Killing fields of constant length on homogeneous Riemannian manifolds

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Introduction

Killing vector fields (simply Killing fields for short) on a Riemannian manifold (M, g) generate the full connected isometry group of (M, g).
Killing fields of constant length are a natural generalization of vector fields, generating vector translations in Euclidean spaces.

We are going to discuss some special problems concerning
Killing vector fields of constant length on homogeneous Riemannian manifolds.
This talk is based on very recent paper

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Recall that a vector field X on a Riemannian manifold (M, g) is called Killing if $L_X g = 0$. Every such vector field could be characterized by the property that it generate 1-dimensional isometry groups of (M, g).
In this talk, we discuss Killing vector fields of constant length on homogeneous Riemannian manifolds. A detailed study of Killing vector fields of constant length was started in the papers [1, 2, 3], although many of the results in this direction have long been known (see a detailed exposition in [1]). It should be noted that there exists a connection between Killing vector fields of constant length and Clifford-Wolf translations in a Riemannian manifold (M, g).

Recall that a Clifford-Wolf translation in (M, g) is an isometry s moving all points in M one and the same distance, i. e. $\rho_g\big(x, s(x)\big) \equiv \mathrm{const}$ for all $x \in M$, where ρ_g means the inner (length) metric generated by the Riemannian metric tensor g on M. Clifford-Wolf translations naturally appear in the investigation of homogeneous Riemannian coverings of homogeneous Riemannian manifolds [15]. Clifford-Wolf translations are studied in various papers, we refer to [4] and [17] for a detailed discussion .

If a 1-parameter isometry group $\gamma(t)$ on (M, g), generated by a Killing vector field Z, consists of Clifford-Wolf translations, then Z obviously has constant length.

This assertion can be partially inverted:

If a Riemannian manifold (M, g) has the injectivity radius, bounded from below by some positive constant (in particularly, this condition is satisfied for every compact or homogeneous manifold), and Z is a Killing vector field of constant length on (M, g), then the isometries $\gamma(t)$ from the 1-parameter isometry group, generated by the vector field Z, are Clifford-Wolf translations at least for sufficiently small $|t|$ [2].

A metric space (M, ρ) is Clifford-Wolf homogeneous if for any points $x, y \in M$ there exists an isometry f, Clifford-Wolf translation, of the space (M, ρ) onto itself such that $f(x) = y$.
A connected Riemannian manifold (M, g) is Clifford-Wolf homogeneous if it is Clifford-Wolf homogeneous relative to its inner metric ρ_g.

In addition, it is G-Clifford-Wolf homogeneous if one can take isometries f from the Lie (sub)group G of isometries of (M, g) in the above definition of the Clifford-Wolf homogeneity.

Clifford-Wolf homogeneous simply connected Riemannian manifolds are classified in [4]:

A simply connected Riemannian manifold is Clifford-Wolf homogeneous if and only if it is a direct metric product of an Euclidean space, odd-dimensional spheres of constant curvature and simply connected compact simple Lie groups supplied with bi-invariant Riemannian metrics [4].

Note that every geodesic γ in a Clifford-Wolf homogeneous Riemannian manifold (M, g) is an integral curve of a Killing vector field of constant length on (M, g) [4].

In a recent paper [17], Ming Xu and Joseph A. Wolf obtained the classification of normal Riemannian homogeneous space G/H with nontrivial Killing vector field of constant length, where G is compact and simple. Every of these spaces with $\dim(G) > \dim(H) > 0$ is locally symmetric and its universal Riemannian cover is either an odd-dimensional sphere of constant curvature, or a Riemannian symmetric space $SU(2n)/Sp(n)$.

This result is very important in the context of the study of general Riemannian homogeneous manifolds with nonzero Killing fields of constant length. In a very recent paper [16] by the same authors and Fabio Podestà, this result was extended to the class of pseudo-Riemannian normal homogeneous spaces.

In this talk, we present some structural results on the Lie algebras of transitive isometry groups of a general compact homogenous Riemannian manifold with nontrivial Killing vector fields of constant length, see [13].

Examples

Let us consider some examples of Killing fields of constant length.

It is well known that the group of left translations, as well as the group of right translation, of a compact Lie group G, supplied with a bi-invariant Riemannian metric μ, consists of Clifford-Wolf translations. Therefore, Killing fields that generate these groups have constant length on (G, μ). Of course, every direct metric product of (G, μ) with any Riemannian manifolds also has Killing fields of constant length.

Now, let us consider more interesting examples.

Let F be a connected compact simple Lie group, $k \in \mathbb{N}$. Consider a so-called Ledger – Obata space $F^k / \operatorname{diag}(F)$ (note that for $k = 2$ we get irreducible symmetric spaces). We supply it with any invariant Riemannian metric g. The structure of invariant Riemannian metrics on $F^k / \operatorname{diag}(F)$ is quite complicated for $k \geq 3$. It should be noted that for any compact Lie group F, a Ledger – Obata space $F^k / \operatorname{diag}(F)$ is diffeomorphic to the Lie group F^{k-1}.

It is easy to see that every copy F in F^k consists of Clifford-Wolf translations on $(F^k / \operatorname{diag}(F), g)$ as well as every copy of the Lie algebra $\mathfrak{f}$ in $\mathfrak{f} \oplus \mathfrak{f} \oplus \cdots \oplus \mathfrak{f} = k \cdot \mathfrak{f}$ consists of Killing fields of constant length for any invariant Riemannian metric g. For example, we may consider g induced with the Killing form of $k \cdot \mathfrak{f}$. For such a choice $(F^k / \operatorname{diag}(F), g)$ is (locally) indecomposable.

This example could be easily generalized for more complicated cases.

Now we consider another type of examples.

We say that a Lie algebra $\mathfrak{g}$ of Killing vector fields is transitive on a Riemannian manifold (M, g) if $\mathfrak{g}$ generates the tangent space to M at every point $x \in M$, or, equivalently, the connected isometry group G with the Lie algebra $\mathfrak{g}$ acts transitively on (M, g).

The following simple observation gives many examples of Killing vector fields of constant length on homogeneous Riemannian manifolds. Suppose that a Lie algebra $\mathfrak{g}$ of Killing vector fields is transitive on a Riemannian manifold (M, g) and the Killing field Z on (M, g) commutes with $\mathfrak{g}$ (in particular, Z is in the center of $\mathfrak{g}$), then Z has a constant length on (M, g).

Indeed, for any $X \in \mathfrak{g}$ we have $X \cdot g(Z, Z) = 2g([X, Z], Z) = 0$. Since $\mathfrak{g}$ is transitive on M, we get $g(Z, Z) = \mathrm{const}$.

Example. Consider the irreducible symmetric space $M = SU(2n)/Sp(n)$, $n \geq 2$. It is known that the subgroup $SU(2n-1) \cdot S^1 \subset SU(2n)$ acts transitively on M, see e.g. [15]. Therefore, the Killing vector Z generated by S^1, is a Killing field of constant length on $M = SU(2n)/Sp(n)$.

Example. Consider the sphere S^{2n-1}, $n \geq 2$, as the symmetric space $S^{2n-1} = SO(2n)/SO(2n-1)$. It is known that the subgroup $U(n) = SU(n) \cdot S^1 \subset SO(2n)$ acts transitively on S^{2n-1}. Therefore, the Killing vector Z generating S^1, is a Killing field of constant length on S^{2n-1}.

Note that the full connected isometry group of the sphere S^{n-1} with the canonical Riemannian metric g_{can} of constant curvature 1, is $SO(n)$, but there are some subgroups G of $SO(n)$ with transitive action on S^{n-1}.

It is interesting that S^{n-1} is G-Clifford-Wolf homogeneous for some of them:
$S^{2n-1} = SO(2n)/SO(2n-1) = U(n)/U(n-1)$ is $SO(2n)$-Clifford-Wolf homogeneous and $U(n)$-Clifford-Wolf homogeneous;
$S^{4n-1} = Sp(n)/Sp(n-1) = Sp(n)\cdot S^1/Sp(n-1)\cdot S^1 = SU(2n)/SU(2n-1)$ is $Sp(n)$-Clifford-Wolf homogeneous, $Sp(n)\cdot S^1$-Clifford-Wolf homogeneous, and $SU(2n)$-Clifford-Wolf homogeneous;
$S^7 = Spin(7)/G_2$ is $Spin(7)$-Clifford-Wolf homogeneous;
$S^{15} = Spin(9)/Spin(7)$ is $Spin(9)$-Clifford-Wolf homogeneous
(see details in [6]).

In fact, every of these results gives an example of Killing vector field of constant length in the Lie algebra $\mathfrak{g}$ corresponded to the group G [6].

New results

Let us consider any Lie group G acting on the Riemannian manifold (M, g) by isometries. The action of a on $x \in M$ will be denoted by $a(x)$. We will identify the Lie algebra $\mathfrak{g}$ of G with the corresponded Lie algebra of Killing vector fields on (M, g) as follows. For any $U \in \mathfrak{g}$ we consider a one-parameter group $\exp(tU) \subset G$ of isometries of (M, g) and define a Killing vector field $\widetilde{U}$ by a usual formula

$$\widetilde{U}(x) = \frac{d}{dt} \exp(tU)(x)\Big|_{t=0}.$$

It is clear that the map $U \to \widetilde{U}$ is linear and injective, but $[\widetilde{U}, \widetilde{V}] = -\widetilde{[U, V]}$.

Let (M, g) be a compact connected Riemannian manifold, G is a transitive isometry group of (M, g). We identify elements of the Lie algebra $\mathfrak{g}$ of G with Killing vector fields on (M, g) as above. Since G is compact, then we have a decomposition

$$\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k,$$

where $\mathfrak{c}$ is the center and $\mathfrak{g}_i$, $i = 1, \dots, k$, are simple ideals in $\mathfrak{g}$.

We are going to state the main results formulated under the above assumptions and notations.

Theorem (Theorem 1)

Let $Z = Z_0 + Z_1 + Z_2 + \cdots + Z_l \in \mathfrak{g}$ *be a Killing vector field of constant length on* (M, g)*, where* $1 \leq l \leq k$, $Z_0 \in \mathfrak{c}$, $Z_i \in \mathfrak{g}_i$ *and* $Z_i \neq 0$ *for* $1 \leq i \leq l$*. Then the following statements hold:*
1) *For every* $i \neq j$, $1 \leq i, j \leq l$*, we have* $g(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ *at every point of* M*. In particular,* $g(Z_i, \mathfrak{g}_j) = 0$ *and* $g(Z_i, Z_j) = 0$.
2) *Every Killing field of the type* $Z_0 + Z_i$, $1 \leq i \leq l$*, has constant length. Conversely, if for every* i, $1 \leq i \leq l$*, the Killing field* $Z_0 + Z_i$ *has constant length, and for every* $i \neq j$, $1 \leq i, j \leq l$*, the equality* $g(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ *holds, then the Killing field* $Z = Z_0 + Z_1 + Z_2 + \cdots + Z_l$ *has constant length on* (M, g).

Corollary (Corollary 1)

If (*under the assumptions of theorem 1*) *we have* $\mathfrak{c} = 0$ *and* $k = l$*, then every* $\mathfrak{g}_i$, $1 \leq i \leq k$*, is a parallel distribution on* (M, g)*. Moreover, if* (M, g) *is simply connected, then it is a direct metric product of* k *Riemannian manifolds.*

Theorem 1 allows to restrict our attention on Killing vector fields of constant length of the following special type: $Z = Z_0 + Z_i$, where Z_0 is in the center $\mathfrak{c}$ of $\mathfrak{g}$ and Z_i is in the simple ideal $\mathfrak{g}_i$ in $\mathfrak{g}$. Without loss of generality we will assume that $i = 1$.

Theorem (Theorem 2)

Let $Z = Z_0 + Z_1 \in \mathfrak{g}$ be a Killing vector field of constant length on (M, g), where $Z_0 \in \mathfrak{c}$, $Z_1 \in \mathfrak{g}_1$ and $Z_1 \neq 0$, and let $\mathfrak{k}$ be the centralizer of Z (and Z_1) in $\mathfrak{g}_1$. Then either any $X \in \mathfrak{g}_1$ is a Killing field of constant length on (M, g), or the pair $(\mathfrak{g}_1, \mathfrak{k})$ is one of the following irreducible Hermitian symmetric pair:
1) $(su(p+q), su(p) \oplus su(q) \oplus \mathbb{R})$, $p \geq q \geq 1$;
2) $(so(2n), su(n) \oplus \mathbb{R})$, $n \geq 5$;
3) $(so(p+2), so(p) \oplus \mathbb{R})$, $p \geq 5$;
4) $(sp(n), su(n) \oplus \mathbb{R})$, $n \geq 2$;
In the latter four cases the center of $\mathfrak{k}$ is a one-dimensional Lie algebra spanned by the vector Z_1.

If a Killing field of constant length $Z = Z_0 + Z_1$ satisfies one of the cases 1)-4) in Theorem 2, we will say that it has Hermitian type. Recall that an element $U \in \mathfrak{g}$ is regular in $\mathfrak{g}$, if its centralizer has minimal dimension among all the elements of $\mathfrak{g}$. For Z of Hermitian type, Z_1 is not a regular element in $\mathfrak{g}_1$, since $\mathfrak{k}$ is not commutative in the cases 1)-4) of Theorem 2. Hence, we get

Corollary (Corollary 2)

If Z_1 is a regular element in the Lie algebra $\mathfrak{g}_1$ (under the assumptions of Theorem 2), then every $X \in \mathfrak{g}_1$ is a Killing vector field of constant length on (M, g).

Moreover, the following result holds.

Theorem (Theorem 3)

If $Z = Z_0 + Z_1 + \cdots + Z_k$ *is a regular element of* $\mathfrak{g}$ *and has constant length on* (M, g)*, then the following assertions hold:*
1) $g(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ *for every* $i \neq j$*,* $i, j = 1, \dots, k$*, at every point of* M*;*
2) $g(Z_0, \mathfrak{g}_i) = 0$ *for every* $i = 1, \dots, k$*, at every point of* M*;*
3) *every* $X \in \mathfrak{g}_{\mathfrak{s}}$*, where* $\mathfrak{g}_{\mathfrak{s}} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ *is a semisimple part of* $\mathfrak{g}$*, has constant length on* (M, g)*.*
Moreover, if M *is simply connected, then* $Z_0 = 0$ *and* (M, g) *is a direct metric product of* (G_i, μ_i)*,* $1 \leq i \leq k$*, where* G_i *is a connected and simply connected compact simple Lie group with the Lie algebra* $\mathfrak{g}_i$ *and* μ_i *is a bi-invariant Riemannian metric on* G_i*.*

The above results can be used for the classification of homogeneous Riemannian manifolds with Killing vector fields of some special kinds. We state some unsolved questions and problems in this direction.

Problem (Problem 1)

Classify homogeneous Riemanninan spaces $(G/H, g)$ with nontrivial Killing vector fields of constant length, where G is simple.

Recall, that normal homogeneous Riemanninan spaces $(G/H, g)$ with Killing fields of constant length, where G is simple, are classified in [17]. On the other hand, the set of G-invariant metrics on a space G/H could have any dimension even for simple G.

Question (Question 1)

Let (M,g) be a Riemannian homogeneous manifold, G be its full connected isometry group, and $Z \in \mathfrak{g}$ be a Killing vector field of constant length on (M,g). Does $\mathfrak{k}$, the centralizer of Z in $\mathfrak{g}$, act transitively on M?

Note, that for symmetric spaces $(M = G/H, g)$ we have an affirmative answer to this question (see Lemma 3 in [2]). Of course, Question 1 is interesting even under an additional assumption that the group G is simple.

There is a natural generalization of normal homogeneous spaces. Recall, that a Riemannian manifold $(M = G/H, g)$, where H is a compact subgroup of a Lie group G and g is a G-invariant Riemannian metric, is called a geodesic orbit space if any geodesic γ of M is an orbit of 1-parameter subgroup of the group G, detailed information on this class of homogeneous Riemannian manifolds one can find e. g. in [5]. It is known that all normal homogeneous spaces are geodesic orbit. The following problem is natural.

Problem (Problem 2)

Classify geodesic orbit Riemannian spaces with nontrivial Killing vector fields of constant length.

Recall the following result of [12]: If $(M = G/H, g)$ is a geodesic orbit Riemannian space, $\mathfrak{g}$ is its Lie algebra of Killing fields, and $\mathfrak{a}$ is an abelian ideal in $\mathfrak{g}$, then every $X \in \mathfrak{a}$ has constant length on (M, g).

Hence it suffices to study only homogeneous spaces G/H with semisimple G in Problem 2.

Note that there exist examples that cover all possibilities in Theorem 2, excepting the case $3)$, where we have examples only for specific values of p.

Question (Question 2)

Is there a Riemannian homogeneous space $(M = G/H, g)$ with a Killing vector field of constant length $Z \in \mathfrak{g}$, corresponded to the case $3)$ in Theorem 2, i.e. $(\mathfrak{g}_1, \mathfrak{k}) = (so(p+2), so(p) \oplus \mathbb{R})$, for $p = 6$ and $p \geq 8$?

Note, that the suitable examples for $p = 5$ and $p = 7$ are known, see [13]. Note also that there are serious restrictions on a Riemannian manifold $(M = G/H, g)$ as in Question 2, see details in [13].

Thank you for your time and attention!

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