GRADIENT AND DIVERGENCE ON SYMPLECTIC MANIFOLDS

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Introduction

Let (M, ω) be a symplectic manifold of dimension 2n and let x be a point of this manifold. Denote by $T_x M$ and $T_x^* M$ the tangent space and the cotangent space at x, respectively.
The symplectic form can be transmited to the bilinear form acting on covectors, in the following way

$$\omega(\eta, \xi) = \omega(\eta^\sharp, \xi^\sharp),$$

where $\eta, \xi \in T_x^* M$ and for any one form η, $\eta^\sharp$ is defined by

$$\eta(v) = \omega(\eta^\sharp, v), \quad v \in T_x M.$$

This form induces bilinear pairings on each $\Lambda^p T_x^* M$, $p = 1, \dots 2n$, also denoted by ω and defined by

$$\omega(\eta_1 \wedge \cdots \wedge \eta_p, \xi_\wedge \cdots \wedge \xi_p) = det(\omega(\eta_i, \xi_j)_{i,j=1,\dots,p}),$$

where $\eta_i, \xi_j \in T_x^* M$ for any $i, j = 1, \dots, p$.

Introduction

Consider two spaces:

$\Lambda^p = C^\infty(\Lambda^p T^*M)$ - the space of scalar-valued p-forms,

$\vec{\Lambda}^p = C^\infty(\Lambda^p T^*M \otimes TM)$ - the space of vector-valued p-forms.

Now, the symplectic form can be extended to the space of vector-valued forms by the formula

$$\omega(\varphi \otimes X, \psi \otimes Y) = \omega(\varphi, \psi)\omega(X, Y),$$

for any $\varphi \otimes X, \psi \otimes Y \in \vec{\Lambda}^p$. Additionally,

(i) $\omega(\varphi \otimes X, Y) = \omega(X, Y) \cdot \varphi,$ *for any* $\varphi \otimes X \in \vec{\Lambda}^p$ *and* $Y \in \Gamma(TM)$,

(ii) $\omega(\eta \otimes \varphi, \xi \otimes \psi) = \omega(\eta, \xi) \cdot \omega(\varphi, \psi),$ *for any* $\eta \otimes \varphi, \xi \otimes \psi \in C^\infty(T^*M \otimes \Lambda^p T^*M)$.

Define three exterior products (all denoted by the same symbol):

$\wedge : \Lambda^p \times \Lambda^q \to \Lambda^{p+q} \qquad \varphi \wedge \psi;$

$\wedge : \Lambda^p \times \vec{\Lambda}^q \to \vec{\Lambda}^{p+q} \qquad \varphi \wedge (\psi \otimes X) = (\varphi \wedge \psi) \otimes X;$

$\wedge : \vec{\Lambda}^p \times \vec{\Lambda}^q \to \Lambda^{p+q} \qquad (\varphi \otimes X) \wedge (\psi \otimes Y) = (\varphi \wedge \psi) \cdot \omega(X, Y).$

Differential operators on symplectic manifolds

Definition 1

A symplectic covariant derivative on (M, ω) is a smooth linear covariant derivative ∇ such that:

$$T^{\nabla} = 0, \quad \nabla\omega = 0.$$

Take ∇ any symplectic covariant derivative. This operator can be extended to the tensor algebra and the next to the space of vector-valued p-forms for any vector field Y according to the formula

$$\nabla_Y : \vec{\Lambda}^p \ni \varphi \otimes X \longmapsto \nabla_Y \varphi \otimes X + \varphi \otimes \nabla_Y X \in \vec{\Lambda}^p.$$

Differential operators on symplectic manifolds

Let $d : \Lambda^p \to \Lambda^{p+1}$ be the operator of exterior derivation. It can be extended to the space of vector-valued forms according to the formula

$$d : \vec{\Lambda}^p \ni \varphi \otimes X \longmapsto d\varphi \otimes X + (-1)^p \varphi \wedge \nabla X \in \vec{\Lambda}^{p+1}.$$

Proposition 1

(1) *For any* $\varphi \in \Lambda^p$ *and* $\psi \in \Lambda^q$, $d(\varphi \wedge \psi) = d\varphi \wedge \psi + (-1)^p \varphi \wedge d\psi$.

(2) *For any* $\varphi \in \Lambda^p$ *and* $\Psi \in \vec{\Lambda}^q$, $d(\varphi \wedge \Psi) = d\varphi \wedge \Psi + (-1)^p \varphi \wedge d\Psi$.

(3) *For any* $\Phi \in \vec{\Lambda}^p$ *and* $\Psi \in \vec{\Lambda}^q$, $d(\Phi \wedge \Psi) = d\Phi \wedge \Psi + (-1)^p \Phi \wedge d\Psi$.

Differential operators on symplectic manifolds

Let $e_1, \ldots, e_{2n}$ be a local symplectic base on M and let $e^1, \ldots, e^{2n}$ be a dual base to $e_1, \ldots, e_{2n}$. Note, that M is oriented, so

$$\Omega = \frac{1}{n!}\omega^n = (-1)^{\frac{n(n-1)}{2}} e^1 \wedge e^2 \wedge \cdots \wedge e^{2n}.$$

Definition 2

The linear operator $* : \Lambda^p \to \Lambda^{2n-p}$ *defined by*

$$\varphi \wedge *\psi = \omega(\varphi, \psi)\Omega, \qquad \textit{for all } \varphi, \psi \in \Lambda^p$$

is called the symplectic Hodge star operator.

The symplectic Hodge star operator can be extended to the space of vector-valued forms according to the formula

$$* : \vec{\Lambda}^p \ni \varphi \otimes X \longmapsto (*\varphi) \otimes X \in \vec{\Lambda}^{2n-p}.$$

Definition 3

The linear operator $tr : \Lambda^p \to \Lambda^{p-2}$ defined by

$$tr\varphi = \sum_{k=1}^{n} \imath_{e_{n+k}} \imath_{e_k} \varphi \qquad \text{for all } \varphi \in \Lambda^p$$

is called the symplectic trace operator, where $e_1, \ldots, e_{2n}$ is the local symplectic base on M.

Moreover, we also define the symplectic trace operator acting on the space of vector-valued forms:

$$Tr : \vec{\Lambda}^p \ni \varphi \otimes X \longmapsto \imath_X \varphi \in \Lambda^{p-1}.$$

Gradient and divergence in the Rummler sense

Definition 4

Define the linear operator $j : \Lambda^p \to \vec{\Lambda}^{p-1}$ *by the formula*

$$\omega(j\varphi, X) = \imath_X \varphi,$$

for any $\varphi \in \Lambda^p, X \in \Gamma(TM)$.

Proposition 2

Let $e_1, ..., e_{2n}$ *be a local symplectic base of* M *on a neighborhood* U. *For any* $\varphi \in \Lambda^p$, *we have*

$$j\varphi = \sum_{k=1}^{n} \imath_{e_{n+k}} \varphi \otimes e_k - \imath_{e_k} \varphi \otimes e_{n+k} \quad \text{in} \quad U.$$

Proposition 3

For any $\varphi \in \Lambda^p$ *and* $\psi \in \Lambda^q$, $j(\varphi \wedge \psi) = j\varphi \wedge \psi + (-1)^p \varphi \wedge j\psi$.

Gradient and divergence in the Rummler sense

Definition 5

Define the linear operator $\alpha : \vec{\Lambda}^p \to \Lambda^{p+1}$ *by the formula*

$$\alpha(\varphi \otimes X) = X^\flat \wedge \varphi,$$

for any $\varphi \otimes X \in \vec{\Lambda}^p$, *where one form* $X^\flat$ *is defined by*

$$X^\flat(Y) = \omega(X, Y), \text{ for any vector field } Y.$$

Proposition 4

For any $\varphi \in \Lambda^p$ *and* $\Psi \in \vec{\Lambda}^p$, *we have* $\omega(j\varphi, \Psi) = \omega(\varphi, \alpha\Psi)$.

Moreover, we have

Proposition 5

For any $\varphi \in \Lambda^p$, $\alpha j \varphi = p\varphi$.

Gradient and divergence in the Rummler sense

The definitions of symplectic trace operators imply the following

Proposition 6

For any $\varphi \in \Lambda^p$,

$$Trj\varphi = -2tr\varphi.$$

Moreover, the symplectic trace operator acting on vector exterior forms satisfies

Proposition 7

For any $\Phi \in \vec{\Lambda}^p$,

$$Tr\Phi = (-1)^p * \alpha * \Phi.$$

Gradient and divergence in the Rummler sense

Now, two first order linear operators grad and div will be introduced. The first one acts on scalar exterior forms.

Definition 6

The differential operator grad $: \Lambda^p \to \vec{\Lambda}^p$ *defined by*

$$grad = jd + dj$$

is called the gradient.

Remark 1

For any $f \in \Lambda^0$ *and* $X \in \vec{\Lambda}^0$, $\omega(gradf, X) = X(f)$.

Gradient and divergence in the Rummler sense

Definition 7

The differential operator $div : \vec{\Lambda}^p \to \Lambda^p$ *defined by*

$$div = Trd + dTr$$

is called the divergence.

Proposition 8

For any $\Phi \in \vec{\Lambda}^p$, *we have*

$$div(\Phi) = Tr\, \nabla(\Phi).$$

In particular, $div(X) = Tr(\nabla X)$.

Proposition 9

For any $\varphi \otimes X \in \vec{\Lambda}^p$, *we have* $div(\varphi \otimes X) = \nabla_X \varphi + \varphi \cdot div(X)$.

Gradient and divergence in the Rummler sense

Theorem 1

(1) *For any* $\varphi \in \Lambda^p$ *and* $\psi \in \Lambda^q$,

$$grad(\varphi \wedge \psi) = grad\varphi \wedge \psi + \varphi \wedge grad\psi.$$

(2) *For any* $\varphi \in \Lambda^p$ *and* $\Psi \in \vec{\Lambda}^q$,

$$div(\varphi \wedge \Psi) = grad\varphi \wedge \Psi + \varphi \wedge div\Psi.$$

(3) *For any* $\varphi \in \Lambda^p$ *and* $\Psi \in \vec{\Lambda}^q$,

$$Tr(\varphi \wedge \Psi) = j\varphi \wedge \Psi + (-1)^p \varphi \wedge Tr\Psi.$$

Gradient and divergence in the Rummler sense

Theorem 2

Let $e_1, ..., e_{2n}$ be a local symplectic base of M on a neighborhood U. Let $e^1, ..., e^{2n}$ be its dual base. For any $\varphi \in \Lambda^p$, we have

$$grad\ \varphi = \sum_{k=1}^{n} \nabla_{e_{n+k}}\varphi \otimes e_k - \nabla_{e_k}\varphi \otimes e_{n+k} \text{ in } U.$$

Proposition 10

For any vector field X and for any $\varphi \in \Lambda^p$, we have

$$grad\varphi \wedge X = \nabla_X \varphi.$$

Proposition 11

(1) *For any $f \in \Lambda^0$, $grad\ df = d\ grad f$.*

(2) *For any $\varphi \in \Lambda^p$, $grad\ d\varphi = d\ grad\varphi + d^2 j\varphi$.*

Gradient and divergence in the Rummler sense

It holds the following relations:

Theorem 3

For any vector field X,

(1) $j\nabla_X = \nabla_X j$.

(2) $*\nabla_X = \nabla_X *$.

Theorem 4

(1) $*\,grad = grad\,*$;

(2) $*\,div = div\,*$.

Formally adjoint operators

For any $\theta \in \Lambda^p$ and $\vartheta \in \Lambda^p$, we define the global product θ and ϑ by

$$(\theta, \vartheta) = \int_M \theta \wedge *\vartheta,$$

or equivalently

$$(\theta, \vartheta) = \int_M \omega(\theta, \vartheta) \cdot \Omega,$$

if the integral on the right-hand side exists.
It is defined by analogy for vector exterior forms.

Definition 8

The operator L is called formally adjoint to $L^\dagger$, if the following condition is satisfied

$$(L\theta, \vartheta) = (\theta, L^\dagger \vartheta),$$

if only θ or ϑ has the compact support.

Formally adjoint operators

Definition 9

Define the differential operator $\delta : \Lambda^p \to \Lambda^{p-1}$ *by*

$$\delta\varphi = (-1)^p(*d*)\varphi, \quad \textit{for all } \varphi \in \Lambda^p.$$

Between, the operators grad, α and classical operators *, d and δ, there hold the following relations:

Proposition 12

(1) $d = \alpha\, grad$.

(2) $\delta = Tr\, grad$.

Formally adjoint operators

It turns out, that the following operators are formally adjoint:

Theorem 5

δ is formally adjoint to d, i.e.

$$(d\varphi, \psi) = (\varphi, \delta\psi),$$

for any $\varphi \in \Lambda^{p-1}$ and $\psi \in \Lambda^p$, if only φ or ψ has a compact support.

Theorem 6

-div is formally adjoint to grad, i.e.

$$(grad\varphi, \Psi) = (\varphi, -div\Psi),$$

for any $\varphi \in \Lambda^p$ and $\Psi \in \vec{\Lambda}^p$, if only φ or Ψ has a compact support.

Formally adjoint operators

Definition 10

Define the differential operator $\nabla^* : C^\infty(T^*M \otimes \Lambda^p T^*M) \to \Lambda^p$ *by*

$$
\begin{aligned}
(\diamondsuit)\ \ \nabla^*(\Xi) &= \sum_{k=1}^{n} \nabla_{e_{n+k}} \imath_{e_k} \Xi - \nabla_{e_k} \imath_{e_{n+k}} \Xi \\
&- div(e_k) \cdot \imath_{e_{n+k}} \Xi + div(e_{n+k}) \cdot \imath_{e_k} \Xi,
\end{aligned}
$$

for any $\Xi \in C^\infty(T^*M \otimes \Lambda^p T^*M)$ *and for any local symplectic base* $e_1, \ldots, e_{2n}$ *on M.*

Formally adjoint operators

Proposition 13

For any $\xi \otimes \varphi \in C^\infty(T^*M \otimes \Lambda^p T^*M)$,

$$\nabla^*(\xi \otimes \varphi) = -\nabla_{\xi^\sharp}\varphi - \varphi \cdot div\xi^\sharp.$$

Proposition 14

For any covariant derivative on M such, that $\nabla\omega = 0$ *(not necessarily* $T = 0$*) and for any compactly supported vector field X on M*

$$\int_M (divX + \omega(X, \tau))\Omega = 0,$$

where for any local symplectic base $e_1, ..., e_{2n}$ *on M,*

$$\tau = \sum_{k=1}^{n} T(e_k, e_{n+k}).$$

Formally adjoint operators

With respect to the global products,

Theorem 7

∇^* is formally adjoint to ∇ i.e.

$$(\nabla\varphi, \Xi) = (\varphi, \nabla^*\Xi),$$

for any $\varphi \in \Lambda^p$ and $\Xi \in C^\infty(T^*M \otimes \Lambda^p T^*M)$, if only φ or Ξ has a compact support.

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