Symplectic and hyperkahler implosion

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SYMPLECTIC MANIFOLDS

Closed nondegenerate 2-form ω on M^{2n}

Local model : $\sum dp_i \wedge dq_i$ on $\mathbb{R}^{2n}$

EXAMPLES

$\mathbb{C}^n$, cotangent bundles T^*N

coadjoint orbit of G (flag manifold)

Kähler manifolds:

metric g, complex structure $I : TM \to TM$

$$\omega(X,Y) = g(IX,Y).$$

HYPERKAHLER MANIFOLDS

Metric g, complex structures I, J, K with quaternionic relations

$$IJ = K = -JI \quad \text{etc}$$

Now three symplectic forms

$$\omega_1(X,Y) = g(IX,Y) \ : \ \omega_2(X,Y) = g(JX,Y)$$

$$\omega_3(X,Y) = g(KX,Y)$$

Riemannian data ; no local model

Examples?

MOMENT MAPS AND SYMPLECTIC REDUCTION

(M^{2n}, ω) with S^1 action: Killing field X

$$1 - \text{form} \quad \omega(X, .) = \iota_X \omega$$

$$0 = L_X \omega = d\iota_X \omega + \iota_X d\omega = d\iota_X \omega.$$

In good cases:

$$i_X \omega = d\mu$$

where

$$\mu : M \to \mathbb{R}$$

is S^1-invariant.

μ is the *moment map*

Now $M//_\epsilon S^1 = \mu^{-1}(\epsilon)/S^1$ is a symplectic manifold of dimension $\dim M - 2$.

Example. Flat $\mathbb{C}$ with standard action of S^1

$$z \mapsto e^{i\theta} z.$$

Moment map is

$$\phi : z \mapsto |z|^2$$

Symplectic reduction

$M//_\epsilon S^1 = \mu^{-1}(\epsilon)/S^1$

(where $\epsilon > 0$) gives $\mathbb{CP}^{n-1}$.

For action of general G, μ takes values in $Lie(G)^*$ and is G-equivariant. Take ϵ in centre; dim of symplectic quotient $M//_\epsilon G$ is $\dim M - 2\dim G$.

HYPERKAHLER QUOTIENTS

Action of G on M hyperkähler. Moment map

$$\mu = (\mu_1, \mu_2, \mu_3) : M \to Lie(G)^* \otimes \mathbb{R}^3$$

Hyperkähler quotient is

$$M///_\epsilon G = \mu^{-1}(\epsilon_1, \epsilon_2, \epsilon_3)/G$$

of dimension $\dim M - 4 \dim G$

Example. Flat $\mathbb{H}$ with standard action of S^1

$$(z, w) \mapsto (e^{i\theta} z, e^{-i\theta} w)$$

Moment map is

$$\phi : (z, w) \mapsto (\frac{1}{2}(|z|^2 - |w|^2), \mathrm{Re}\ zw, \mathrm{Im}\ zw).$$

Singular ϵ-locus typically codim 3 in target; no wall-crossing.

Other examples use:

infinite-dimensional spaces (connexions, Higgs fields), and

infinite dimensional groups (gauge groups).

Moment maps are (reductions of) Self-Dual Yang-Mills equations.

Can give finite-dimensional moduli spaces with hyperkahler structure

e.g. moduli spaces for

instantons, monopoles, Nahm equations, Higgs pairs

Examples include

T^*G_C

complex coadjoint orbits

(cf. symplectic case)

Implosion

Implosion of M is "abelianisation"

In symplectic case this means

$M//_\lambda G = M_{\mathrm{impl}}//_\lambda T$

(T max torus in G)

Universal example: $M = T^*G$ so want

$M_{\mathrm{impl}}//_\lambda T = \mathcal{O}_\lambda$ coadjoint orbit

Example. $G = SU(2)$, $\mathcal{O}_\lambda = S^2$ or $*$, so

$M_{\mathrm{impl}} = \mathbb{C}^2$.

In general: take $G \times \bar{\mathfrak{t}}^*_+$ and collapse by commutator of stabiliser of $t \in \bar{\mathfrak{t}}^*_+$.

eg for $SU(2)$ take $SU(2) \times [0, \infty)$ and collapse by $SU(2)$ at origin, to obtain $\mathbb{C}^2$.

For higher rank groups we obtain a stratified space, not smooth in general.

Top stratum is $G \times \mathfrak{t}^*_+$, where no collapsing occurs as stabiliser is just T.

Algebro-geometric description

$G_{\mathbb{C}}//N$: N maximal unipotent

eg for $SU(2)$ we have $SL(2,\mathbb{C})//N$

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} & x_{12}+nx_{11} \\ x_{21} & x_{22}+nx_{21} \end{pmatrix}$$

Invariants x_{11}, x_{21}, so $SL(2,\mathbb{C})//N = \mathbb{C}^2$.

Strata are $G_{\mathbb{C}}/[P,P]$ where P are parabolics

So top stratum is $G_{\mathbb{C}}/N = GA = G \times \mathfrak{t}^*_+$ via Iwasawa decomposition

Hyperkahler version?

$M = T^*G_{\mathbb{C}}$. Want

$M_{\mathsf{impl}}///_\lambda T \sim$ complex coadjoint orbit

Consider complex-symplectic quotient by N, in GIT sense ; this is

$(G_{\mathbb{C}} \times \mathfrak{b})//N$

Exists in general by results of Ginzburg-Riche

For $SU(n)$, we can describe via quiver varieties, as hyperkahler quotient

Affine completion of 'top stratum' $G_{\mathbb{C}} \times_N \mathfrak{b}$

In general torus reductions will give :

Kostant varieties (level sets of the collection of invariant polynomials)

eg regular semisimple orbits

in general, closure of a regular orbit. Union of orbits. Semisimple orbit is lowest stratum

eg nilpotent variety is reduction at level zero.

Complex coadjoint orbits may also be described as moduli spaces of solutions to Nahm's equations on the half-line (Kronheimer, Biquard, Kovalev)

This gives an alternative approach to implosion using Nahm data