QCH Kähler surfaces and complex foliations on Kähler surfaces

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Dirac operators in differential geometry and global analysis - in memory of Thomas Friedrich

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The aim of the present talk is to describe connected QCH Kähler surfaces (M, g, J).

Definition

The QCH Kähler surface is the Kähler surface with the property: the holomorphic curvature $K(\pi) = R(X, JX, JX, X)$ of any J-invariant 2-plane $\pi \subset T_xM$, where $X \in \pi$ and $g(X, X) = 1$, depends only on the point x and the number $|X_{\mathcal{D}}| = \sqrt{g(X_{\mathcal{D}}, X_{\mathcal{D}})}$, where $X_{\mathcal{D}}$ is an orthogonal projection of X on $\mathcal{D}$ where $\mathcal{D}$ is a J-invariant 2-dimensional distribution.

In this case we have

$$R(X, JX, JX, X) = \phi(x, |X_{\mathcal{D}}|)$$

where $\phi(x, t) = a(x) + b(x)t^2 + c(x)t^4$ and a, b, c are smooth functions on M. Also $R = a\Pi + b\Phi + c\Psi$ for certain curvature tensors $\Pi, \Phi, \Psi \in \bigotimes^4 \mathfrak{X}^*(M)$ of Kähler type. The investigation of such manifolds, called QCH Kähler manifolds, was started by G. Ganchev and V. Mihova in [G-M-1],[G-M-2]. (M, g, J) is a QCH Kähler surface if and only if the antiselfdual Weyl tensor W^- is degenerate and there exist a negative almost complex structure $\overline{J}$ which preserves the Ricci tensor Ric of (M, g, J) i.e. $Ric(\overline{J}., \overline{J}.) = Ric(.,.)$ and such that $\overline{\omega} = g(\overline{J}.,.)$ is an eigenvector of W^- corresponding to simple eigenvalue of W^-. Equivalently (M, g, J) is a QCH Kähler surface iff it admits a negative almost complex structure $\overline{J}$ satisfying the Gray second condition $R(X, Y, Z, W) - R(\overline{J}X, \overline{J}Y, Z, W) = R(\overline{J}X, Y, \overline{J}Z, W) + R(\overline{J}X, Y, Z, \overline{J}W)$. Hence (M, g, J) satisfies the second Gray condition if J preserves the Ricci tensor and W^+ is degenerate. In [A-C-G-1] Apostolov, Calderbank and Gauduchon have classified weakly selfdual Kähler surfaces, extending the result

of Bryant who classified self-dual Kähler surfaces [B]. Weakly self-dual Kähler surfaces turned out to be of Calabi type and of orthotoric type or surfaces with parallel Ricci tensor.
We show that any Calabi type Kähler surface and every orthotoric Kähler surface is a QCH manifold. In both cases the opposite complex structure $\overline{J}$ is conformally Kähler. We also classify generalized Calabi type Kähler surfaces.

Definition

By the generalized Calabi type Kähler surface we mean the QCH Kähler surface such that the opposite Hermitian structure is determined by a complex foliation on complex curves.

Let (M, g, J) be a 4-dimensional Kähler manifold with a 2-dimensional J-invariant distribution $\mathcal{D}$. By ω we shall denote the Kähler form of (M, g, J) i.e. $\omega(X, Y) = g(JX, Y)$. Let (M, g, J)

be a QCH Kähler surface with respect to $J-$ *invariant* 2-dimensional distribution $\mathcal{D}$. Let us denote by $\mathcal{E}$ the distribution $\mathcal{D}^{\perp}$, which is a 2-dimensional, J-invariant distribution. By h, m respectively we shall denote the tensors $h = g \circ (p_{\mathcal{D}} \times p_{\mathcal{D}}), m = g \circ (p_{\mathcal{E}} \times p_{\mathcal{E}})$, where $p_{\mathcal{D}}, p_{\mathcal{E}}$ are the orthogonal projections on $\mathcal{D}, \mathcal{E}$ respectively. It follows that $g = h + m$. Let us define almost complex structure $\overline{J}$ by $\overline{J}_{|\mathcal{E}} = -J_{|\mathcal{E}}$ and $\overline{J}_{|\mathcal{D}} = J_{|\mathcal{D}}$. Let $\theta(X) = g(\xi, X)$ and $J\theta = -\theta \circ J$ which means that $J\theta(X) = g(J\xi, X)$. For every almost Hermitian manifold (M, g, J) the self-dual Weyl tensor W^{+} decomposes under the action of the unitary group $U(2)$. We have $\bigwedge^{*} M = \mathbb{R} \oplus LM$ where $LM = [[\bigwedge^{(0,2)} M]]$ and we can write W^{+} as a matrix with respect to this block decomposition

$$W^{+} = \begin{pmatrix} \frac{\kappa}{6} & W_2^{+} \\ (W_2^{+})^{*} & W_3^{+} - \frac{\kappa}{12} Id_{|LM} \end{pmatrix}$$

where κ is the conformal scalar curvature of (M, g, J) (see [A-A-D]). The selfdual Weyl tensor W^{+} of (M, g, J) is called degenerate if $W_2 = 0, W_3 = 0$. In general the self-dual Weyl tensor of 4-manifold (M, g) is called degenerate if it has at most two eigenvalues as an endomorphism $W^{+} : \bigwedge^{+} M \to \bigwedge^{+} M$. We shall denote by Ric_0 and ρ_0 the trace free part of the Ricci tensor Ric and the Ricci form ρ respectively. An ambikähler structure on a real 4-manifold consists of a pair of Kähler metrics $(g_{+}, J_{+}, \omega_{+})$ and $(g_{-}, J_{-}, \omega_{-})$ such that g_{+} and g_{-} are conformal metrics and J_{+} gives an opposite orientation to that given by J_{-} (i.e the volume elements $\frac{1}{2}\omega_{+} \wedge \omega_{+}$ and $\frac{1}{2}\omega_{-} \wedge \omega_{-}$ have opposite signs).

We shall recall some results from [G-M-1]. Let

$$R(X,Y)Z = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z \tag{1}$$

and let us write

$$R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

If R is the curvature tensor of a QCH Kähler manifold (M,g,J), then there exist functions $a,b,c \in C^\infty(M)$ such that

$$R = a\Pi + b\Phi + c\Psi, \tag{2}$$

where Π is the standard Kähler tensor of constant holomorphic curvature i.e.

$$\Pi(X,Y,Z,U)=\frac{1}{4}(g(Y,Z)g(X,U)-g(X,Z)g(Y,U) \tag{3}$$

$$+g(JY,Z)g(JX,U)-g(JX,Z)g(JY,U)-2g(JX,Y)g(JZ,U)),$$

the tensor Φ is defined by the following relation

$$\Phi(X,Y,Z,U)=\frac{1}{8}(g(Y,Z)h(X,U)-g(X,Z)h(Y,U) \tag{4}$$

$$+g(X,U)h(Y,Z)-g(Y,U)h(X,Z)+g(JY,Z)h(JX,U)$$

$$-g(JX,Z)h(JY,U)+g(JX,U)h(JY,Z)-g(JY,U)h(JX,Z)$$

$$-2g(JX,Y)h(JZ,U)-2g(JZ,U)h(JX,Y))$$

and finally

$$\Psi(X,Y,Z,U) = -h(JX,Y)h(JZ,U) = -(h_J \otimes h_J)(X,Y,Z,U). \tag{5}$$

where $h_J(X,Y) = h(JX,Y)$. Let $V = (V,g,J)$ be a real $2n$ dimensional vector space with complex structure J which is skew-symmetric with respect to the scalar product g on V. Let assume further that $V = D \oplus E$ where D is a 2-dimensional, J-invariant subspace of V, E denotes its orthogonal complement in V. Note that the tensors Π, Φ, Ψ given above are of Kähler type. It is easy to check that for a unit vector $X \in V$ $\Pi(X,JX,JX,X) = 1, \Phi(X,JX,JX,X) = |X_D|^2, \Psi(X,JX,JX,X) = |X_D|^4$, where X_D means an orthogonal projection of a vector X on the subspace D and $|X| = \sqrt{g(X,X)}$. It follows that for a tensor (2.2) defined on V we have

$$R(X,JX,JX,X) = \phi(|X_D|)$$

where $\phi(t) = a + bt^2 + ct^4$.
Let $J, \overline{J}$ be hermitian, opposite orthogonal structures on a Riemannian 4-manifold (M, g) such that J is a positive almost complex structure. Let $\mathcal{E} = ker(J\overline{J} - Id), \mathcal{D} = ker(J\overline{J} + Id)$ and let the tensors Π, Φ, Ψ be defined as above where $h = g(p_{\mathcal{D}}, p_{\mathcal{D}})$.

Proposition

Let (M, g, J) be a Kähler surface which is a QCH manifold with respect to the distribution $\mathcal{D}$. Then (M, g, J) is also QCH manifold with respect to the distribution $\mathcal{E} = \mathcal{D}^{\perp}$ and if Φ', Ψ' are the above tensors with respect to $\mathcal{E}$ then

$$R = (a + b + c)\Pi - (b + 2c)\Phi' + c\Psi' \tag{6}$$

.

Proof.

Let us assume that

$$X \in TM, |X| = 1$$

. Then if $\alpha = |X_{\mathcal{D}}|, \beta = ||X_{\mathcal{E}}||$ then $1 = \alpha^2 + \beta^2$. Hence $R(X, JX, JX, X) = a + b\alpha^2 + c\alpha^4 = a + b(1-\beta^2) + c(1-\beta^2)^2 = a + b + c - (b+2c)\beta^2 + c\beta^4$. □

If (M, g, J) is a QCH Kähler surface then one can show that the Ricci tensor ρ of (M, g, J) satisfies the equation

$$\rho(X, Y) = \lambda m(X, Y) + \mu h(X, Y) \tag{7}$$

where $\lambda = \frac{3}{2}a + \frac{b}{4}, \mu = \frac{3}{2}a + \frac{5}{4}b + c$ are eigenvalues of ρ (see [G-M-1], Corollary 2.1 and Remark 2.1.) In particular the distributions $\mathcal{E}, \mathcal{D}$ are eigendistributions of the tensor ρ

corresponding to the eigenvalues λ, μ of ρ. The Kulkarni-Nomizu product of two symmetric $(2,0)$-tensors $h, k \in \bigotimes^2 TM^*$ we call a tensor $h \oslash k$ defined as follows:

$$h \oslash k(X,Y,Z,T) = h(X,Z)k(Y,T) + h(Y,T)k(X,Z)$$

$$-h(X,T)k(Y,Z) - h(Y,Z)k(X,T).$$

Similarly we define the Kulkarni-Nomizu product of two 2-forms ω, η

$$\omega \oslash \eta(X,Y,Z,T) = \omega(X,Z)\eta(Y,T) + \omega(Y,T)\eta(X,Z)$$

$$-\omega(X,T)\eta(Y,Z) - \omega(Y,Z)\eta(X,T).$$

Then $b(\omega \oslash \eta) = -\frac{2}{3}\omega \wedge \eta$ where b is the Bianchi operator. In fact

$$3b(\omega\oslash\eta)(X,Y,Z,T)=\omega(X,Z)\eta(Y,T)+\omega(Y,T)\eta(X,Z)-\omega(X,T)\eta(Y$$

$$-\omega(Y,Z)\eta(X,T)+\omega(Y,X)\eta(Z,T)+\omega(Z,T)\eta(Y,X)$$

$$-\omega(Y,T)\eta(Z,X)-\omega(Z,X)\eta(Y,T)+\omega(Z,Y)\eta(X,T)$$

$$+\omega(X,T)\eta(Z,Y)-\omega(Z,T)\eta(X,Y)-\omega(X,Y)\eta(Z,T)$$

$$=-2\omega\wedge\eta(X,Y,Z,T).$$

Note that

$$\Pi=-\frac{1}{4}(\frac{1}{2}(g\oslash g+\omega\oslash\omega)+2\omega\otimes\omega)), \tag{8}$$

$$\Phi=-\frac{1}{8}(h\oslash g+h_J\oslash\omega+2\omega\otimes h_J+2h_J\otimes\omega), \tag{9}$$

$$\Psi = -h_J \otimes h_J, \tag{10}$$

where $\omega = g(J.,.)$ is the Kähler form. Note that $b(\Psi) = \frac{1}{3} h_J \wedge h_J = 0$ since $h_J = e_1 \wedge e_2$ is primitive, where e_1, e_2 is an orthonormal basis in $\mathcal{D}$.

Theorem

Let (M, g, J) be a Kähler surface. If (M, g, J) is a QCH manifold then $W^- = c(\frac{1}{6}\Pi - \Phi + \Psi)$ and W^- is degenerate. The 2-form $\overline{\omega}$ is an eigenvector of W^- corresponding to a simple eigenvalue of W^- and $\overline{J}$ preserves the Ricci tensor. On the other hand let us assume that (M, g, J) admits a negative almost complex structure $\overline{J}$ such that $Ric(\overline{J}, \overline{J}) = Ric$. Let $\mathcal{E} = ker(J\overline{J} - Id), \mathcal{D} = ker(J\overline{J} + Id)$. If $W^- = \frac{\kappa}{2}(\frac{1}{6}\Pi - \Phi + \Psi)$ or equivalently if the half-Weyl tensor W^- is degenerate and $\overline{\omega}$ is an eigenvector of W^- corresponding to a simple eigenvalue of W^- then (M, g, J) is a QCH manifold.

Note that for a Kähler surface (M, g, J) the Bochner tensor coincides with W^- and we have

$$R = -\frac{\tau}{12}(\frac{1}{4}(g \oslash g + \omega \oslash \omega) + \omega \otimes \omega)$$

$$-\frac{1}{4}(\frac{1}{2}(Ric_0 \oslash g + \rho_0 \oslash \omega) + \rho_0 \otimes \omega + \omega \otimes \rho_0) + W^-.$$

If (M, g, J) is a QCH Kähler surface then $Ric = \lambda m + \mu h$ where $\lambda = \frac{3}{2}a + \frac{b}{4}, \mu = \frac{3}{2}a + \frac{5}{4}b + c$. Consequently $Ric_0 = -\frac{b+c}{2}m + \frac{b+c}{2}h = \delta h - \delta m$ where $\delta = \frac{b+c}{2}$. Hence $Ric_0 = 2\delta h - \delta g$. Hence we have

$$R = -\frac{\tau}{12}(\frac{1}{4}(g \oslash g + \omega \oslash \omega) + \omega \otimes \omega)$$

$$-\frac{1}{4}(\frac{1}{2}((2\delta h - \delta g) \oslash g + (2\delta h_J - \delta\omega) \oslash \omega) + (2\delta h_J - \delta\omega) \otimes \omega +$$

$$\omega \otimes (2\delta h_J - \delta\omega)) + W^-.$$

Consequently

$$R = \frac{\tau}{6}\Pi + 2\delta\Phi - \delta\Pi + W^- = (a - \frac{c}{6})\Pi + (b + c)\Phi + W^-$$

and $a\Pi + b\Phi + c\Psi = (a - \frac{c}{6})\Pi + (b + c)\Phi + W^-$ hence $W^- = c(\frac{1}{6}\Pi - \Phi + \Psi)$. It follows that W^- is degenerate and $\overline{\omega}$ is an eigenvalue of W^- corresponding to the simple eigenvalue of W^-. It is also clear that $Ric(\overline{J}, \overline{J}) = Ric$.
On the other hand let us assume that a Kähler surface (M, g, J) admits a negative almost complex structure $\overline{J}$ preserving the Ricci tenor Ric and such that W^- is degenerate with eigenvector $\overline{\omega}$ corresponding to the simple eigenvalue of W^-. Equivalently it means that $\overline{J}$ satisfies the second Gray condition of the curvature i.e. $R(X, Y, Z, W) - R(\overline{J}X, \overline{J}Y, Z, W) = R(\overline{J}X, Y, \overline{J}Z, W) + R(\overline{J}X, Y, Z, \overline{J}W)$. Then $W^- = \frac{\kappa}{2}((\frac{1}{6}\Pi - \Phi + \Psi)$. If $Ric_0 = \delta(h - m)$ then as above

$R = \frac{\tau}{6}\Pi + 2\delta\Phi - \delta\Pi + W^-$. Consequently
$R = (\frac{\tau}{6} - \delta)\Pi + 2\delta\Phi + \frac{\kappa}{2}(\frac{1}{6}\Pi - \Phi + \Psi)$ and consequently

$$R = (\frac{\tau}{6} - \delta + \frac{\kappa}{12})\Pi + (2\delta - \frac{\kappa}{2})\Phi + \frac{\kappa}{2}\Psi. \tag{11}$$

Remark

Note that κ is the conformal scalar curvature of $(M, g, \overline{J})$. The Bochner tensor of QCH manifold was first identified in [G-M-2].

Corollary

A Kähler surface (M, g, J) is a QCH manifold iff it admits a negative almost complex structure $\overline{J}$ satisfying the second Gray condition of the curvature i.e.

$$R(X,Y,Z,W) - R(\overline{J}X,\overline{J}Y,Z,W) =$$

$$R(\overline{J}X,Y,\overline{J}Z,W) + R(\overline{J}X,Y,Z,\overline{J}W)$$

The J-invariant distribution $\mathcal{D}$ with respect to which (M, g, J) is a QCH manifold is given by $\mathcal{D} = \ker(J\overline{J} - Id)$ or by $\mathcal{D} = \ker(J\overline{J} + Id)$.

Theorem

Let us assume that (M, g, J) is a Kähler surface admitting a negative Hermitian structure $\overline{J}$ such that $Ric(\overline{J}, \overline{J}) = Ric$. Then (M, g, J) is a QCH manifold.

Proof.

If a Hermitian manifold (M, g, J) has a J-invariant Ricci tensor *Ric* then the tensor W^+ is degenerate (see [A-G]). □

Remark

If a Kähler surface (M, g, J) is compact and admits a negative Hermitian structure $\overline{J}$ as above then $(M, g, \overline{J})$ is locally conformally Kähler and hence globally conformally Kähler if $b_1(M)$ is even. Thus (M, g, J) is ambiKähler since $b_1(M)$ is even.

Now we give examples of QCH Kähler surfaces. First we give (see [A-C-G-1])

Definition

A Kähler surface (M, g, J) is said to be of Calabi type if it admits a non-vanishing Hamiltonian Killing vector field ξ such that the almost Hermitian pair (g, I) -with I equal to J on the distribution spanned by ξ and $J\xi$ and $-J$ on the orthogonal distribution - is conformally Kähler.

Every Kähler surface of Calabi type is given locally by

$$g = (az - b)g_{\Sigma} + w(z)dz^2 + w(z)^{-1}(dt + \alpha)^2, \qquad (12)$$

$$\omega = (az - b)\omega_{\Sigma} + dz \wedge (dt + \alpha), d\alpha = a\omega_{\Sigma}$$

where $\xi = \frac{\partial}{\partial t}$.
The Kähler form of Hermitian structure I is given by $\omega_I = (az - b)\omega_{\Sigma} - dz \wedge (dt + \alpha)$ and the Kähler metric

corresponding to I is $g_- = (az-b)^2 g$.
If $a \neq 0$ then the metric (*) is a product metric. If $a \neq 0$ then we set $a=1, b=0$ and write $w(z) = \frac{z}{V(z)}$ hence

$$g = zg_\Sigma + \frac{z}{V(z)} zdz^2 + \frac{V(z)}{z}(dt+\alpha)^2, \qquad (13)$$

$$\omega = z\omega_\Sigma + dz \wedge (dt+\alpha), d\alpha = \omega_\Sigma$$

It is known that for a Kähler surface of Calabi type of non-product type we have $\rho_0 = \delta\omega_I$ where $\delta = -\frac{1}{4z}(\tau_\Sigma + (\frac{V_z}{z^2})_z z^2)$ (see [A-C-G-1]) and consequently $Ric(I,I) = Ric$. This last relation remains true in the product case metric. Hence we have

Theorem

Every Kähler surface of Calabi type is a QCH Kähler surface.

Definition

A Kähler surface (M, g, J) is ortho-toric if it admits two independent Hamiltonian Killing vector fields with Poisson commuting momentum maps $\xi\eta$ and $\xi + \eta$ such that $d\xi$ and $d\eta$ are orthogonal.

An explicit classification of ortho-toric Kähler metrics is given in [A-C-G]. We have (this Proposition is proved in [A-C-G], Prop.8)

Proposition

The almost Hermitian structure (g, J, ω) *defined by*

$$g = (\xi - \eta)\left(\frac{d\xi^2}{F(\xi)} - \frac{d\eta^2}{G(\eta)}\right) + \frac{1}{\xi - \eta}\left(F(\xi)(dt + \eta dz)^2 - G(\eta)(dt + \xi dz)^2\right) \tag{14}$$

$$J d\xi = \frac{F(\xi)}{\xi - \eta}(dt + \eta dz), \; Jdt = -\frac{\xi d\xi}{F(\xi)} - \frac{\eta d\eta}{G(\eta)} \tag{15}$$

$$J d\eta = -\frac{G(\eta)}{\xi - \eta}(dt + \xi dz), \; Jdz = \frac{d\xi}{F(\xi)} + \frac{d\eta}{G(\eta)},$$

$$\omega = d\xi \wedge (dt + \eta dz) + d\eta \wedge (dt + \xi dz) \tag{16}$$

is orthotoric where F, G *are any functions of one variable. Every orthotoric Kähler surface* (M, g, J) *is of this form.*

Theorem

Every orthotoric Kähler surface is a QCH Kähler surface.

Note that both Calabi type and orthotoric Kähler surfaces are ambikähler. On the other hand we have

Theorem

Let (M, g, J) be ambi-Kähler surface which is a QCH manifold. Then locally (M, g, J) is orthotoric or of Calabi type or a product of two Riemannian surfaces or is an anti-selfdual Einstein-Kähler surface.

Now we give a classification of locally homogeneous QCH Kähler surfaces.

Proposition

Let (M, g, J) be a QCH locally homogeneous manifold. Then the following cases occur:
(a) (M, g, J) has constant holomorphic curvature (hence is locally symmetric and self-dual)
(b) (M, g, J) is locally a product of two Riemannian surfaces of constant scalar curvature
(c) (M, g, J) is locally isometric to a unique 4-dimensional proper 3-symmetric space.

Remark

A Riemannian 3-symmetric space is a manifold (M, g) such that for each $x \in M$ there exists an isometry $\theta_x \in Iso(M)$ such that $\theta_x^3 = Id$ and x is an isolated fixed point. On a such manifold there is a natural canonical g-ortogonal almost complex structure $\overline{J}$ such that all θ_x are holomorphic with respect to $\overline{J}$. Such structure in dimension 4 is almost Kähler and satisfies the Gray condition G_2. The example of 3-symmetric 4-dimensional Riemannian space with non-integrable structure $\overline{J}$ was constructed by O. Kowalski in [Ko],Th.VI.3. This is the only proper generalized symmetric space in dimension 4.

Remark

This example is defined on $\mathbb{R}^4 = \{x, y, u, v\}$ by the metric

$$g = (-x + \sqrt{x^2 + y^2 + 1})du^2 + (x + \sqrt{x^2 + y^2 + 1})dv^2 - 2ydu \odot dv$$
$$+ [\frac{(1 + y^2)dx^2 + (1 + x^2)dy^2 - 2xydx \odot dy}{1 + x^2 + y^2}] \tag{17}$$

It admits a Kähler structure J in an opposite orientation.

Remark

Every QCH Kähler surface is a holomorphically pseudosymmetric Kähler manifold. (see [O],[J-1]). In fact from [J-1] it follows that $R.R = (a + \frac{b}{2})\Pi.R$. Hence in the case of QCH Kähler surfaces we have

$$R.R = \frac{1}{6}(\tau - \kappa)\Pi.R \tag{18}$$

where τ is the scalar curvature of (M, g, J) and κ is the conformal scalar curvature of $(M, g, \overline{J})$. Note that (2.19) is the obstruction for a Kähler surface to have a negative almost complex $\overline{J}$ structure satisfying the Gray condition (G_2). In an extremal situation where $(M, g, \overline{J})$ is Kähler we have $R.R = 0$.

Now we recall some results from [J-5].

Lemma

Let (M, g, J) be a Hermitian 4-manifold. Let us assume that $|\nabla J| \neq 0$ on M. Then for any local orthonormal oriented basis $\{E_1, E_2\}$ of $\mathcal{D}^\perp$ there exists a global oriented orthonormal basis $\{E_3, E_4\}$ of $\mathcal{D}$ independent of the choice of $\{E_1, E_2\}$ such that

$$\nabla\Omega = \alpha(\theta_1 \otimes \Phi + \theta_2 \otimes \Psi) \tag{19}$$

where $\Phi = \theta_1 \wedge \theta_3 - \theta_2 \wedge \theta_4$, $\Psi = \theta_1 \wedge \theta_4 + \theta_2 \wedge \theta_3$,$\alpha = \frac{1}{2\sqrt{2}}|\nabla J|$ and $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ is a cobasis dual to $\{E_1, E_2, E_3, E_4\}$. Moreover $\delta\Omega = -2\alpha\theta_3, \theta = -\alpha\theta_4$ where the Lee form θ of (M, g, J) is defined by the equality $d\Omega = 2\theta \wedge \Omega$

By $\mathcal{D}$ we denote the distribution spanned by the fields $\{E_3, E_4\}$. Throughout all the paper we shall assume that $|\nabla J| \neq 0$, hence

$\alpha \neq 0$. Then $\mathcal{D}$ is a foliation. If the natural opposite almost Hermitian structure is Hermitian then $\mathcal{D}$ is totally geodesic (see [J-4]). Next we recall some results from [J-5].

Lemma

Let (M, g, J) be a Hermitian surface with J-invariant Ricci tensor (i.e., $\mathcal{R}(LM) \subset \wedge^+ M$). Let $\{E_1, E_2, E_3, E_4\}$ be a local orthonormal frame such that (2.1) holds. Then

$$\Gamma^3_{11} = \Gamma^3_{22} = E_3 \ln \alpha,$$
$$\Gamma^3_{44} = \Gamma^4_{21} = -\Gamma^4_{12} = -E_3 \ln \alpha,$$
$$\Gamma^3_{21} = -\Gamma^3_{12},\ \Gamma^4_{11} = \Gamma^4_{22},$$
$$-\Gamma^3_{21} + \Gamma^4_{22} = \alpha,$$
$$\Gamma^4_{33} = -E_4 \ln \alpha + \alpha,$$

where $\nabla_X E_i = \sum \omega^j_i(X) E_j$ and $\Gamma^i_{kj} = \omega^i_j(E_k)$.

Lemma

Let (M, g, J) be a Hermitian surface with J-invariant Ricci tensor. Then $\Gamma^4_{13} = -E_2 \ln \alpha$, $\Gamma^4_{23} = E_1 \ln \alpha$ and $d\theta$ is anti-self-dual.

If (M, g, J) is a Hermitian surface with $|\nabla J| \neq 0$ on M, then the distributions $\mathcal{D}, \mathcal{D}^\perp$ define a natural opposite almost Hermitian structure $\overline{J}$ on M. This structure is defined as follows $\overline{J}|_{\mathcal{D}} = -J|_{\mathcal{D}}, \overline{J}|_{\mathcal{D}^\perp} = J|_{\mathcal{D}^\perp}$. In the special basis we just have: $\overline{J}E_1 = E_2, \overline{J}E_3 = -E_4$.

Lemma

Let (M, g, J) be a Hermitian 4-manifold with Hermitian Ricci tensor and Kähler natural opposite structure. Let us assume that $|\nabla J| \neq 0$ on M. Then
(a) $\mathcal{D}$ is a totally geodesic foliation,
(b)$E_3 \ln \alpha = 0$,
(c) $\nabla_{E_4} E_4 = 0$,
(d) $d\theta = -E_2\alpha\bar{\Phi} - E_1\alpha\bar{\Psi}$ where
$\bar{\Phi} = \theta_1 \wedge \theta_3 + \theta_2 \wedge \theta_4, \bar{\Psi} = \theta_1 \wedge \theta_4 - \theta_2 \wedge \theta_3$.

Hence, if the opposite structure is Kähler, then $E_3\alpha = 0, \Gamma^3_{11} = \Gamma^3_{22} = \Gamma^3_{44} = \Gamma^4_{21} = \Gamma^4_{12} = 0$ and $-\Gamma^3_{21} = \Gamma^3_{12} = \Gamma^4_{11} = \Gamma^4_{22} = \frac{1}{2}\alpha$. We also have $\Gamma^4_{13} = -E_2 \ln \alpha, \Gamma^4_{23} = E_1 \ln \alpha$.

Lemma

Let (M, g, J) be a Hermitian 4-manifold with Hermitian Ricci tensor and Kähler natural opposite structure. Let us assume that $|\nabla J| \neq 0$ on M. If $(M, g, \overline{J})$ is a semi-symmetric surface foliated by Euclidean spaces, then $E_4 \ln \alpha = \frac{1}{2}\alpha$. On the other hand if $(M, g, \overline{J})$ is a QCH surface with Hermitian opposite structure J such that $\overline{J}$ is a natural opposite structure for J and $E_4 \ln \alpha = \frac{1}{2}\alpha$, then $(M, g, \overline{J})$ is semi-symmetric.

Let us assume that $\theta_1 = fdx, \theta_2 = fdy$. We will later show that we can always assume that $\theta_1 = fdx, \theta_2 = fdy$ where (x, y, z, t) is a local foliated coordinate system i.e. $\mathcal{D} = kerdx \cap kerdy$. Then $\Gamma^1_{32} = \frac{\alpha}{2}, \Gamma^2_{41} = 0$. Then we have (see [J-5])

$$[E_1, E_4] = -\frac{\alpha}{2}E_1 + E_2 \ln \alpha E_3, \tag{20}$$

$$[E_2, E_4] = -\frac{\alpha}{2}E_2 - E_1 \ln \alpha E_3, \tag{21}$$

$$[E_1, E_3] = -E_2 \ln \alpha E_4, \tag{22}$$

$$[E_2, E_3] = E_1 \ln \alpha E_4, \tag{23}$$

$$[E_3, E_4] = -(-E_4 \ln \alpha + \alpha)E_3, \tag{24}$$

$$[E_1, E_2] = \Gamma^1_{12}E_1 - \Gamma^2_{21}E_2 + \alpha E_3. \tag{25}$$

We also have (equations (2.3))

$$d\theta_1 = \Gamma^2_{11}\theta_1 \wedge \theta_2 + \frac{1}{2}\alpha\theta_1 \wedge \theta_4, d\theta_2 = -\Gamma^1_{22}\theta_1 \wedge \theta_2 + \frac{1}{2}\alpha\theta_2 \wedge \theta_4,$$

$$d\theta_3 = -\alpha\theta_1 \wedge \theta_2 - E_2 \ln \alpha\theta_1 \wedge \theta_4$$

$$+E_1 \ln \alpha\theta_2 \wedge \theta_4 + (-E_4 \ln \alpha + \alpha)\theta_3 \wedge \theta_4,$$

$$d\theta_4 = E_2 \ln \alpha\theta_1 \wedge \theta_3 - E_1 \ln \alpha\theta_2 \wedge \theta_3.$$

Lemma

The two almost Hermitian structures $J, \overline{J}$ on M, such that the $(1,0)$ distribution of J is given by $\theta_1 + i\theta_2 = 0, \theta_3 + i\theta_4 = 0$ and the $(1,0)$ distribution of $\overline{J}$ by $\theta_1 + i\theta_2 = 0, \theta_3 - i\theta_4 = 0$, are integrable if equations (2.3) are satisfied. The form $d\theta = -d(\alpha\theta_4)$ is self-dual with respect to the orientation given by $\overline{J}$.

Definition

A foliation $\mathcal{F}$ on a Riemannian manifold (M, g) is called conformal if, for any V tangent to the leaves od $\mathcal{F}$, the equation

$$L_V g = \theta(V) g$$

holds on $T\mathcal{F}^{\perp}$, where θ is a one form vanishing on $T\mathcal{F}^{\perp}$. A foliation $\mathcal{F}$ is called homothetic if it is conformal and $d\theta = 0$.

Definition

A complex distribution $\mathcal{D}$ on a complex manifold (M, g, J) is called holomorphic if $L_\xi J(TM) \subset \mathcal{D}$ for any $\xi \in \Gamma(\mathcal{D})$.

Theorem

Let (M, g, J) be a Kähler surface and $\mathcal{D}$ be a complex foliation, $dim\mathcal{D} = 2$. Then the almost Hermitian structure I given by $I_{|\mathcal{D}} = J_{|\mathcal{D}}, I_{|\mathcal{D}^\perp} = -J_{|\mathcal{D}^\perp}$ is integrable if and only if $\mathcal{D}$ is holomorphic and totally geodesic.

Let $(M, g, \bar{J})$ be a Kähler surface with foliation $\mathcal{D}$ whose leaves are complex curves such that the natural opposite almost Hermitian structure J determined by $\mathcal{D}$ is Hermitian. Then $\mathcal{D}$ is conformal and $d\Omega = 2\theta \wedge \Omega$ and $L_V g = \theta(V)g$ on $\mathcal{D}^\perp$ for any $V \in \mathcal{D}$. It is shown in [J-3] that $d\theta(X, Y) = 0$ for all $X, Y \in \mathcal{D}$.
Note that Calabi type surfaces (M, g, J) are Kähler QCH surfaces with a foliation $\mathcal{D}$ which determines an opposite Hermitian locally conformally Kähler structure. Hence $\mathcal{D}$ is conformal and homothetic and we have $d\theta = 0$ and hence $E_1\alpha = E_2\alpha = E_3\alpha = 0$.

Hence if α is not constant we may assume that $E_4\alpha \neq 0$ on M, we just consider an open submanifold U given by $U = \{x \in M : E_4\alpha \neq 0\}$. We construct special coordinates on M.

Theorem

Let $X \in \mathcal{D}$ where $\mathcal{D}$ is a 2-dimensional foliation on the manifold M, dim $M = 4$. Then there exist foliated coordinates (x, y, z, t) on M such that $X = \frac{\partial}{\partial z}$.

Proof.

Assume (x', y', z, t) are coordinates on M such that $X = \frac{\partial}{\partial z}$ and $(dz, dt) : \mathcal{D} \to \mathbb{R}^2$ is a surjection. As a new coordinates let us take (x, y, z, t), where (x, y, z', t') are foliated coordinates,i.e. $\mathcal{D} = kerdx \cap \ker dy$. Then $dx(X) = dy(X) = dt(X) = 0, dz(X) = 1$ and 1-forms dx, dy, dz, dt are linearly independent. Hence (x, y, z, t) is a new foliated coordinate system and in these coordinates $X = \frac{\partial}{\partial z}$. $\square$

Theorem

Let $X, Y \in \mathcal{D}$ be linearly independent vector fields, where $\mathcal{D}$ is a two-dimensional foliation on M, $\dim M = 4$. Then there exist foliated coordinates (x, y, z, t) on M such that $X = f\frac{\partial}{\partial z}, Y = g\frac{\partial}{\partial t}$.

Proof.

Let (x', y', z', t) be coordinates, such that $X = \frac{\partial}{\partial z'}$ and $(dz', dt) : \mathcal{D} \to \mathbb{R}^2$ is a surjection. Let (x'', y'', z, t') be coordinates, such that $Y = \frac{\partial}{\partial t'}$ and $(dz, dt') : \mathcal{D} \to \mathbb{R}^2$ is a surjection. As a new coordinates let us take (x, y, z, t), where (x, y, z_1', t_1') are foliated coordinates such that $\mathcal{D} = kerdx \cap \ker dy$. Then $dx(X) = dy(X) = dt(X) = 0$ and $dx(Y) = dy(Y) = dz(Y) = 0$. The 1-forms dx, dy, dz, dt are linearly independent. Hence (x, y, z, t) is a new foliated coordinate system and in new coordinates $X = f\frac{\partial}{\partial z}$, $Y = g\frac{\partial}{\partial t}$. $\square$

Theorem

Let $\alpha \in C^\infty(M)$ be a function such that $X\alpha \neq 0$, where $X \in \mathcal{D}$. Then there exist foliated coordinates (x, y, z, t) for which $\alpha = \alpha(z)$ and $X = f\frac{\partial}{\partial z}$ for certain function f.

Proof.

Let us take coordinates (x, y, z, t) given by Th.A and let $x' = x, y' = y, z' = g(\alpha(x, y, z, t)),\ t' = t$ for some smooth function g on $\mathbb{R}$ with $g' \neq 0$. Then α is a function of z' only and $X = \frac{\partial}{\partial z} = g'(\alpha(x, y, z, t))\frac{\partial \alpha}{\partial z}(x, y, z, t)\frac{\partial}{\partial z'} = f\frac{\partial}{\partial z'}$ where

$$f = g'(\alpha(x, y, z, t))\frac{\partial \alpha}{\partial z}(x, y, z, t).$$

$\square$

Theorem

Let $a, b : \mathbb{R}^4 \to \mathbb{R}$ be smooth functions such that $\partial_y a = \partial_x b$. Then there exists a smooth function $f : \mathbb{R}^4 \to \mathbb{R}$ such that $\partial_x f = a, \partial_y f = b$.

Proof.

Let us define f by the formula $f(x, y, z, t) = A(z, t) + \int_\gamma a(x, y, z, t)dx + b(x, y, z, t)dy$ where A is some smooth function and γ is a smooth curve contained in the plane $z = const, t = const$ which joins a point $(0, 0, z, t)$ with a point (x, y, z, t). Then f is well defined since the form $\omega = adx + bdy$ is closed (hence exact) in the plane $z = const, t = const$. It is easy to see that $\partial_x f = a, \partial_y f = b$. $\square$

Lemma

Let θ be a 1-form on $\mathbb{R}^4$ such that $d\theta(\partial_x, \partial_y) = 0$. Then there exists a function $f : \mathbb{R}^4 \to \mathbb{R}$ such that $\theta(\partial_x) = \partial_x f, \theta(\partial_y) = \partial_y f$.

Proof.

Let $a = \theta(\partial_x), b = \theta(\partial_y)$. Then $\partial_y a = \partial_x b$ and we can apply Th D. $\square$

Theorem

Let (M, g, J) be a Kähler surface with conformal 2-dimensional foliation $\mathcal{F}$, such that $d\theta(X, Y) = 0$ for $X, Y \in T\mathcal{F}$. Then for horizontal vector fields $Z, Y \in (T\mathcal{F})^{\perp}$ we have $g(Y, Z) = fp^{*}h(Y, Z)$ where $p : M \to N$ is a local submersion, whose fibers are leaves of a foliation $\mathcal{F}$ and f is a certain function on M.

Proof.

Let $L_V g(Y,Z) = \theta(V)g(Y,Z)$. We have $\theta(X) = d\ln f(X)$ for all $X \in T\mathcal{F}$ for a certain positive function f (see Lemma G). Define a metric h on N by the formula $h(X,Y) = \frac{1}{f}g(X^*,Y^*)$ where X^*, Y^* are horizontal lifts of X, Y i.e. $p(X^*) = X, p(Y^*) = Y$ and $X^*, Y^* \in (T\mathcal{F})^{\perp}$. We will show that h is well defined. It is enough to show that for $V \in T(\mathcal{F})$ we have $V(\frac{1}{f}g(X^*,Y^*)) = 0$. But $V(\frac{1}{f}g(X^*,Y^*)) = -\frac{Vf}{f^2}g(X^*,Y^*) + \frac{1}{f}(L_V g)(X^*,Y^*) + \frac{1}{f}g([V,X^*],Y^*) + \frac{1}{f}g(X^*,[V,Y^*]) = -\frac{(d\ln f)V}{f}g(X^*,Y^*) + \frac{1}{f}(d\ln f)Vg(X^*,Y^*) = 0$ since the fields $[V,X^*],[V,Y^*]$ are tangent to the leaves of a foliation. $\square$

Remark

In the case of QCH Kähler surfaces with Hermitian opposite almost Hermitian structure given by complex foliation $\mathcal{F}$ we always have $d\theta(X, Y) = 0$ for $X, Y \in T\mathcal{F}$

(see [J-8], Remark 3.3, p.234).

Theorem

Let (M, g, J) be a QCH Kähler surface with a complex foliation $\mathcal{D}$ determining a Hermitian opposite natural structure. Then there exists an orthonormal frame $\{E_1, E_2\}$ on $\mathcal{D}^{\perp}$, such that $\theta_1 = fdx, \theta_2 = fdy$ where f is a positive function and (x, y, z, t) is a foliated coordinate system on M.

Proof.

Let us take isothermal coordinates on N for h. Then $h = k(dx^2 + dy^2)$ and $g(Y,Z) = f_1 k(dx^2 + dy^2)$ for $Y, Z \in T\mathcal{D}^{\perp}$ (see Th.E). Now take $f = \sqrt{f_1 k}$ □

Now we show that for Calabi surfaces with nonconstant α we can take local coordinates for which $E_1 = \frac{1}{f}\partial_x + l\partial_t, E_2 = \frac{1}{f}\partial_y + n\partial_t, E_3 = g\partial_t, E_4 = \partial_z$ where g is $g = \beta g_1$ where $\alpha\beta = e^A$, $A = \int \alpha$ and $g_1 = g_1(x, y, t)$. We assume that $E_4\alpha \neq 0$ on M. Note that in our case $E_1\alpha = E_2\alpha = E_3\alpha = 0$ hence we assume that α is nonconstant. From the above theorems it follows that we can take foliated coordinates such that $E_4 = h\partial_z$ and $E_3 \in span(\partial_z, \partial_t) = \mathcal{D}$. Since $E_3\alpha = 0$ and $\alpha_z \neq 0$ it follows that $E_3 = g\partial_t$ for some function $g(x, y, z, t)$. Now we show that we can take $h = 1$. Note that

$kerdz = span(E_1, E_2, E_3) = span(\partial_x, \partial_y, \partial_t)$. $E_4\alpha$ depends only on z since using the equations for Lie brackets we obtain

$$E_1 E_4 \alpha = [E_1, E_4]\alpha = -\frac{\alpha}{2} E_1 \alpha = 0 \tag{26}$$

and so on. Hence $E_4\alpha = \phi(z)$. Now $E_4\alpha = h\alpha'(z) = \phi(z)$. Hence h is a function of z only and taking appropriate transformation of coordinates we can assume that $h = 1$. Now from $[E_3, E_4] = -(-\frac{\alpha'}{\alpha} + \alpha)E_3$ it is easy to check that $g = \beta g_1(x, y, t)$ where $\alpha\beta = e^A$. For any function $g_1(x, y, t)$ we define $g = \beta g_1$. Now we change coordinates by $x' = x, y' = y, t' = \int \frac{1}{g_1(x,y,t)} dt, z' = z$. Then $\partial_{z'} = \partial_z, \partial_{t'} = g_1 \partial_t$. Then in new coordinates $g_1 = 1$ and $E_4 = \partial_{z'}, E_3 = \beta(z')\partial_{t'}$. These new coordinates we again write as (x, y, z, t). Note that this system of coordinates is foliated. In the case $\alpha = const \neq 0$ we use Th.B to find foliated coordinates such

that $E_4 = h\partial_z, E_3 = g\partial_t$. Now it is easy to see that $\partial_t h = 0$. Let us take new coordinates $x' = x, y' = y, t' = t, z' = \int \frac{1}{h(x,y,z)} dz$. Then in new coordinates we have $E_4 = \partial_z, E_3 = g\partial_t$. Now again $g = \beta(z)g_1(x,y,t)$. Using the new coordinates transformation $x' = x, y' = y, t' = \int \frac{1}{g_1(x,y,t)} dt, z' = z$ we can assume that $E_4 = \partial_z, E_3 = \beta(z)\partial_t$ and new coordinates are foliated.
Now we consider the case of QCH Kähler surfaces with the opposite Hermitian structure which is not locally conformally Kähler. Let us take foliated coordinates such that
$\theta_1 = fdx, \theta_2 = fdy$ and
$E_1 = \frac{1}{f}\partial_x + k\partial_z + l\partial_t, E_2 = \frac{1}{f}\partial_y + m\partial_z + n\partial_t, E_4 = r\partial_z, E_3 = g\partial_t$.
It is easy to see that $\partial_t r = 0$ hence $r = r(x,y,z)$. Let us change coordinates by $x' = x, y' = y, z' = \int \frac{1}{r(x,y,z)} dz, t' = t$. Then in new coordinates $\alpha = \alpha(x,y,z)$ and $E_4 = \partial_z, E_3 = g(x,y,z,t)\partial_t$.
Let $\beta(x,y,z)$ satisfies equation $\partial_z \ln\beta + \partial_z \ln\alpha = A$ where

$A = A(x, y, z)$ satisfies $\partial_z A = \alpha$. Hence $\alpha\beta = e^A$. Note that $\partial_z \ln g = \partial_z \ln \beta$ and hence $g = \beta g_1(x, y, t)$. Now changing coordinates by $x' = x, y' = y, z' = z, t' = \int \frac{1}{g_1(x,y,t)} dt$ we obtain new foliated coordinates in which $E_4 = \partial_z, E_3 = \beta\partial_t$. Since $\theta_1 = fdx, \theta_2 = fdy$ from our equations we get

$$[\frac{1}{f}\partial_x + k\partial_z + l\partial_t, \partial_z] = -\frac{\alpha}{2}(\frac{1}{f}\partial_x + k\partial_z + l\partial_t) + (\frac{1}{f}\partial_y \ln\alpha + m\partial_z \ln\alpha)\beta\partial_t.$$

Hence $\partial_z f = -\frac{\alpha}{2} f, \partial_z k = \frac{\alpha}{2} k$ and $f = e^{-\frac{A}{2}} h(x, y)$ and $k = e^{\frac{A}{2}} k_1(x, y, t)$. Analogously $\partial_z m = \frac{\alpha}{2} m$ and $m = e^{\frac{A}{2}} m_1(x, y, t)$. We also have

$$[\frac{1}{f}\partial_x + k\partial_z + l\partial_t, \beta\partial_t] = -(\frac{1}{f}\partial_y \ln\alpha + m\partial_z \ln\alpha)\partial_z.$$

It follows that $\partial_t k = \frac{1}{f\beta}\partial_y \ln\alpha + m\frac{\partial_z \ln\alpha}{\beta}$. Consequently we obtain

$$\partial_{zt} k = \frac{\alpha}{2} k_t = \frac{\alpha}{2}\left(\frac{1}{f\beta}\partial_y \ln\alpha + m\frac{\partial_z \ln\alpha}{\beta}\right).$$

On the other hand

$$\partial_{zt} k = \left(\frac{1}{f\beta}\partial_y \ln\alpha\right)_z + m\alpha\frac{\partial_z \ln\alpha}{2\beta} + m\left(\frac{\partial_z \ln\alpha}{\beta}\right)_z.$$

Note also that $\partial_t k = \frac{1}{f\beta}\partial_y \ln\alpha + m\frac{\partial_z \ln\alpha}{g}$ and $\partial_t m = \frac{1}{f\beta}\partial_x \ln\alpha + k\frac{\partial_z \ln\alpha}{g}$. Hence if $m_t = 0$ then $k_t = 0$ and $m = -\frac{1}{f\partial_z \ln a}\partial_y \ln\alpha, k = -\frac{1}{f\partial_z \ln a}\partial_x \ln\alpha$. It easy to check that in that case $E_1 \ln\alpha = E_2 \ln\alpha = 0$ and consequently $d\theta = 0$ which means that the opposite structure is locally conformally Kähler. Hence $m_t \neq 0, k_t \neq 0$. Hence since f, β, α do not depend on t and

m is a nonconstant function depending on t we get $(\frac{\partial_z \ln \alpha}{\beta})_z = 0$. It follows that $\beta = C(x,y)(\ln \alpha)_z$ and thus since $\partial_z(\ln \beta) = -\partial_z \ln \alpha + \alpha$ we obtain $\alpha'' = \alpha\alpha'$ and $\alpha' = \frac{1}{2}\alpha^2 + D(x,y)$. Now in the case $D > 0$ we get $\alpha = 2a\tan(a(z+\phi(x,y)))$ where $a = a(x,y) = \sqrt{\frac{D}{2}}$. Taking new coordinates with $z' = z + \phi$ we can assume that $\alpha = 2a\tan(az)$. In the case $D < 0$ we get $\alpha = -2a\coth(az)$. If $D = 0$ in some open subset of M we get $\alpha = -\frac{2}{z+\phi(x,y)}$ and changing the coordinates by $x' = x, y' = y, z' = z + \phi(x,y), t' = t$ we can assume that $\alpha = -\frac{2}{z}$. It follows that

$$\frac{1}{f\beta}\partial_y \ln \alpha = e^{\frac{A}{2}}c(x,y)$$

for some function c. We shall consider the case $\alpha = 2a\tan(az)$.

Then

$$\alpha_y = 2a_y \tan az + 2a \frac{1}{\cos^2 az} a_y z$$

and $(\ln \alpha)_y = (\ln a)_y + \frac{a_x z}{\sin az \cos az}$. Note that $\beta = \frac{1}{a \sin 2az}$ and we get

$$\frac{\partial_y \ln \alpha}{h\beta} = a_y\left(\frac{1}{a^2 h} \sin 2az + \frac{z}{2ha}\right) = c(x, y).$$

We obtain $a_y = 0$. Similarly we show that $a_x = 0$ and hence $a = const$. The other case is similar. It follows that α, β are functions of z only in the introduced coordinates and $\alpha = -\frac{2}{z}, \alpha = 2a \tan az, \alpha = -2a \coth az$ where $a \in \mathbb{R}, a \neq 0$.

Next we classify using our method Calabi type Kähler surfaces already classified in [A-C-G]. Let $\alpha(z)$ be any smooth nonvanishing function defined on an open subset $V \subset \mathbb{R}$ and $A = \int \alpha, \beta\alpha = e^A$.

Theorem

Let $U \subset \mathbb{R}^2$ be an open set and let $g_\Sigma = h^2(dx^2 + dy^2)$ be a Riemannian metric on U, where $h : U \to \mathbb{R}$ is a positive function $h = h(x, y)$. Let $\omega_\Sigma = h^2 dx \wedge dy$ be a volume form of $\Sigma = (U, g_\Sigma)$. Let $M = U \times N$, where $N = V \times \mathbb{R}$. Let us define the metric g on M by $g(X, Y) = \exp(-A)g_\Sigma(X, Y) + \theta_3(X)\theta_3(Y) + \theta_4(X)\theta_4(Y)$, where $\theta_3 = \frac{1}{\beta}(dt - l_2(x, y)dx - n_2(x, y)dy)$, $A = \int \alpha, \beta\alpha = e^A$ and $\theta_4 = dz$ and $d(l_2 dx + n_2 dy) = \omega_\Sigma$. Then (M, g) admits a Kähler structure $\overline{J}$ with the Kähler form $\overline{\Omega} = e^{-A}\omega_\Sigma + \theta_4 \wedge \theta_3$ and a Hermitian structure J with the Kähler form $\Omega = e^{-A}\omega_\Sigma + \theta_3 \wedge \theta_4$. The Ricci tensor of (M, g) is J-invariant and J is not locally conformally Kähler. The Lee form of (M, g, J) is $\theta = -\alpha\theta_4$. The scalar curvature of (M, g) is

$$\tau = 2\left(-\frac{(\Delta \ln h)e^A}{h^2} - 2\alpha^2 + 2\alpha' + \frac{\beta''}{\beta} - 2\left(\frac{\beta'}{\beta}\right)^2\right).$$

Proof. We shall use equations (2.2). Let us take a coordinate system such that $\theta_1 = fdx, \theta_2 = fdy$, $\alpha = \alpha(z)$ and assume $E_1\alpha = E_2\alpha = 0$. Then
$E_1 = \frac{1}{f}\partial_x + l\partial_t, E_2 = \frac{1}{f}\partial_y + n\partial_t, E_3 = \beta\partial_t, E_4 = \partial_z$. Then
$\theta_1 = fdx, \theta_2 = fdy, \theta_4 = dz, \theta_3 = \frac{1}{\beta}(dt - lfdx - nfdy)$. Then
$E_4 \ln\alpha = \frac{\alpha'}{\alpha}$ and

$$E_1 \ln\alpha = 0, E_2 \ln\alpha = 0.$$

We have $[E_1, E_4] = \frac{f_z}{f^2}\partial_x - l_z\partial_t = -\frac{\alpha}{2f}\partial_x - \frac{\alpha}{2}l\partial_t$. Hence $l_z = \frac{\alpha}{2}l, f_z = -\frac{\alpha}{2}f$. This implies $f = e^{-\frac{A}{2}}h(x,y)$. On the other hand $[E_1, E_3] = -\beta l_t\partial_t = 0$ and

$$l_t = 0.$$

This yields

$$l = l_1(x,y)e^{\frac{A}{2}}.$$

Similarly $[E_2, E_3] = -gn_t\partial_t = 0$

$$n_t = 0.$$

Since $[E_2, E_4] = -n_z\partial_t + \frac{f_z}{f^2}\partial_y = -\frac{\alpha}{2f}\partial_y - \frac{1}{2}\alpha n\partial_t$, we get $n_z = \frac{1}{2}\alpha n$. Hence

$$n = n_1(x, y)e^{\frac{A}{2}}.$$

Hence $\theta_3 = \frac{1}{\beta}(dt - l_2(x, y)dx - n_2(x, y)dy$ and $\theta_4 = dz$. Now we prove that

$$d\theta_3 = -\alpha\theta_1 \wedge \theta_2 +$$

$$(-\frac{\alpha'}{\alpha} + \alpha)\theta_3 \wedge \theta_4 = -\alpha\theta_1 \wedge \theta_2$$

$$+\frac{\beta_z}{\beta}\theta_3 \wedge \theta_4$$

if $d(l_2 dx + n_2 dy) = \omega_\Sigma$.
In fact,

$$d\theta_3 = d(\frac{1}{\beta}(dt - l_2(x,y)dx - n_2(x,y)dy) =$$

$$-\alpha\theta_1 \wedge \theta_2 + \frac{\beta_z}{\beta}\theta_3 \wedge \theta_4$$

after some easy computation if we assume that $d((l_2 dx + n_2 dy) = \omega_\Sigma = h^2 dx \wedge dy$.
Now we have $d\theta_1 \wedge \theta_2 = d(e^{-A}h^2 dx \wedge dy) =$ $-e^{-A}\alpha dz \wedge h^2 dx \wedge dy = -\alpha dz \wedge \theta_1 \wedge \theta_2 = -\alpha\theta_4 \wedge \theta_1 \wedge \theta_2$. From (2.3) it is also clear that $d(\theta_3 \wedge \theta_4) = d(\theta_3) \wedge \theta_4 = -\alpha\theta_1 \wedge \theta_2 \wedge \theta_4$.
Hence $d\overline{\Omega} = 0$, where $\overline{\Omega}(X,Y) = g(\overline{J}X, Y)$, and $\overline{\Omega} = \theta_1 \wedge \theta_2 - \theta_3 \wedge \theta_4$. Hence in view of Lemma F $(M, g, \overline{J})$ is a Kähler surface and the Lee form of (M, g, J) is $\theta = -\alpha\theta_4$.

Now we show that ∂_t is a real holomorphic vector field. Note that the opposite Kähler structure $\overline{J}$ satisfies $\overline{J}\partial_z = \beta\partial_t, \overline{J}\partial_t = -\frac{1}{\beta}\partial_z, \overline{J}\partial_x = \partial_y + fn\partial_t + \frac{1}{\beta}fl\partial_z, \overline{J}\partial_y = -\partial_x - fl\partial_t + \frac{1}{\beta}fn\partial_z$. It is clear that $(L_{\partial_t}\overline{J}) = 0$. The field $X = \partial_t + i\frac{1}{\beta}\partial_z$ is a holomorphic vector field. The form $\psi = \frac{1}{2}(dt - i\beta dz)$ satisfies $\psi(X) = 1$ and $\psi(\partial_t - i\frac{1}{\beta}\partial_z) = 0$. If Ψ is a holomorphic $(1,0)$ form such that $\Psi(X) = 1$, then $\Psi - \psi = 0 mod\{dx, dy\}$. If $\overline{\Omega}$ is a Kähler form of Kähler manifold $(M, \overline{J}, g)$, then $\frac{1}{2}\overline{\Omega}^2 = e^{-A}h^2\frac{1}{\beta}dx \wedge dy \wedge dz \wedge dt$. We also have
$\frac{i^2}{4}(dx + idy) \wedge \overline{(dx + idy)} \wedge \Psi \wedge \overline{\Psi} = \frac{1}{4}\beta dx \wedge dy \wedge dz \wedge dt$. Hence the Ricci form of $(M, g, \overline{J})$ is
$\rho = -\frac{1}{2}dd^c\ln(e^{-A}) - dd^c\ln h + dd^c\ln\beta =$
$\frac{1}{2}dd^cA - d((\ln h)_x dy - (\ln h)_y dx) + d(\frac{\beta'}{\beta}\theta_3) =$
$-\Delta\ln h\frac{e^A}{h^2}\theta_1\wedge\theta_2 + (\alpha' - \frac{3}{2}\alpha^2)\theta_1\wedge\theta_2 + (\frac{\beta''}{\beta} - 2(\frac{\beta'}{\beta})^2 + \frac{1}{2}\alpha' - \frac{\alpha\beta'}{2\beta})\theta_4\wedge\theta_3.$

Consequently E_1, E_2, E_3, E_4 are eigenfields of the Ricci tensor. Since

$$\rho = (-(\Delta \ln h)e^A \frac{1}{h^2} - \frac{3}{2}\alpha^2 +$$
$$\alpha')\theta_1 \wedge \theta_2 + (\frac{1}{2}\alpha' + \frac{\beta''}{\beta}$$
$$-2(\frac{\beta'}{\beta})^2 - \frac{\beta'\alpha}{2\beta})\theta_4 \wedge \theta_3$$

, it follows that $\frac{1}{2}\tau = -\frac{(\Delta \ln h)e^A}{h^2} - 2\alpha^2 + 2\alpha' + \frac{\beta''}{\beta} - 2(\frac{\beta'}{\beta})^2$.

Remark

It is easy to show that $X = \partial_t$ is a Hamiltonian Killing vector field. It has also special Kähler Ricci potential in the sense of Derdziński-Maschler. ([D-M-1], [D-M-2]). In fact $\bar{J}X = -\frac{1}{\beta}E_4 = -\nabla u$ where u is a potential of X so $X = \bar{J}\nabla u$ and $u(z) = \int \frac{1}{\beta}dz$. Now since $\nabla_{E_4}E_4 = 0, \nabla_{E_3}E_4 = -(-\frac{\alpha'}{\alpha} + \alpha)E_3$ it follows that X, JX are eigenvector fields of the Hessian H^u.

The semi-symmetric Kähler surfaces are classified in [J-5]. Note that we can take $kf = H\sin\frac{1}{2}(t + \phi(x,y)), mf = H\cos\frac{1}{2}(t + \phi(x,y))$ for a certain function ϕ and changing coordinates $x' = x, y' = y, z' = z, t' = t + \phi(x,y)$ we can assume that $kf = H\sin\frac{1}{2}t, fm = H\cos\frac{1}{2}t$. We also have in these coordinates $\alpha = -\frac{2}{z}$.

Theorem

Let $U \subset \mathbb{R}^2$ and let $g_\Sigma = h^2(dx^2 + dy^2)$ be a Riemannian metric on U, where $h : U \to \mathbb{R}$ is a positive function $h = h(x, y)$. Let $\omega_\Sigma = h^2 dx \wedge dy$ be a volume form of $\Sigma = (U, g)$. Let $M = U \times N$, where $N = \{(z, t) \in \mathbb{R}^2 : z < 0\}$. Let us define the metric g on M by $g(X, Y) = z^2 g_\Sigma(X, Y) + \theta_3(X)\theta_3(Y) + \theta_4(X)\theta_4(Y)$, where $\theta_3 = -\frac{z}{2}dt + (\cos\frac{1}{2}tH(x, y) + zl_2(x, y))dx + (-\sin\frac{1}{2}tH(x, y) + zn_2(x, y))dy$ and $\theta_4 = dz - \sin\frac{1}{2}tH(x, y)dx - \cos\frac{1}{2}tH(x, y)dy$ and the function H satisfies the equation

$\Delta \ln H = (\ln H)_{xx} + (\ln H)_{yy} = 2h^2$ *on* U,

$l_2 = -(\ln H)_y, n_2 = (\ln H)_x$.

Theorem

Then (M,g) admits a Kähler structure $\overline{J}$ with the Kähler form $\overline{\Omega} = z^2\omega_\Sigma + \theta_4 \wedge \theta_3$ and a Hermitian structure J with the Kähler form $\Omega = z^2\omega_\Sigma + \theta_3 \wedge \theta_4$. The Ricci tensor of (M,g) is J-invariant and J is not locally conformally Kähler. The Lee form of (M,g,J) is $\theta = -\alpha\theta_4$, where $\alpha = -\frac{2}{z}$. The scalar curvature of (M,g) is $\tau = -\frac{2\Delta \ln h}{z^2h^2} - \frac{8}{z^2}$ and is equal to conformal curvature κ of (M,g,J). $(M,g,\overline{\Omega})$ is a semi-symmetric QCH Kähler surface.

Proof Let us take a coordinate system such that
$E_1 = \frac{1}{f}\partial_x + k\partial_z + l\partial_t, E_2 = \frac{1}{f}\partial_y + m\partial_z + n\partial_t, E_3 = \alpha\partial_t, E_4 = \partial_z$.
Then $\theta_1 = fdx, \theta_2 = fdy, \theta_4 = dz - (fk)dx - (fm)dy, \theta_3 =$

$\frac{1}{\alpha}dt - (\frac{1}{\alpha}lf)dx - (\frac{1}{\alpha}nf)dy$. Let $\alpha = -\frac{2}{z}$. Then $E_4 \ln \alpha = \frac{1}{2}\alpha$ and

$$E_1 \ln \alpha = -\frac{k}{z}, E_2 \ln \alpha = -\frac{m}{z}.$$

We have
$[E_1, E_4] = \frac{f_z}{f^2}\partial_x - k_z\partial_z - l_z\partial_t = -\frac{\alpha}{2f}\partial_x - \frac{\alpha}{2}k\partial_z - \frac{\alpha}{2}l\partial_t - \alpha\frac{m}{z}\partial_t$.
Hence $k_z = -\frac{k}{z}, l_z = -\frac{l}{z} - \frac{2m}{z^2}, f_z = \frac{f}{z}$. This implies
$f = zh(x, y), k = \frac{k_1(x,y,t)}{z}$. On the other hand
$[E_1, E_3] = -\alpha k_t\partial_z - \alpha l_t\partial_t + k\alpha_z\partial_t = \frac{m}{z}\partial_z$ and

$$l_t = -\frac{k}{z}, 2k_t = m.$$

This yields

$$l = \frac{2m}{z} + \frac{l_1(x, y)}{z}.$$

Similarly $[E_2, E_3] = m\alpha_z\partial_t - \alpha m_t\partial_z - \alpha n_t\partial_t = -\frac{k}{z}\partial_z$

$$2m_t = -k, n_t = -\frac{m}{z}.$$

Since
$[E_2, E_4] = -m_z\partial_z - n_z\partial_t + \frac{f_z}{f^2}\partial_y = -\frac{\alpha}{2f}\partial_y - \frac{1}{2}\alpha m\partial_z - \frac{1}{2}\alpha n\partial_t + \alpha\frac{k}{z}\partial_t$,
we get $m_z = -\frac{m}{z}$. Hence $m = \frac{m_1(x,y,t)}{z}$ and

$$n = -\frac{2k}{z} + \frac{n_1(x,y)}{z}.$$

Let us take $kf = \sin\frac{1}{2}tH(x,y), mf = \cos\frac{1}{2}tH(x,y)$, where
$f = zh(x,y)$ and $\theta_1 \wedge \theta_2 = z^2h^2 dx \wedge dy$. Then
$lf = \frac{2}{z}\cos\frac{1}{2}tH(x,y) + 2l_2(x,y), nf = -\frac{2}{z}\sin\frac{1}{2}tH(x,y) + 2n_2(x,y)$.
Hence

$$\theta_3 = -\frac{z}{2}dt + (\cos\frac{1}{2}tH(x,y) + zl_2(x,y))dx + (-\sin\frac{1}{2}tH(x,y) + zn_2(x,y))dy$$

and

$$\theta_4 = dz - \sin\frac{1}{2}tH(x,y)dx - \cos\frac{1}{2}tH(x,y)dy.$$

Now we prove that

$$d\theta_3 = -\alpha\theta_1 \wedge \theta_2 - E_2 \ln\alpha\theta_1 \wedge \theta_4$$
$$+E_1 \ln\alpha\theta_2 \wedge \theta_4 + \frac{1}{2}\alpha\theta_3 \wedge \theta_4$$

and $d\theta_4 = E_2 \ln\alpha\theta_1 \wedge \theta_3 - E_1 \ln\alpha\theta_2 \wedge \theta_3$ if
$l_2 = -(\ln H)_y, n_2 = (\ln H)_x$, where $\Delta \ln H = 2h^2$.
In fact,

$$-\alpha\theta_1 \wedge \theta_2 - E_2 \ln\alpha\theta_1 \wedge \theta_4 + E_1 \ln\alpha\theta_2 \wedge \theta_4$$

$$+\frac{1}{2}\alpha\theta_3 \wedge \theta_4 = -\alpha f^2 dx \wedge dy$$

$$+\frac{mf}{z}dx \wedge (dz - \sin\frac{1}{2}tHdx - \cos\frac{1}{2}tHdy)$$

$$-\frac{kf}{z}dy \wedge (dz - \sin\frac{1}{2}tHdx - \cos\frac{1}{2}tHdy) + \frac{1}{2}\alpha(-\frac{z}{2}dt +$$

$$(\cos\frac{1}{2}tH + zl_2)dx + (-\sin\frac{1}{2}tH + zn_2)dy) \wedge (dz - \sin\frac{1}{2}tHdx$$

$$-\cos\frac{1}{2}tHdy) = \frac{2f^2}{z}dx \wedge dy - \frac{1}{z}\cos^2\frac{1}{2}tH^2 dx \wedge dy$$

$$+\frac{1}{z}\sin^2\frac{1}{2}tH^2 dy \wedge dx - \frac{1}{z}\sin\frac{1}{2}tHdy \wedge dz$$

$$+\frac{1}{z}\cos\frac{1}{2}tHdx \wedge dz - \frac{1}{z}(-\frac{z}{2}dt \wedge dz + \frac{z}{2}\sin\frac{1}{2}tHdt \wedge dx$$

$$+\frac{z}{2}\cos\frac{1}{2}tHdt\wedge dy+\cos\frac{1}{2}tHdx\wedge dz+$$

$$zl_2dx\wedge dz-\cos^2\frac{1}{2}tH^2dx\wedge dy$$

$$-zl_2\cos\frac{1}{2}tHdx\wedge dy-\sin\frac{1}{2}tHdy\wedge dz+zn_2dy\wedge dz+$$

$$\sin^2\frac{1}{2}tH^2dy\wedge dx-zn_2\sin\frac{1}{2}tHdy\wedge dx)=$$

$$\frac{2f^2}{z}dx\wedge dy+\frac{1}{2}dt\wedge dz-\frac{1}{2}\sin\frac{1}{2}tHdt\wedge dx$$

$$-\frac{1}{2}\cos\frac{1}{2}tHdt\wedge dy-l_2dx\wedge dz+l_2\cos\frac{1}{2}tHdx\wedge dy$$

$$-n_2dy\wedge dz+n_2\sin\frac{1}{2}tHdy\wedge dx.$$

On the other hand

$$d\theta_3 = -\frac{1}{2}dz \wedge dt - \frac{1}{2}\sin\frac{1}{2}tHdt \wedge dx +$$

$$\cos\frac{1}{2}tH_y dy \wedge dx + l_2 dz \wedge dx$$

$$+ zl_{2y}dy \wedge dx - \frac{1}{2}\cos\frac{1}{2}tHdt \wedge dy$$

$$-\sin\frac{1}{2}tH_x dx \wedge dy + n_2 dz \wedge dy + zn_{2x}dx \wedge dy.$$

It is clear that θ_3 satisfies (2.3) if $l_2 = -(\ln H)_y, n_2 = (\ln H)_x$ and $\Delta \ln H = 2h^2$, where $\Delta f = f_{xx} + f_{yy}$, which means that $d(l_2 dx + n_2 dy) = 2\omega_\Sigma$.

Now

$E_2 \ln \alpha\theta_1 \wedge \theta_3 - E_1 \ln \alpha\theta_2 \wedge \theta_3 = -\frac{\cos\frac{1}{2}tH}{z}dx \wedge (-\frac{z}{2}dt - \sin\frac{1}{2}tHdy + zn_2 dy) + \frac{\sin\frac{1}{2}tH}{z}dy \wedge (-\frac{z}{2}dt + \cos\frac{1}{2}tHdx + zl_2 dx) = \frac{1}{2}\cos\frac{1}{2}tHdx \wedge$

$dt - \cos\frac{1}{2}tHn_2 dx \wedge dy - \frac{1}{2}\sin\frac{1}{2}tHdy \wedge dt + \sin\frac{1}{2}tHl_2 dy \wedge dx$.
On the other hand
$d\theta_4 = -\frac{1}{2}\cos\frac{1}{2}tHdt \wedge dx - \sin\frac{1}{2}tH_y dy \wedge dx + \frac{1}{2}\sin\frac{1}{2}tHdt \wedge dy - \cos\frac{1}{2}tH_x dx \wedge dy$.
It is clear that θ_4 satisfies (2.3) if $l_2 = -(\ln H)_y, n_2 = (\ln H)_x$.
Now we have $d\theta_1 \wedge \theta_2 = d(z^2h^2 dx \wedge dy) = 2zdzh^2 dx \wedge dy = \frac{2}{z}dz \wedge \theta_1 \wedge \theta_2 = -\alpha\theta_4 \wedge \theta_1 \wedge \theta_2$. From (2.3) it is also clear that $d(\theta_3 \wedge \theta_4) = -\alpha\theta_4 \wedge \theta_1 \wedge \theta_2$. Hence $d\overline{\Omega} = 0$, where $\overline{\Omega}(X, Y) = g(\overline{J}X, Y)$, and $\overline{\Omega} = \theta_1 \wedge \theta_2 - \theta_3 \wedge \theta_4$. Hence in view of Lemma $(M, g, \overline{J})$ is a Kähler surface and the Lee form of (M, g, J) is $\theta = -\alpha\theta_4$.
Now we show that ∂_t is a real holomorphic vector field. Note that

the opposite Kähler structure $\overline{J}$ satisfies

$$\overline{J}\partial_z = \alpha\partial_t, \tag{27}$$

$$\overline{J}\partial_t = -\frac{1}{\alpha}\partial_z, \tag{28}$$

$$\overline{J}\partial_x = \partial_y + fm\partial_z + fn\partial_t - \alpha fk\partial_t + \frac{1}{\alpha}fl\partial_z, \tag{29}$$

$$\overline{J}\partial_y = -\partial_x - fk\partial_z - fl\partial_t - \alpha fm\partial_t + \frac{1}{\alpha}fn\partial_z \tag{30}$$

. It is clear that $(L_{\partial_t}\overline{J})\partial_z = (L_{\partial_t}\overline{J})\partial_t = 0$. We also have $(L_{\partial_t}\overline{J})\partial_x = fm_t\partial_z + fn_t\partial_t - \alpha fk_t\partial_t + \frac{1}{\alpha}fl_t\partial_z = 0$ and similarly $(L_{\partial_t}\overline{J})\partial_y = 0$. Thus $L_{\partial_t}\overline{J} = 0$ and $X = \partial_t + i\frac{1}{\alpha}\partial_z$ is a holomorphic vector field such that $(dt - i\alpha dz)(X) = 2$. The form $\psi = \frac{1}{2}(dt - i\alpha dz)$ satisfies $\psi(X) = 1$ and $\psi(\partial_t - i\frac{1}{\alpha}\partial_z) = 0$. If Ψ is a holomorphic $(1,0)$ form such that $\Psi(X) = 1$, then

$\Psi - \psi = 0 mod\{dx, dy\}$. If $\overline{\Omega}$ is a Kähler form of Kähler manifold $(M, \overline{J}, g)$, then $\frac{1}{2}\overline{\Omega}^2 = z^2 h^2 \frac{1}{\alpha} dx \wedge dy \wedge dz \wedge dt$. We also have $\frac{i^2}{4}(dx + idy) \wedge \overline{(dx + idy)} \wedge \Psi \wedge \overline{\Psi} = \frac{\alpha}{4} dx \wedge dy \wedge dz \wedge dt$. Hence the Ricci form of (M, g, J) is
$\rho = -\frac{1}{2} dd^c \ln \frac{z^2 h^2 \frac{1}{\alpha} dx \wedge dy \wedge dz \wedge dt}{\frac{\alpha}{4} dx \wedge dy \wedge dz \wedge dt} = -\frac{1}{2} dd^c \ln h^2 - \frac{1}{2} dd^c \ln z^4 =$
$-dd^c \ln h - 2dd^c \ln z = -d\overline{J}d \ln h - 2d\overline{J}d \ln z =$
$-d((\ln h)_x dy - (\ln h)_y dx) - 2d(-(\ln H)_y dx + \ln H_x dy) =$
$-\Delta \ln h dx \wedge dy - 2\Delta(\ln H) dx \wedge dy$. Consequently E_1, E_2, E_3, E_4 are eigenfields of the Ricci tensor and E_3, E_4 corresponds to the eigenvalue 0. Since
$\rho = -(\Delta \ln h + 4h^2) dx \wedge dy = -\frac{\Delta \ln h + 4h^2}{z^2 h^2} z^2 h^2 dx \wedge dy$, it follows that $\frac{1}{2}\tau = -\frac{\Delta \ln h}{z^2 h^2} - \frac{4}{z^2}$.
Now we classify the remaining cases. First we assume $\alpha = 2a \tan az, a \in \mathbb{R}, a \neq 0$.

Theorem

Let $U \subset \mathbb{R}^2$ and let $g_\Sigma = h^2(dx^2 + dy^2)$ be a Riemannian metric on U, where $h : U \to \mathbb{R}$ is a positive function $h = h(x, y)$. Let $\omega_\Sigma = h^2 dx \wedge dy$ be a volume form of $\Sigma = (U, g)$. Let $M = U \times N$, where $N = \{(z, t) \in \mathbb{R}^2 : |z| < \frac{\pi}{2|a|}\}$. Let us define the metric g on M by $g(X, Y) = (\cos az)^2 g_\Sigma(X, Y) + \theta_3(X)\theta_3(Y) + \theta_4(X)\theta_4(Y)$, where $\theta_3 = \sin 2az dt - (\cos 2at \cos 2az H(x, y) +$ $\sin 2az l_2(x, y))dx - (-\sin 2at \cos 2az H(x, y) + \sin 2az n_2(x, y))dy$,

Theorem

$$\theta_4 = dz - \sin 2atH(x,y)dx - \cos 2atH(x,y)dy$$

and the function H satisfies the equation
$\Delta \ln H = (\ln H)_{xx} + (\ln H)_{yy} = 2a^2h^2 - 4a^2H^2$ *on* U,
$l_2 = -\frac{1}{2a}(\ln H)_y, n_2 = \frac{1}{2a}(\ln H)_x$. *Then* (M,g) *admits a Kähler structure* $\overline{J}$ *with the Kähler form* $\overline{\Omega} = (\cos az)^2\omega_\Sigma + \theta_4 \wedge \theta_3$ *and a Hermitian structure* J *with the Kähler form*
$\Omega = (\cos az)^2\omega_\Sigma + \theta_3 \wedge \theta_4$. *The Ricci tensor of* (M,g) *is* J*-invariant and* J *is not locally conformally Kähler. The Lee form of* (M,g,J) *is* $\theta = -\alpha\theta_4$, *where* $\alpha = 2a\tan az$.

Proof. Let us take a coordinate system such that
$E_1 = \frac{1}{f}\partial_x + k\partial_z + l\partial_t, E_2 = \frac{1}{f}\partial_y + m\partial_z + n\partial_t, E_3 = \frac{1}{\beta}\partial_t, E_4 = \partial_z$.
Then $\theta_1 = fdx, \theta_2 = fdy, \theta_4 = dz - (fk)dx - (fm)dy, \theta_3 =$

$\beta dt-(\beta lf)dx-(\beta nf)dy$. Let $\alpha=2a\tan az,\beta=\sin 2az$. Then

$$E_1\ln\alpha=\frac{2ak}{\sin 2az},E_2\ln\alpha=\frac{2am}{\sin 2az}.$$

We have
$[E_1,E_4]=\frac{f_z}{f^2}\partial_x-k_z\partial_z-l_z\partial_t=-\frac{\alpha}{2f}\partial_x-\frac{\alpha}{2}k\partial_z-\frac{\alpha}{2}l\partial_t+\frac{2am}{\sin^2 2az}\partial_t$.
Hence $k_z=a(\tan az)k,l_z=a(\tan az)l-\frac{2am}{sin^2 2az},f_z=-a\tan azf$.
This implies

$$f=\cos azh(x,y),k=\frac{k_1(x,y,t)}{\cos az}.$$

On the other hand
$[E_1,E_3]=-\frac{1}{\beta}(k_t\partial_z+l_t\partial_t)-k\frac{\beta_z}{\beta^2}\partial_t=-\frac{2am}{\sin 2az}\partial_z$ and

$$l_t=-2ak\cot 2az,k_t=2am.$$

This yields

$$l = m\cot 2az + \frac{l_1(x,y)}{\cos az}.$$

Similarly $[E_2, E_3] = -m\frac{\beta_z}{\beta^2}\partial_t - \frac{1}{\beta}(m_t\partial_z + n_t\partial_t) = -\frac{2ak}{\sin 2az}\partial_z$,

$$m_t = -2ak, n_t = -2am\cot 2az.$$

Since $[E_2, E_4] = -m_z\partial_z - n_z\partial_t + \frac{f_z}{f^2}\partial_t =$
$-\frac{\alpha}{2f}\partial_y - \frac{1}{2}\alpha m\partial_z - \frac{1}{2}\alpha n\partial_t - \frac{2ak}{\sin^2 2az}\partial_t$, we get $m_z = a\tan azm$.
Hence $m = \frac{m_1(x,y,t)}{\cos az}$ and

$$n = -k\cot 2az + \frac{n_1(x,y)}{\cos az}.$$

Let us take coordinates in which
$kf = \sin 2atH(x,y), mf = \cos 2atH(x,y)$, where $f = \cos azh(x,y)$

and $\theta_1 \wedge \theta_2 = (\cos az)^2 h^2 dx \wedge dy$. Then

$$lf = \cot 2az \cos 2atH(x,y) + l_2(x,y),$$
$$nf = -\cot 2az \sin 2atH(x,y) + n_2(x,y).$$

Hence

$$\theta_3 = \sin 2azdt - (\cos 2at \cos 2azH(x,y) + \sin 2azl_2(x,y))dx - (-\sin 2at \cos 2azH(x,y) + \sin 2azn_2(x,y))dy$$

and

$$\theta_4 = dz - \sin 2atH(x,y)dx - \cos 2atH(x,y)dy.$$

Now we prove that

$$d\theta_3 = -\alpha\theta_1 \wedge \theta_2 - E_2 \ln\alpha\theta_1 \wedge \theta_4 + E_1 \ln\alpha\theta_2 \wedge \theta_4 + (-E_4 \ln\alpha + \alpha)\theta_3 \wedge \theta_4$$

and $d\theta_4 = E_2 \ln \alpha \theta_1 \wedge \theta_3 - E_1 \ln \alpha \theta_2 \wedge \theta_3$ if $l_2 = -\frac{1}{2a}(\ln H)_y$, $n_2 = \frac{1}{2a}(\ln H)_x$, where $\Delta \ln H = -4a^2H^2 + 2a^2h^2$.
In fact,

$$
\begin{gathered}
-\alpha\theta_1 \wedge \theta_2 - E_2 \ln \alpha \theta_1 \wedge \theta_4 + E_1 \ln \alpha \theta_2 \wedge \theta_4 + \\
(-E_4 \ln \alpha + \alpha)\theta_3 \wedge \theta_4 = \\
-2a \tan azf^2 dx \wedge dy - \frac{2amf}{\sin 2az} dx \wedge (dz - \sin 2atHdx - \cos 2atHdy) + \\
\frac{2akf}{\sin 2az} dy \wedge (dz - \sin 2atHdx - \cos 2atHdy) \\
-\frac{2a \cos 2az}{\sin 2az}(\sin 2azdt - \cos 2az \cos 2atHdx \\
-\sin 2azl_2 dx + \sin 2at \cos 2azHdy - \sin 2azn_2 dy) \\
\wedge(dz - \sin 2atHdx - \cos 2atHdy)
\end{gathered}
$$

$$= -a\sin 2azh^2 dx \wedge dy +$$

$$\frac{2a}{\sin 2az}\cos^2 2atH^2 dx \wedge dy$$

$$-\frac{2a}{\sin 2az}\sin^2 2atH^2 dy \wedge dx + \frac{2a\sin 2atH}{\sin 2az} dy \wedge dz -$$

$$\frac{2a\cos 2atH}{\sin 2az} dx \wedge dz - \frac{2a\cos 2az}{\sin 2az}(\sin 2azdt \wedge dz$$

$$-\sin 2az \sin 2atHdt \wedge dx - \sin 2az\cos 2atHdt \wedge dy$$

$$-\cos 2az\cos 2atHdx \wedge dz - \sin 2azl_2 dx \wedge dz$$

$$+\cos 2az\cos^2 2atH^2 dx \wedge dy + \sin 2azl_2\cos 2atHdx \wedge dy +$$

$$\cos 2az\sin 2atHdy \wedge dz$$

$$-\sin 2azn_2 dy \wedge dz - \cos 2az\sin^2 2atH^2 dy \wedge dx$$

$$+n_2\sin 2az\sin 2atHdy \wedge dx).$$

On the other hand

$$d\theta_3 =$$
$$2a\cos 2az dz \wedge dt + 2a\sin 2az\cos 2at H dz \wedge dx +$$
$$2a\cos 2az\sin 2at H dt \wedge dx -$$
$$\cos 2az\cos 2at H_y dy \wedge dx - 2a l_2 \cos 2az dz \wedge dx - \sin 2az l_{2y} dy \wedge dx$$
$$+2a\cos 2az\cos 2at H dt \wedge dy - 2a\sin 2at\sin 2az H dz \wedge dy +$$
$$\sin 2at\cos 2az H_x dx \wedge dy - 2a\cos 2az n_2 dz \wedge dy$$
$$-\sin 2az n_{2x} dx \wedge dy.$$

It is clear that θ_3 satisfies (2.3) if $l_2 = -\frac{1}{2a}(\ln H)_y, n_2 = \frac{1}{2a}(\ln H)_x$ and $\Delta \ln H = 2a^2h^2 - 4a^2H^2$, where $\Delta f = f_{xx} + f_{yy}$.

Now

$$E_2 \ln \alpha\theta_1 \wedge \theta_3 - E_1 \ln \alpha\theta_2 \wedge \theta_3 = \frac{2a \cos 2atH}{\sin 2az} dx$$
$$\wedge(\sin 2azdt + \sin 2at \cos 2azHdy - \sin 2azn_2 dy) - \frac{2a \sin 2atH}{\sin 2az} dy$$
$$\wedge(\sin 2azdt - \cos 2az \cos 2atHdx - \sin 2azl_2 dx)$$
$$= 2a(\cos 2atHdx \wedge dt - \cos 2atHn_2 dx \wedge dy$$
$$+ \frac{\cos 2at \sin 2at \cos 2azH^2}{\sin 2az} dx \wedge dy -$$
$$\sin 2atHdy \wedge dt + \frac{\sin 2at \cos 2at \cos 2azH^2}{\sin 2az} dy \wedge dx + \sin 2atHl_2 dy \wedge dx).$$

On the other hand
$d\theta_4 = -2a \cos 2atHdt \wedge dx - \sin 2atH_y dy \wedge dx + 2a \sin 2atHdt \wedge dy - \cos 2atH_x dx \wedge dy.$

It is clear that θ_4 satisfies (2.3) if $l_2 = -\frac{1}{2a}(\ln H)_y, n_2 = \frac{1}{2a}(\ln H)_x$.
Now we have
$d\theta_1 \wedge \theta_2 = d(\cos az^2 h^2 dx \wedge dy) = -2a \sin az \cos azdzh^2 dx \wedge dy = -2a \tan azdz \wedge \theta_1 \wedge \theta_2 = -\alpha\theta_4 \wedge \theta_1 \wedge \theta_2$. From (2.3) it is also clear that $d\theta_3 \wedge \theta_4 = -\alpha\theta_4 \wedge \theta_1 \wedge \theta_2$. Hence $d\overline{\Omega} = 0$, where $\overline{\Omega}(X, Y) = g(\overline{J}X, Y)$ and $\overline{\Omega} = \theta_1 \wedge \theta_2 - \theta_3 \wedge \theta_4$. Hence in view of Lemma F $(M, g, \overline{J})$ is a Kähler surface and the Lee form of (M, g, J) is $\theta = -\alpha\theta_4$.
Now we show that $\mathcal{D} = span\{E_3, E_4\}$ is a conformal foliation.
Note that the opposite Kähler structure $\overline{J}$ satisfies
$\overline{J}\partial_z = \frac{1}{\beta}\partial_t, \overline{J}\partial_t = -\beta\partial_z, \overline{J}\partial_x = \partial_y + fm\partial_z + fn\partial_t - \frac{1}{\beta}fk\partial_t + \beta fl\partial_z, \overline{J}\partial_y = -\partial_x - fk\partial_z - fl\partial_t - \frac{1}{\beta}fm\partial_t + \beta fn\partial_z$. Hence
$(L_{\partial_t}\overline{J})\partial_x, (L_{\partial_z}\overline{J})\partial_x, (L_{\partial_t}\overline{J})\partial_y, (L_{\partial_z}\overline{J})\partial_y \in \mathcal{D}$, which means that $L_\xi\overline{J}(\mathcal{D}^\perp) \subset \mathcal{D}$ for $\xi \in \Gamma(\mathcal{D})$. Thus foliation $\mathcal{D}$ is conformal and since $d\theta^- = 0$, it follows from [J-3] that $(M, g, \overline{J})$ is a QCH Kähler

surface.

Next we consider the remaining case $\alpha = -2a \coth az, a \in \mathbb{R}, a \neq 0.$

Theorem

Let $U \subset \mathbb{R}^2$ *and let* $g_\Sigma = h^2(dx^2 + dy^2)$ *be a Riemannian metric on* U*, where* $h : U \to \mathbb{R}$ *is a positive function* $h = h(x, y)$*. Let* $\omega_\Sigma = h^2 dx \wedge dy$ *be a volume form of* $\Sigma = (U, g)$*. Let* $M = U \times N$*, where* $N = \{(z, t) \in \mathbb{R}^2 : z < 0\}$*. Let us define the metric* g *on* M *by* $g(X, Y) = (\sinh az)^2 g_\Sigma(X, Y) + \theta_3(X)\theta_3(Y) + \theta_4(X)\theta_4(Y)$*, where*

$$\theta_3 = \sinh 2azdt - (-\sin 2at \cosh 2azH(x, y) + \sinh 2azl_2(x, y))dx - (\cos 2at \cosh 2azH(x, y) + \sinh 2azn_2(x, y))dy$$

and

$$\theta_4 = dz - \cos 2atH(x, y)dx - \sin 2atH(x, y)dy$$

and the function H *satisfies the equation*
$\Delta \ln H = (\ln H)_{xx} + (\ln H)_{yy} = 2a^2 h^2 + 4a^2 H^2$ *on* U*,*

Theorem

Then (M, g) admits a Kähler structure $\overline{J}$ with the Kähler form

$$\overline{\Omega} = (\sinh az)^2 \omega_\Sigma + \theta_4 \wedge \theta_3$$

and a Hermitian structure J with the Kähler form $\Omega = (\sinh az)^2 \omega_\Sigma + \theta_3 \wedge \theta_4$. The Ricci tensor of (M, g) is J-invariant and J is not locally conformally Kähler. The Lee form of (M, g, J) is $\theta = -\alpha\theta_4$, where $\alpha = -2a \coth az$.

Proof.Let us now take a coordinate system such that $E_1 = \frac{1}{f}\partial_x + k\partial_z + l\partial_t, E_2 = \frac{1}{f}\partial_y + m\partial_z + n\partial_t, E_3 = \frac{1}{\beta}\partial_t, E_4 = \partial_z$. Then $\theta_1 = fdx, \theta_2 = fdy, \theta_4 = dz - (fk)dx - (fm)dy, \theta_3 = \beta dt - (\beta lf)dx - (\beta nf)dy$. Let $\alpha = -2a \coth az, \beta = \sinh 2az$. Then

$$E_1 \ln \alpha = -\frac{2ak}{\sinh 2az}, E_2 \ln \alpha = -\frac{2am}{\sinh 2az}.$$

We have
$[E_1, E_4] = \frac{f_z}{f^2}\partial_x - k_z\partial_z - l_z\partial_t = -\frac{\alpha}{2f}\partial_x - \frac{\alpha}{2}k\partial_z - \frac{\alpha}{2}l\partial_t - \frac{2am}{\sinh^2 2az}\partial_t$.
Hence
$k_z = -a\coth azk, l_z = -a\coth azl + \frac{2am}{\sinh^2 2az}, f_z = a\coth azf$. This implies $f = \sinh azh(x,y), k = \frac{k_1(x,y,t)}{\sinh az}$. On the other hand
$[E_1, E_3] = -\frac{1}{\beta}(k_t\partial_z + l_t\partial_t) - k\frac{\beta_z}{\beta^2}\partial_t = \frac{2am}{\sinh 2az}\partial_z$ and

$$l_t = -2ak\coth 2az, k_t = -2am.$$

This yields

$$l = -m\coth 2az + \frac{l_1(x,y)}{\sinh az}.$$

Similarly $[E_2, E_3] = -m\frac{\beta_z}{\beta^2}\partial_t - \frac{1}{\beta}(m_t\partial_z + n_t\partial_t) = -\frac{2ak}{\sinh 2az}\partial_z$

$$m_t = 2ak, n_t = -2am\coth 2az.$$

Since

$$[E_2, E_4] = -m_z\partial_z - n_z\partial_t + \frac{f_z}{f^2}\partial_y =$$
$$-\frac{\alpha}{2f}\partial_y - \frac{1}{2}\alpha m\partial_z - \frac{1}{2}\alpha n\partial_t$$
$$+\frac{2ak}{\sinh^2 2az}\partial_t$$

, we get $m_z = -a\coth azm$. Hence $m = \frac{m_1(x,y,t)}{\sinh az}$ and

$$n = k\coth 2az + \frac{n_1(x,y)}{\sinh az}.$$

Let us take $kf = \cos 2atH(x,y), mf = \sin 2atH(x,y)$, where $f = \sinh azh(x,y)$ and $\theta_1 \wedge \theta_2 = (\sinh az)^2 h^2 dx \wedge dy$. Then

$$lf\beta = -\sin 2at \cosh 2azH + \sinh 2azl_2(x,y),$$
$$nf\beta = \cosh 2az \cos 2atH(x,y) + \sinh 2azn_2(x,y).$$

Hence

$$\theta_3 =$$
$$\sinh 2azdt - (-\sin 2at \cosh 2azH(x,y) + \sinh 2azl_2(x,y))dx -$$
$$(\cos 2at \cosh 2azH(x,y) + \sinh 2azn_2(x,y))dy$$

and

$$\theta_4 = dz - \cos 2atH(x,y)dx - \sin 2atH(x,y)dy.$$

Now we prove that

$$d\theta_3 =$$
$$-\alpha\theta_1 \wedge \theta_2 - E_2 \ln \alpha\theta_1 \wedge \theta_4$$
$$+E_1 \ln \alpha\theta_2 \wedge \theta_4 + (-E_4 \ln \alpha + \alpha)\theta_3 \wedge \theta_4$$

and $d\theta_4 = E_2 \ln \alpha\theta_1 \wedge \theta_3 - E_1 \ln \alpha\theta_2 \wedge \theta_3$ if
$l_2 = \frac{1}{2a}(\ln H)_y, n_2 = -\frac{1}{2a}(\ln H)_x$, where $\Delta \ln H = (4a^2 H^2 + 2a^2 h^2)$.

In fact,

$$-\alpha\theta_1 \wedge \theta_2 - E_2 \ln \alpha\theta_1 \wedge \theta_4 + E_1 \ln \alpha\theta_2 \wedge \theta_4 + (-E_4 \ln \alpha + \alpha)\theta_3 \wedge \theta_4 =$$

$$2a \coth azf^2 dx \wedge dy + \frac{2amf}{\sinh 2az} dx \wedge (dz - \sin 2atHdy) - \frac{2akf}{\sinh 2az} dy$$

$$\wedge(dz - \cos 2atHdx) - \frac{2a \cosh 2az}{\sinh 2az}(\sinh 2azdt +$$

$$\cosh 2az \sin 2atHdx - \sinh 2azl_2 dx - \cos 2at \cosh 2azHdy - \sinh 2azn_2)dy$$

$$\wedge(dz - \cos 2atHdx - \sin 2atHdy) = a \sinh 2azh^2 dx \wedge dy -$$

$$\frac{2a}{\sinh 2az} \sin^2 2atH^2 dx \wedge dy + \frac{2a}{\sinh 2az} \cos^2 2atH^2 dy \wedge dx$$

$$-\frac{2a\cos 2atH}{\sinh 2az}dy\wedge dz+\frac{2a\sin 2atH}{\sinh 2az}dx\wedge dz-$$
$$\frac{2a\cosh 2az}{\sinh 2az}(\sinh 2azdt\wedge dz$$
$$-\sinh 2az\cos 2atHdt\wedge dx-\sinh 2az\sin 2atHdt\wedge dy$$
$$+\cosh 2az\sin 2atHdx\wedge dz-\sinh 2azl_2dx\wedge dz$$
$$-\cosh 2az\sin^2 2atH^2dx\wedge dy+$$
$$\sinh 2azl_2\sin 2atHdx\wedge dy-\cosh z\cos 2atHdy\wedge dz$$
$$-\sinh 2azn_2dy\wedge dz+\cosh 2az\cos^2 2atH^2dy\wedge dx$$
$$+n_2\sinh 2az\cos 2atHdy\wedge dx).$$

On the other hand

$$
\begin{gathered}
d\theta_3 \\
= 2a\cosh 2az dz \wedge dt + 2a\sinh 2az \sin 2at H dz \wedge dx + \\
2a\cosh 2az \cos 2at H dt \wedge dx + \cosh 2az \sin 2at H_y dy \wedge dx \\
-2al_2 \cosh 2az dz \wedge dx - \sinh 2az l_{2y} dy \wedge dx + 2a\cosh 2az \sin 2at H dt \wedge dy \\
-2a\cos 2at \sinh 2az H dz \wedge dy - \cos 2at \cosh 2az H_x dx \wedge dy \\
-2a\cosh 2az n_2 dz \wedge dy - \sinh 2az n_{2x} dx \wedge dy .
\end{gathered}
$$

It is clear that θ_3 satisfies (2.3) if $l_2 = \frac{1}{2a}(\ln H)_y, n_2 = -\frac{1}{2a}(\ln H)_x$ and $\Delta \ln H = 2a^2h^2 + 4a^2H^2$, where $\Delta f = f_{xx} + f_{yy}$.

Now

$$E_2 \ln \alpha \theta_1 \wedge \theta_3 - E_1 \ln \alpha \theta_2 \wedge \theta_3 =$$

$$-2a\frac{\sin 2atH}{\sinh 2az} dx \wedge (\sinh 2azdt - \cos 2at \cosh 2azHdy - \sinh 2azn_2 dy)$$
$$+\frac{2a \cos 2atH}{\sinh 2az} dy \wedge (\sinh 2azdt + \cosh 2az \sin 2atHdx$$
$$- \sinh 2azl_2 dx)$$

$$= 2a(-\sin 2atHdx \wedge dt + \sin 2atHn_2 dx \wedge dy$$
$$+\frac{\cos 2at \sin 2at \cosh zH^2}{\sinh 2az} dx \wedge dy$$
$$+\cos 2atHdy \wedge dt + \frac{\sin 2at \cos 2at \cosh 2azH^2}{\sinh 2az} dy \wedge dx$$
$$-\cos 2atHl_2 dy \wedge dx) = -2a \sin 2atHdx \wedge dt$$
$$+2a \sin 2atHn_2 dx \wedge dy + 2a \cos 2atHdy \wedge dt$$
$$-2a \cos 2atHl_2 dy \wedge dx.$$

On the other hand
$d\theta_4 = 2a \sin 2atHdt \wedge dx - \cos 2atH_y dy \wedge dx - 2a \cos 2atHdt \wedge dy - \sin 2atH_x dx \wedge dy$.
It is clear that θ_4 satisfies (2.3) if $l_2 = \frac{1}{2a}(\ln H)_y, n_2 = -\frac{1}{2a}(\ln H)_x$.
Now we have
$d\theta_1 \wedge \theta_2 = d((\sinh az)^2 h^2 dx \wedge dy) = 2a \sinh az \cosh azh^2 dz \wedge dx \wedge$

$dy = -(-2a\coth az)dz \wedge \theta_1 \wedge \theta_2 = -\alpha\theta_4 \wedge \theta_1 \wedge \theta_2$. From (2.3) it is also clear that $d\theta_3 \wedge \theta_4 = -\alpha\theta_4 \wedge \theta_1 \wedge \theta_2$. Hence $d\overline{\Omega} = 0$, where $\overline{\Omega}(X, Y) = g(\overline{J}X, Y)$ and $\overline{\Omega} = \theta_1 \wedge \theta_2 - \theta_3 \wedge \theta_4$. Hence in view of Lemma H $(M, g, \overline{J})$ is a Kähler surface and the Lee form of (M, g, J) is $\theta = -\alpha\theta_4$. As above one can show that $(M, g, \overline{J})$ is a QCH Kähler surface.

Remark

Note that the generalized Calabi type Kähler surfaces which are not of Calabi type are fibered bundles over Σ and the fibers which are leafs of the foliation $\mathcal{D}$ have constant sectional curvature : 0 in the semi-symmetric case, positive $4a^2$ if $\alpha = 2a\tan az$, negative $-4a^2$ if $\alpha = -2a\coth az$.

References.

[B] Bryant R. *Bochner-Kähler metrics* J. Amer. Math. Soc.14 (2001) , 623-715. [A-C-G-1]V. Apostolov,D.M.J. Calderbank, P.

Gauduchon *The geometry of weakly self-dual Kähler surfaces* Compos. Math. 135, 279-322, (2003)

[A-C-G-2] V. Apostolov,D.M.J. Calderbank, P. Gauduchon*Ambitoric geometry I: Einstein metrics and extremal ambikähler structures* arxiv

[A-G]V. Apostolov, P. Gauduchon The Riemannian Goldberg-Sachs Theorem Internat. J. Math. vol.8, No.4, (1997),421-439

[Bes] A. L. Besse *Einstein manifolds*, Ergebnisse, ser.3, vol. 10, Springer-Verlag, Berlin-Heidelberg-New York, 1987.

[D-1] A. Derdziński, *Self-dual Kähler manifolds and Einstein* *manifolds of dimension four* , Compos. Math. 49,(1983),405-433

[D-2] A. Derdziński, *Examples of Kähler and Einstein self-dual metrics on complex plane* Seminar Arthur Besse 1978/79. par
[D-M-1] A. Derdziński, G. Maschler *Special Kähler-Ricci potentials*

on compact Kähler manifolds, J. reine angew. Math. 593 (2006), 73-116.

[D-M-2] A. Derdziński, G. Maschler *Local classification of conformally-Einstein Kähler metrics in higher dimensions*, Proc. London Math. Soc. (3) 87 (2003), no. 3, 779-819.

[G-M-1] G.Ganchev, V. Mihova *Kähler manifolds of quasi-constant holomorphic sectional curvatures*, Cent. Eur. J. Math. 6(1),(2008), 43-75.

[G-M-2] G.Ganchev, V. Mihova *Warped product Kähler manifolds* *and Bochner-Kähler metrics*, J. Geom. Phys. 58(2008), 803-824.

[J-1] W. Jelonek, Compact holomorphically pseudosymmetric Kähler manifolds Coll. Math.117,(2009),No.2,243-249.

[J-2] W.Jelonek *Kähler manifolds with quasi-constant holomorphic curvature*, Ann. Glob. Anal. and Geom, vol.36, p. 143-159,(2009)

[J-3] W. Jelonek, *Holomorphically pseudosymmetric Kähler metrics on* $\mathbb{CP}^n$ Coll. Math.127,(2012),No.1,127-131.

[J-4] W. Jelonek *Einstein Hermitian and anti-Hermitian 4-manifolds* Ann. Polon. Math. 81.1 (2003), 7-24.

[J-5] W. Jelonek*QCH Kähler surfaces II* arxiv

[J-6] W. Jelonek *Kähler surfaces with quasi constant holomorphic curvature* Glasgow Math.J.58,(2016), 503-512.

[J-7]W. Jelonek *Semi-symmetric Kähler surfaces* Colloq. Math.J.148,(2017), 1-12.

[J-8] W. Jelonek *Complex foliations and Kähler QCH surfaces* Colloq. Math. 156,(2019),229-242.

[J-M] W. Jelonek, E. Mulawa *Generalized Calabi type Kähler surfaces* arxiv (2019)

[Jen] G.R.Jensen *Homogeneous Einstein manifolds of dimension four* J. Diff. Geom. 3,(1969) 309-349.

[Ko] O. Kowalski *Generalized symmetric spaces* Lecture Notes in Math. 805, Springer, New York,1980.

[O] Z. Olszak, *Bochner flat Kählerian manifolds with a certain condition on the Ricci tensor* Simon Stevin 63, (1989),295-303

[K-N] S. Kobayashi and K. Nomizu *Foundations of Differential*

Geometry, vol.2, Interscience, New York 1963