'_{4,\lambda,-\lambda/2}$ admit an LCK structure.

Idea of the proof:

In order to do this we consider different cases according to the eigenvalues of the matrix $B\in\mathfrak{sl}(2,\mathbb{R})$ and the matrix M. And we use the clasification of 4-dimensional solvable Lie algebras in [ABDO] and the following Lemma

Lemma

Two Lie algebras $\mathfrak{g}_A=\mathbb{R}\ltimes_A\mathbb{R}^n$ and $\mathfrak{g}_B=\mathbb{R}\ltimes_B\mathbb{R}^n$ are isomorphic if and only if there exists $c\neq 0$ such that A and cB are conjugate.

Lattices

Theorem (Milnor)

If G admits a lattice then it is unimodular, that is, $\operatorname{tr}\operatorname{ad}_X = 0$ for all $X \in \mathfrak{g}$.

For nilpotent Lie groups:

Theorem (Malcev)

Let G be a simple connected nilpotent Lie group, then there exists a lattice on G if and only if the Lie algebra $\mathfrak{g}$ admits a basis such that the structure constants in this basis are rational.

For almost abelian Lie groups:

Proposition (Bock)

Let $G = \mathbb{R} \ltimes_\phi \mathbb{R}^{2n+1}$ be an almost abelian Lie group. Then G admits a lattice if and only if there exists a $t_0 \neq 0$ such that $\phi(t_0)$ can be conjugated to an integer matrix.

Lattices in dimension 4:

Theorem

Let G be a simply connected 4-dimensional unimodular almost abelian Lie group with a left-invariant LCS structure, and let $\mathfrak{g}$ denote its Lie algebra. If G admits lattices then $\mathfrak{g}$ is isomorphic to one of the following Lie algebras:

* $\mathbb{R}^4$
* $\mathfrak{h}_3 \times \mathbb{R}$
* $\mathfrak{n}_4$
* $\mathfrak{r}_{3,-1} \times \mathbb{R}$
* $\mathfrak{r}_{4,\lambda,-(1+\lambda)}$ *for countably many values of* $\lambda > 1$
* $\mathfrak{r}'_{3,0} \times \mathbb{R}$
* $\mathfrak{r}'_{4,\lambda,-\lambda/2}$ *for countably many values of* $\lambda > 0$.

Lattices in dim ≥ 6:

Recall that for a unimodular almost abelian Lie group $G = \mathbb{R} \ltimes_{\phi} \mathbb{R}^{2n+1}$, equipped with a left-invariant LCS structure, the action is given by $\phi(t) = e^{t\,\mathrm{ad}_{f_1}}$, where $\mathbb{R}$ is generated by f_1.

$$\phi(t) = e^{t\,\mathrm{ad}_{f_1}|_{\mathbb{R}^{2n+1}}} = \left(\begin{array}{c|c} e^{t\mu} & \\ \hline & e^{-\frac{t\mu}{2n}} e^{tB} \end{array}\right),$$

for some $B \in \mathfrak{sp}(n, \mathbb{R})$. We may assume $w = 0$.

LCK case: G admits no lattices. This is a consequence of:

Lemma

Let $p(x) = x^{2n+1} - m_{2n}x^{2n} + m_{2n-1}x^{2n-1} + \cdots + m_1 x - 1$ be a polynomial with $m_j \in \mathbb{Z}$ and $n > 1$. Let $x_0, \ldots, x_{2n}$ be the roots of p, where $x_0 \in \mathbb{R}$ is a simple root. If $|x_1| = \cdots = |x_{2n}|$, then $x_0 = 1$.

LCS case: Let $\mathfrak{g}$ be an almost abelian Lie algebra given by $\mathfrak{g} = \mathbb{R}f_1 \ltimes_M \mathbb{R}^{2n+1}$ with

$$M = \left(\begin{array}{c|ccccccc} 1 & & & & & & & \\ \hline & 0 & & & & & & \\ & & -\frac{1}{n} & & & & & \\ & & & \frac{1}{n} & & & & \\ & & & & -\frac{2}{n} & & & \\ & & & & & \frac{2}{n} & & \\ & & & & & & \ddots & \\ & & & & & & & -1 \end{array}\right).$$

$\mathfrak{g}$ admits an LCS form.

$G = \mathbb{R} \ltimes_\phi \mathbb{R}^{2n+1}$

$$\phi(t) = e^{tM} = \left(\begin{array}{c|ccccccc} e^t & & & & & & & \\ \hline & 1 & & & & & & \\ & & e^{-\frac{t}{n}} & & & & & \\ & & & e^{\frac{t}{n}} & & & & \\ & & & & e^{-\frac{2t}{n}} & & & \\ & & & & & e^{\frac{2t}{n}} & & \\ & & & & & & \ddots & \\ & & & & & & & e^{-t} \end{array}\right).$$

Given $m \in \mathbb{Z}, m > 2$ we take

$$t_m = n \operatorname{arccosh}(\frac{m}{2}) > 0$$

then $\phi(t_m)$ is conjugated to an integer matrix.

The lattice is

$$\Gamma_m := t_m\mathbb{Z} \ltimes P^{-1}\mathbb{Z}^{2n+1},$$

where P satisfies that $P\phi(t_m)P^{-1}$ is an integer matrix.

Therefore $\Gamma_m\backslash G$ is a solvmanifold with an LCS structure.

Two questions:

1. Are the solvmanifolds $\Gamma_{m_1}\backslash G$ and $\Gamma_{m_2}\backslash G$ diffeomorphic?
2. Are they the only examples in dimension 6?

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Thank you for you attention!!

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