Canonical structures on homogeneous Φ-spaces and their applications

Vitaly V. Balashchenko
Belarusian State University, Minsk, Belarus
balashchenko@bsu.by; vitbal@tut.by

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Contents

1. Homogeneous Φ-spaces and canonical structures

Researchers who founded this theory: V.I.Vedernikov, N.A.Stepanov, A.Ledger, A.Gray, J.A.Wolf, A.S.Fedenko, O.Kowalski, L.V.Sabinin, V.Kac ...

Definition 1. *Let G be a connected Lie group, Φ its (analytic) automorphism, G^{Φ} the subgroup of all fixed points of Φ, and G^{Φ}_{o} the identity component of G^{Φ}. Suppose a closed subgroup H of G satisfies the condition*

$$G^{\Phi}_{o} \subset H \subset G^{\Phi}.$$

Then G/H is called a homogeneous Φ-space.

Homogeneous Φ-spaces include homogeneous symmetric spaces ($\Phi^2 = id$) and, more general, homogeneous k-symmetric spaces ($\Phi^k = id$).

For any homogeneous Φ-space G/H one can define the mapping

$$S_o = D\colon\ G/H \to G/H,\ xH \to \Phi(x)H.$$

It is evident that in view of homogeneity the "symmetry" S_p can be defined at any point $p \in G/H$.

We dwell on homogeneous k-symmetric spaces G/H only.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the corresponding Lie algebras for G and H, $\varphi = d\Phi_e$ the automorphism of $\mathfrak{g}$, where $\varphi^k = id$. Consider the linear operator $A = \varphi - id$. It is known (N.A.Stepanov) that G/H is a reductive space for which the corresponding *canonical reductive decomposition* is of the form:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},\ \mathfrak{m} = A\mathfrak{g}.$$

Besides, this decomposition is obviously φ–invariant. Denote by θ the restriction of φ to $\mathfrak{m}$. As usual, we identify $\mathfrak{m}$ with the tangent space $T_o(G/H)$ at the point $o = H$.

Definition 2 (VVB, N.A.Stepanov, 1991)**.** *An invariant affinor structure F (i.e. a tensor field of type $(1,1)$) on a homogeneous k-symmetric space G/H is called canonical if its value at the point $o = H$ is a polynomial in θ.*

Denote by $\mathcal{A}(\theta)$ the set of all canonical affinor structures on G/H. It is easy to see that $\mathcal{A}(\theta)$ is a commutative subalgebra of the algebra $\mathcal{A}$ of all invariant affinor structures on G/H.

Important property: All canonical structures are, in addition, invariant with respect to the ”symmetries” $\{S_p\}$ of G/H.

The first remarkable example: the canonical almost complex structure $J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$ on any homogeneous 3-symmetric space (N.A.Stepanov, J.Wolf, A.Gray, 1967-1968), and many-many applications up to now ...

The main conjecture: For homogeneous k-symmetric spaces $(k \geq 3)$ the algebra $\mathcal{A}(\theta)$ contains a rich collection of classical structures. – YES!!!

Concentrate on the following affinor structures of classical types:

almost complex structures J $(J^2 = -1)$;

almost product structures P $(P^2 = 1)$;

f-structures $(f^3 + f = 0)$ (K.Yano, 1963);

h-structures $(h^3 - h = 0)$ (V.F.Kirichenko, 1983).

f-structures and h-structures generalize the structures J and P.

We use the notation: $s = [\frac{k-1}{2}]$ (integer part), $u = s$ (for odd k), and $u = s + 1$ (for even k).

Theorem 1 (VVB, N.A.Stepanov, 1991, 1998)**.** *Let G/H be a homogeneous k-symmetric space.*

(1) *All non-trivial canonical f-structures on G/H can be given by the operators*

$$f = \frac{2}{k} \sum_{m=1}^{u} \left(\sum_{j=1}^{u} \zeta_j \sin \frac{2\pi m j}{k} \right) \left(\theta^m - \theta^{k-m} \right),$$

where $\zeta_j \in \{-1; 0; 1\}$, $j = 1, 2, \ldots, u$, and not all coefficients ζ_j are zero. In particular, suppose that $-1 \notin \operatorname{spec} \theta$. Then the polynomials f define canonical almost complex structures J iff all $\zeta_j \in \{-1; 1\}$.

(2) *All canonical h-structures on G/H can be given by the polynomials* $h = \sum_{m=0}^{k-1} a_m \theta^m$, *where:*

(a) *if* $k = 2n+1$, *then*

$$a_m = a_{k-m} = \frac{2}{k} \sum_{j=1}^{u} \xi_j \cos \frac{2\pi m j}{k};$$

(b) *if* $k = 2n$, *then*

$$a_m = a_{k-m} = \frac{1}{k} \left(2 \sum_{j=1}^{u} \xi_j \cos \frac{2\pi m j}{k} + (-1)^m \xi_n \right)$$

Here the numbers ξ_j *take their values from the set* $\{-1; 0; 1\}$ *and the polynomials* h *define canonical structures* P *iff all* $\xi_j \in \{-1; 1\}$.

We particularize the results for orders 3, 4, and 6 only.

1. For a homogeneous 3-symmetric space there are (up to sign) the following canonical structures of classical type on G/H:

$$J = \frac{1}{\sqrt{3}}(\theta - \theta^2), \; P = 1.$$

2. On a homogeneous 4-symmetric space there are (up to sign) the following canonical classical structures:

$$P = \theta^2, \; f = \frac{1}{2}(\theta - \theta^3), \; h_1 = \frac{1}{2}(1 - \theta^2), \; h_2 = \frac{1}{2}(1 + \theta^2).$$

3. On a homogeneous 6-symmetric space there are (up to sign) the following canonical f–structures:

$$f_1 = \frac{\sqrt{3}}{6}(\theta + \theta^2 - \theta^4 - \theta^5), \; f_2 = \frac{\sqrt{3}}{6}(\theta - \theta^2 + \theta^4 - \theta^5), \; f_3 = f_1 + f_2, \;\; f_4 = f_1 - f$$

We give another explanation for canonical structures f and P.

Let us write the corresponding φ-invariant decomposition of the Lie algebra $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus ... \oplus \mathfrak{m}_u,$$

where the subspaces $\mathfrak{m}_1, \ldots, \mathfrak{m}_u$ correspond to the spectrum of the operator θ.

Denote by f_i, where $i = 1, 2, \ldots, s$, the base canonical f-structure whose image is the subspace $\mathfrak{m}_i$. All the other canonical f-structures are algebraic sums of some base canonical f-structures.

The base canonical almost product structure P_i has $\mathfrak{m}_i$ as a $(+1)$-subspace, the others $\mathfrak{m}_j$, $j \neq i$ form (-1)-subspace.

2. The generalized Hermitian geometry.

2.1. Metric f-structures.

Let (M, g, J) be an almost Hermitian manifold. We recall main classes of *almost Hermitian structures*:

K	*Kähler structure*:	$\nabla J = 0$;
H	*Hermitian structure*:	$\nabla_X(J)Y - \nabla_{JX}(J)JY = 0$;
$\mathbf{G}_1$	AH-structure of class G_1, or G_1-*structure*:	$\nabla_X(J)X - \nabla_{JX}(J)JX = 0$;
NK	*nearly Kähler structure*, or NK-*structure*:	$\nabla_X(J)X = 0$.

It is well known (see, for example, Gray-Hervella 16 classes, 1980) that

$$\mathbf{K} \subset \mathbf{H} \subset \mathbf{G_1}; \quad \mathbf{K} \subset \mathbf{NK} \subset \mathbf{G_1}.$$

Now we consider some classes of *metric f-structures*.
A fundamental role in the geometry of metric f-manifolds is played by the *composition tensor* T (V.F.Kirichenko, 1986):

$$T(X,Y) = \frac{1}{4} f(\nabla_{fX}(f)fY - \nabla_{f^2X}(f)f^2Y),$$

where ∇ is the Levi-Civita connection of a (pseudo)Riemannian manifold (M,g), $\quad X,Y \in \mathfrak{X}(M)$.

Using this tensor T, the algebraic structure of a so-called *adjoint Q-algebra* in $\mathfrak{X}(M)$ can be defined by the formula:

$$X * Y = T(X,Y).$$

It gives the opportunity to introduce some classes of metric f-structures in terms of natural properties of the adjoint Q-algebra.

Kf	*Kähler f–structure*:	$\nabla f = 0$;
Hf	*Hermitian f–structure*:	$T(X, Y) = 0$, i.e. $\mathfrak{X}(M)$ is an abelian Q-algebra;
$\mathbf{G_1 f}$	f-structure of class G_1, or $G_1 f$*-structure*:	$T(X, X) = 0$, i.e. $\mathfrak{X}(M)$ is an anticommutative Q-algebr
Kill f	*Killing f-structure*:	$\nabla_X(f)X = 0$;
NKf	*nearly Kähler f-structure*, or $NK f$*-structure*:	$\nabla_{fX}(f)fX = 0$.

The following relationships between the classes mentioned are evident:

$$\mathbf{Kf} \subset \mathbf{Hf} \subset \mathbf{G_1 f}; \quad \mathbf{Kf} \subset \mathbf{Kill\ f} \subset \mathbf{NKf} \subset \mathbf{G_1 f}.$$

It is important to note that in the special case $f = J$ we obtain the corresponding classes of almost Hermitian structures (16 Gray-Hervella classes).

In particular, for $f = J$ the classes **Kill f** and **NKf** coincide with the well-known class **NK** of *nearly Kähler structures*.

2.3. Invariant metric f-structures on homogeneous manifolds

Any invariant metric f-structure on a reductive homogeneous space G/H determines the orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ such that $\mathfrak{m}_1 = Im\ f$, $\mathfrak{m}_2 = Ker\ f$.

Theorem 2. *(2001) Any invariant metric f-structure on a naturally reductive space $(G/H, g)$ is a $G_1 f$-structure.*

As a special case ($Ker\ f = 0$), it follows the theorem of E.Abbena-S.Garbiero).
We stress the particular role of homogeneous 4- and 5-symmetric spaces.

Theorem 3. *The canonical f-structure $f = \frac{1}{2}(\theta - \theta^3)$ on any naturally reductive 4-symmetric space $(G/H, g)$ is both a Hermitian f-structure and a nearly Kähler f-structure. Moreover, the following conditions are equivalent:*

1) f is a Kähler f-structure; 2) f is a Killing f-structure; 3) f is a quasi-Kähler f-structure; 4) f is an integrable f-structure; 5) $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}$; 6) $[\mathfrak{m}_1, \mathfrak{m}_2] = 0$; 7) G/H is a locally symmetric space: $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$.

Theorem 4. *Let* $(G/H, g)$ *be a naturally reductive* 5*-symmetric space,* f_1 *and* f_2*,* J_1 *and* J_2 *the canonical structures on this space. Then* f_1 *and* f_2 *belong to both classes* **Hf** *and* **NKf**. *Moreover, the following conditions are equivalent:*

1) f_1 *is a Kähler* f*-structure; 2)* f_2 *is a Kähler* f*-structure; 3)* f_1 *is a Killing* f*-structure; 4)* f_2 *is a Killing* f*-structure; 5)* f_1 *is a quasi-Kähler* f*-structure; 6)* f_2 *is a quasi-Kähler* f*-structure; 7)* f_1 *is an integrable* f*-structure; 8)* f_2 *is an integrable* f*-structure; 9)* J_1 *and* J_2 *are* NK*-structures; 10)* $[\mathfrak{m}_1, \mathfrak{m}_2] = 0$ *(here* $\mathfrak{m}_1 = Im\, f_1 = Ker\, f_2, \mathfrak{m}_2 = Im\, f_2 = Ker\, f_1$*); 11)* G/H *is a locally symmetric space:* $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$.

We formulate several recent general results:

Theorem 5 (A.Samsonov, 2011)**.** *Let* $(G/H, g)$ *be a homogeneous k-symmetric space with any "diagonal" metric g. Then, any base canonical f-structure f_i, with $i = 1, 2, \ldots, s$ on G/H is a nearly Kähler f-structure.*

Theorem 6 (A.Samsonov, 2011)**.** *Let* $(G/H, g)$ *be a homogeneous k-symmetric space with any "diagonal" metric g. Then, for any base canonical f-structure f_i on M, the following assertions hold:*

1) *if* $3i \neq k$, *then* f_i *belongs to the class* $\mathbf{Hf}$;
2) *if* $3i = k$, *then* $f_i \in \mathbf{Hf} \Leftrightarrow [\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{h}$.

Note that the above theorems generalize some known results obtained earlier for orders $k = 3, 4, 5$ (including the classical results of N.A.Stepanov and A.Gray for homogeneous 3-symmetric spaces).

Besides, there are invariant NKf-structures and Hf-structures on homogeneous spaces $(G/H, g)$, where the metric g is not naturally reductive. The example of such a kind can be realized on the 6-dimensional generalized Heisenberg group (N, g). These groups were introduced by A.Kaplan and studied by F.Tricerri, L.Vanhecke, J.Berndt and others.

Theorem 7. *The 6-dimensional generalized Heisenberg group* (N, g) *with respect to the canonical f-structure* $f = \frac{1}{2}(\theta - \theta^3)$ *of a homogeneous* Φ*-space of order* 4 *is both Hf- and NKf-manifold. This f-structure is neither Killing nor integrable on* (N, g)*.*

2.2. Left-invariant f-structures on 2-step nilpotent Lie groups.

Some results below were obtained jointly with Pavel Dubovik and Olga Radivanovich. We start with several important examples.

Example 1.

The 6–dimensional generalized Heisenberg group (N, g) (A.Kaplan,1983). It is the Riemannian homogeneous 6–symmetric space (N, g, Φ), and the left-invariant canonical almost Hermitian structure $J = f_3$ is a strictly G_1–structure (i.e. neither nearly Kähler nor Hermitian structure).

It should be mentioned that G_1–structures of such a kind have interesting applications in *heterotic strings* (P.Ivanov, S.Ivanov, 2005).

Example 2. Consider the 5–dimensional Heisenberg group $H(2, 1)$ as a Riemannian homogeneous 6–symmetric space in two ways. Then all the canonical f-structures f_i, $i = 1, \ldots, 4$ are Hermitian f-structures. Besides, the base f-structures f_1 and f_2 are integrable, but the other f-structures f_3 and f_4 are not integrable. In addition, f_1, f_2, f_3 are nearly Kähler f-structures, but f_4 is not.

We notice that the group $H(2,1)$ is used in constructing the 6-dimensional nilmanifold connected with the *heterotic equations* of motion in *string theory* (M.Fernandez, S.Ivanov, L.Ugarte, R.Villacampa, 2009).

General approach. Let G be a 2-step nilpotent Lie group, $\mathfrak{g}$ its Lie algebra, $Z(\mathfrak{g})$ the center of $\mathfrak{g}$. Consider a left-invariant metric f-structure on G with respect to a left-invariant Riemannian metric g on G.

Theorem 8 (VVB, P.Dubovik, 2013)**.** (*i*) *If* $Z(\mathfrak{g}) \subset Ker\, f$ *then* f *is a Hermitian* f*-structure, but it is not a Kähler* f*-structure.*

(*ii*) *If* $Im\, f \subset Z(\mathfrak{g})$ *then* f *is both a Hermitian and a nearly Kähler* f*-structure, but it is not a Kähler* f*-structure.*

Example 3. Let $H(n,1)$ be a $(2n+1)$-dimensional matrix Heisenberg group. We can consider $H(n,1)$ as a Riemannian homogeneous k–symmetric space, where k is even.

As an application of the previous theorem, we obtain

Theorem 9 (VVB, P.Dubovik, 2013)**.** *Any left-invariant canonical f-structure on a $(2n+1)$-dimensional matrix Heisenberg group $H(n,1)$ is a Hermitian f-structure, but it is not a Kähler f-structure.*

3.3. Left-invariant f-structures on filiform Lie groups.

Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension m. Let

$$C^0\mathfrak{g} \supset C^1\mathfrak{g} \supset \cdots \supset C^{m-2}\mathfrak{g} \supset C^{m-1}\mathfrak{g} = 0$$

be the descending central series of $\mathfrak{g}$, where

$$C^0\mathfrak{g} = \mathfrak{g}, C^i\mathfrak{g} = [\mathfrak{g}, C^{i-1}\mathfrak{g}], \quad 1 \leq i \leq m-1.$$

A Lie algebra $\mathfrak{g}$ is called *filiform* (first, M.Vergne; later M.Kerr, T.Payne, 2010) if $dimC^k\mathfrak{g} = m-k-1$ for $k = 1, \ldots, m-1$. A Lie group G is called *filiform* if its Lie algebra is filiform.

Note that the filiform Lie algebras have the maximal possible nilindex, that is $m-1$.

Basic examples of $(n+1)$-dimensional filiform Lie algebras:

1. The Lie algebra L_n:

$[X_0, X_i] = X_{i+1},\ i = 1, \ldots, n-1.$

2. The Lie algebra $Q_n(n = 2k+1)$:
$[X_0, X_i] = X_{i+1},\ i = 1, \dots, n-1,$
$[X_i, X_{n-i}] = (-1)^i X_n,\ i = 1, \dots, k.$

The classification of 6-dimensional nilpotent Lie algebras was obtained by V.V.Morozov (1958), there exist 32 types of such algebras.
We select from this list 5 filiform Lie algebras:
(1) The Lie algebra $\mathfrak{g} = L_5$:
$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_1, e_5] = e_6.$
(2) The Lie algebra $\mathfrak{g} = Q_5$:
$[e_1, e_2] = e_3, [e_1, e_5] = e_6, [e_2, e_3] = e_4, [e_2, e_4] = e_5, [e_3, e_4] = e_6.$
And the other three filiform Lie algebras.

We construct (VVB, P.Dubovik, 2016) a number of left-invariant Hermitian f-structures on all these 6-dimensional filiform Lie groups.

3.4. Left-invariant f-structures on the groups $H(p,r)$.

The following groups were introduced by M.Goze and Y.Haraguchi (1982):

$$H(p,r) = M_{1p} \times M_{pr} \times M_{1r},$$

where matrices M_{ij} have dimensions $1 \times p, p \times r, 1 \times r$ respectively.

The multiplication in $H(p,r)$:

$$(x,y,z)\,(x',y',z') = (x+x', y+y', z+z'+\frac{1}{2}(xy'-x'y)).$$

$H(p,r)$ is a $(rp+r+p)$-dimensional 2-step nilpotent Lie group, which can be equipped with the left-invariant Riemannian metric g. The particular case $H(p,1)$ (i.e. $r=1$) is exactly the matrix Heisenberg group.

Theorem 10. *(P.Piu, M.Goze, 1993) $(H(p,r),g)$ is naturally reductive if and only if $H(p,r)$ is a Heisenberg group (i.e. $r=1$).*

Denote by $\mathfrak{h}(p,r)$ the corresponding Lie algebra.

Question. Are there canonical f-structures on the groups $H(p,r)$?

Example. (VVB, O.Radivanovich, 2017) Consider the case $p = r = 2$, i.e. the 8-dimensional group $H(2,2)$. Lie brackets for the orthonormal basis in $\mathfrak{h}(2,2)$ are:

$$[e_1, e_5] = [e_2, e_7] = e_3, \quad [e_1, e_6] = [e_2, e_8] = e_4.$$

We construct 1-parameter set of metric automorphisms φ_α of order 4 of the Lie algebra $\mathfrak{h}(2,2)$. As a result, $H(2,2)$ is a Riemannian 4-symmetric space for any φ_α. It is interesting to note that all the canonical f_α–structures are non-integrable and belong to no one classes in the generalized Hermitian geometry. This is a first example of such a kind.

3. Homogeneous Riemannian geometry.

Riemannian almost product manifold (M, g, P) naturally admits two complementary mutually orthogonal distributions $\mathbf{V}$ (*vertical*) and $\mathbf{H}$ (*horizontal*) corresponding to the eigenvalues 1 and -1 of P, respectively. In accordance with the Naveira classification there are 36 classes of Riemannian almost product structures (8 types for each of distributions). Here we consider the following types of distributions (in terms of vertical ones):

$\boldsymbol{F}$ (foliation): $\nabla_A(P)B = \nabla_B(P)A$;

$\boldsymbol{AF}$ (anti-foliation): $\nabla_A(P)A = 0$;

$\boldsymbol{TGF}$ (totally geodesic foliation): $\nabla_A P = 0$,

where A and B are vertical vector fields.

It is known (O. Gil-Medrano) that the system of conditions $\boldsymbol{AF}$ and $\boldsymbol{F}$ is equivalent to the condition $\boldsymbol{TGF}$.

Now we concentrate on invariant almost product structures on Riemannian homogeneous manifolds.

Let $(G/H, g = \langle \cdot, \cdot \rangle, P)$ be a naturally reductive homogeneous space. It was proved before (VVB, 1998) that both vertical and horizontal distributions of this structure P are always of type $\boldsymbol{AF}$. Besides, these distributions may be of type $\boldsymbol{F}$ (hence, $\boldsymbol{TGF}$) under simple algebraic criteria.

It means that, in accordance with the Naveira classification, there are exactly three classes of invariant naturally reductive almost product structures. They are $(\mathbf{TGF}, \mathbf{TGF})$, $(\mathbf{TGF}, \mathbf{AF})$, $(\mathbf{AF}, \mathbf{AF})$.

We apply these results for canonical structures P on homogeneous k-symmetric spaces with the "diagonal" metrics.

Let G be a semisimple compact Lie group, B the Killing form of the Lie algebra $\mathfrak{g}$, G/H a homogeneous k-symmetric space. As above, consider the canonical decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus ... \oplus \mathfrak{m}_u,$$

where some subspaces can be trivial. We define the collection of "diagonal" Riemannian metrics on G/H by the formula

$$\langle X, Y \rangle = \lambda_1 B(X_1, Y_1) + ... + \lambda_u B(X_u, Y_u),$$

where $X, Y \in \mathfrak{g}$, $i = \overline{1, u}$, $X_i, Y_i \in \mathfrak{m}_i$ from the above decomposition, $\lambda_i \in \mathbb{R}, \lambda_i < 0$.

Theorem 11. *Any the base canonical distribution $\mathfrak{m}_i$, $\overline{1,u}$ on Riemannian k-symmetric space $(G/H, g = \langle\cdot,\cdot\rangle)$ is of type* $\boldsymbol{AF}$ *for all "diagonal" metrics g.*

Further, the distribution $\mathfrak{m}_i$ belongs to $\boldsymbol{F}$ *(hence,* $\boldsymbol{TGF}$*) if and only if one of the following cases is realized:*

(1) *The subspace $\mathfrak{m}_{2i}$ is trivial.*
(2) *The index i satisfies the condition $k = 3i$.*
(3) $[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{h}$.
(4) *If $k = 2n$, then $i = n$ (i.e. $\mathfrak{m}_n$ belongs to* $\boldsymbol{F}$*).*

It follows that for base canonical distributions the result doesn't depend on the function U. Note that for 4- and 5-symmetric spaces we have the decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$, i.e. all canonical distributions are base.

However, for other canonical distributions (e.g., $\mathfrak{m}_i \oplus \mathfrak{m}_j$) the situation is more complicated.

Example (homogeneous 6-symmetric spaces).

Here the decomposition is the following: $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$.

Theorem 12. *Let G/H be a homogeneous 6-symmetric space, where G is a compact semisimple Lie group. Suppose g is any diagonal Riemannian metric on G/H represented by the collection $(\lambda_1, \lambda_2, \lambda_3)$. Then:*

(1) $\mathfrak{m}_2$ *and* $\mathfrak{m}_3$ *are of type* ***TGF***.
(2) $\mathfrak{m}_1$ *belongs to type* ***TGF*** *if and only if* $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}$.
(3) $\mathfrak{m}_1 \oplus \mathfrak{m}_2$ *is of type* ***AF*** *if and only if any of the following two conditions is satisfied: (a)* $\lambda_1 = \lambda_2$; *(b)* $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1$.

(4) $\mathfrak{m}_1 \oplus \mathfrak{m}_2$ *is of type* $\boldsymbol{F}$ *if and only if* $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1$. *This is also a criterion for type* $\boldsymbol{TGF}$.

(5) $\mathfrak{m}_1 \oplus \mathfrak{m}_3$ *is of type* $\boldsymbol{AF}$ *if and only if any of the following two conditions is satisfied: (a)*$\lambda_1 = \lambda_3$; *(b)* $[\mathfrak{m}_1, \mathfrak{m}_3] = 0$.

(6) $\mathfrak{m}_1 \oplus \mathfrak{m}_3$ *is of type* $\boldsymbol{F}$ *if and only if both the following relations hold:* $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}$, $[\mathfrak{m}_1, \mathfrak{m}_3] = 0$. *This is also a criterion for type* $\boldsymbol{TGF}$.

(7) $\mathfrak{m}_2 \oplus \mathfrak{m}_3$ *is of type* $\boldsymbol{AF}$ *if and only if any of the following two conditions is satisfied: (a)*$\lambda_2 = \lambda_3$; *(b)* $[\mathfrak{m}_2, \mathfrak{m}_3] = 0$.

(8) $\mathfrak{m}_2 \oplus \mathfrak{m}_3$ *is of type* $\boldsymbol{F}$ *if and only if* $[\mathfrak{m}_2, \mathfrak{m}_3] = 0$. *This is also a criterion for type* $\boldsymbol{TGF}$.

This theorem gives the opportunity to characterize the Naveira classes for all combinations of the above canonical distributions. As an example, the canonical structure P_3 belongs to the class $(\boldsymbol{TGF},\ \boldsymbol{TGF})$ if and only if $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1$.

4. Elliptic integrable systems.

The the m-th elliptic integrable systems associated to a k'-symmetric space $N = G/G_o$ were introduced by C.L. Terng [Terng, Chuu-Lian *Geometries and symmetries of soliton equations and integrable elliptic equations* // Surveys on geometry and integrable systems, 401–488, Adv. Stud. Pure Math., 51, Math. Soc. Japan, Tokyo, 2008.] This approach was intensively investigated and greatly developped in the book: I. Khemar, *Elliptic integrable systems: a comprehensive geometric interpretation* // Memoirs of the AMS. 2012. V. 219, no. 1031. x+217 pp. (below we try to keep notations of the author). More exactly, various geometric objects, partially known and completely new, closely related to these systems were intensively investigated.

Let τ be an automorphism of order k' $(k' \geq 1)$ of a real Lie algebra $\mathfrak{g}$, $\mathfrak{g}^{\mathbb{C}} = \oplus_{j \in \mathbb{Z}_{k'}} \mathfrak{g}_j^{\mathbb{C}}$ the corresponding eigenvalue decomposition, where $\mathfrak{g}_j^{\mathbb{C}}$ is the eigenspace of $\tau^{\mathbb{C}}$ with respect to the eigenvalue $\omega_{k'}^j$, and $\omega_{k'}$ is a k'-th primitive root of unity. Using the automorphism τ one can construct (under the known assumptions) a k'-symmetric space $N = G/G_o$. Further, let L be a Riemann surface. Then the m-th elliptic integrable system (briefly, $(m, \mathfrak{g}, \tau)$-system) associated to $(\mathfrak{g}, \tau)$ (or, to $N = G/G_o$) can be written as a zero curvature equation

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0, \ \ \forall \lambda \in \mathbb{C}^*,$$

where $\alpha_\lambda = \sum_{j=0}^m \lambda^{-j} u_j + \lambda^j \bar{u}_j = \sum_{j=-m}^m \lambda^j \hat{\alpha}_j$ is a 1-form on the Riemann surface L taking values in the Lie algebra $\mathfrak{g}$, the "coefficient" u_j

is a 1-form on L with values in the eigenspace $\mathfrak{g}^{\mathbb{C}}_{-j}$. This integer m is called the *order* of the $(m, \mathfrak{g}, \tau)$-system (not the order in the sense of PDE!).

Briefly, there are several types of elliptic integrable systems. As an example, the *first* elliptic integrable system associated to a symmetric space (resp. to a Lie group) is the equation for harmonic maps $f : L \to G/G_o$ into this *symmetric space* (resp. to this Lie group), see e.g. [Dorfmeister, J.; Pedit, F.; Wu, H. Weierstrass type representation of harmonic maps into symmetric spaces. Comm. Anal. Geom. **6** (1998), no. 4, 633–668. ; Uhlenbeck, Karen. Harmonic maps into Lie groups: classical solutions of the chiral model. J. Differential Geom. **30** (1989), no. 1, 1–50.].

The *second* elliptic integrable systems associated to 4-symmetric spaces can be geometrically interpreted in term of vertically harmonic twistor lifts [see e.g. Burstall, Francis E.; Khemar, Idrisse. Twistors, *4-symmetric spaces* and integrable systems. Math. Ann. **344** (2009), no. 2, 451–461].

For geometrical interpretation of arbitrary $(m, \mathfrak{g}, \tau)$-systems the author proposed a number of *new notions, constructions*, methods and applied them to studying these systems. Specifically, a concept of *"generalized" twistor spaces* $\mathcal{Z}_{2k}(M)$ and *twistor lifts* were introduced.

Here the situation is more complicated (the fibre is a $2k$-symmetric space instead of a symmetric space). It should be noted that almost complex structures $\underline{\mathrm{J}}$, $\underline{\mathrm{J}}^*$ (for $k' = 2k + 1$) and f-structures $F^{[m]}$, F, F^* (for $k' = 2k$) effectively used throughout the book are *exactly only the particular examples* of a remarkable collection of the canonical structures on homogeneous k'-symmetric spaces in the above our sense.

Further, the notions of holomorphically harmonic and stringy harmonic maps were introduced. In some sense, these maps correspond to harmonic maps when the Levi-Civita connection is replaced by a metric connection with torsion. In this direction one of the main results is the variational interpretation of stringy harmonicity by means of a sigma model with a Wess-Zumino term.

The author gives new examples of integrable two-dimensional nonlinear sigma models in $(2k+1)$-symmetric spaces that are not symmetric spaces. We note that in this respect nearly Kähler and $\mathcal{G}_1$-structures from the Gray-Hervella classification of almost Hermitian structures are of special interest [see Gray, Alfred; Hervella, Luis M. The sixteen classes of almost Hermitian manifolds and their linear invariants. Ann. Mat. Pura Appl. (4) **123** (1980), 35–58; Friedrich, Thomas; Ivanov, Stefan. Parallel spinors and connections with skew-symmetric torsion in string theory. Asian J. Math. **6** (2002), no. 2, 303–335.].

The author considered generalized harmonic maps into f-manifolds. Using the natural splitting $TN = \mathcal{H} \oplus \mathcal{V}$ on the Riemannian manifold (N, h) with a metric f-structure F as well as an idea of a skew-symmetric torsion, new notions for metric f-manifolds (N, F, h) were introduced and studied. They are: *reductive metric f-manifolds, the extended Nijenhuis tensor, metric f-manifolds of global type $\mathcal{G}_1$, horizontally of type $\mathcal{G}_1$* and some others. This is a *new general approach to generalizing almost Hermitian geometry*. The other notions introduced here such as horizontally Hermitian and horizontally Kähler f-manifolds are close to the generalized Hermitian geometry in the above our sense.

5. Canonical structures of "metallic family".

Recently a new type of affinor structures was introduced. It was initiated by the quadratic equation $x^2 - x - 1 = 0$ for the Golden ratio (Golden section, Golden proportion, Divine ratio, ...). The positive root $\frac{1+\sqrt{5}}{2} = \phi$ of this equation is the Golden ratio (the Phidias number).

Definition 1. (M.Crasmareanu, C.-E.Hretcanu, 2008). Affinor structure F on a manifold M is called a *Golden structure* if $F^2 = F + id$.

This notion is a particular case of a general concept of a *polynomial structure* on M (S.Goldberg, K.Yano, 1970).

Further, fix two positive integers p and q. The positive solution $\sigma_{p,q}$ of the equation $x^2 - px - q = 0$ is called a (p, q)-*metallic number*. These numbers

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}$$

of the *metallic means family* were considered by Vera W. de Spinadel (1997 and later).

Some particular cases of the numbers from the metallic means family:

the golden mean $\phi = \frac{1+\sqrt{5}}{2}$ if $p = q = 1$;

the silver mean $\sigma_{2,1} = 1 + \sqrt{2}$ for $p = 2, q = 1$;

the bronze mean $\sigma_{3,1} = \frac{3+\sqrt{13}}{2}$ for $p = 3, q = 1$;

the copper mean $\sigma_{1,2} = 2$ for $p = 1, q = 2$ and so on.

It should be mentioned that many authors wrote about close relation of some metallic numbers to classical Fibonacci numbers, Pell numbers, design, fractal geometry, dynamical systems, quasicrystals etc.

Definition 2. (M.Crasmareanu, C.-E.Hretcanu, 2013). Affinor structure F on a manifold M is called a *metallic structure* if $F^2 = pF + qI$. For a Riemannian manifold (M, g) the structure F is called a *metallic Riemannian structure* if $g(FX, Y) = g(X, FY)$ for any vector fields X, Y.

Any almost product structure P induces two metallic structures on M:

$$F_1 = \frac{p}{2}I + (\frac{2\sigma_{p,q} - p}{2})P, \quad F_2 = \frac{p}{2}I - (\frac{2\sigma_{p,q} - p}{2})P.$$

Conversely, any metallic structure F on M determines two almost product structures:

$$P = \pm(\frac{2}{2\sigma_{p,q} - p}F - \frac{p}{2\sigma_{p,q} - p}I).$$

Moreover, P is a Riemannian almost product structure on (M, g) if and only if F_1, F_2 are metallic Riemannian structures.

Theorem 13. (2017) All canonical metallic structures F on homogeneous k-symmetric spaces G/H can be described by the formula

$$F = \tfrac{p}{2} I \pm (\tfrac{2\sigma_{p,q} - p}{2}) \sum_{m=0}^{k-1} a_m \theta^m, \text{ where:}$$

(1) if $k = 2n + 1$, then

$$a_m = a_{k-m} = \frac{2}{k} \sum_{j=1}^{u} \xi_j \cos \frac{2\pi m j}{k};$$

(2) if $k = 2n$, then

$$a_m = a_{k-m} = \frac{1}{k} \left(2 \sum_{j=1}^{u} \xi_j \cos \frac{2\pi m j}{k} + (-1)^m \xi_n \right)$$

Here $\xi_j \in \{-1; 1\}$.

Example. Homogeneous 4-symmetric spaces.

Here $P = \theta^2$. It follows that all canonical metallic structures are represented by the formula:

$$F = \frac{p}{2}I \pm (\frac{2\sigma_{p,q} - p}{2})\theta^2.$$

Main conclusion: The properties of the metallic structures F can be obtained from those of the corresponding almost product structures P. It follows that many previous results about invariant distributions and structures on homogeneous k-symmetric spaces and nilpotent Lie groups can be adapted and reformulated in terms of metallic structures.

Linear deformations of almost product structures

Let P be an almost product structure on a manifold M. An affinor structure

$$F = F(a, b) = a\,P + b\,I,$$

where $I = id$, $a, b \in \mathbb{R}$, $a \neq 0$, is called a linear deformation of the structure P.

The structure $F = F(a, b)$ is polynomial of degree 2, more exactly, it satisfies the following equation:

$$F^2 - 2b\,F + (b^2 - a^2)\,I = 0.$$

Evidently, all metallic structures are particular cases of the structures $F = F(a, b)$, namely:

$$a = \pm\frac{\sqrt{p^2 + 4q}}{2}, \quad b = \frac{p}{2}.$$

Theorem 14. *The Nijenhuis tensors of the structures P and $F = F(a,b) = a\,P + b\,I$, are proporsional, more exactly,*

$$N_F(X,Y) = a^2\,N_P(X,Y)$$

for any vector fields X and Y on M. It follows that the structures P and $F = F(a,b)$ are integrable or non-integrable simultaneously.

This theorem generalizes the results of many authors obtained for particular cases of metallic structures.

Other results: Canonical structures $F = F(a,b)$ on homogeneous k-symmetric spaces G/H (full description, compatibility with Riemannian metrics, the corresponding canonical distributions on G/H etc.), left-invariant structures $F = F(a,b)$ on particular nilpotent Lie groups.

Conclusion: Many results for the structures $F = F(a,b)$ can be obtained applying known results for the corresponding almost product structures P.

6. Symplectic geometry.

M a smooth manifold of dimension $2n$, Ω non-singular 2-form, then (M, Ω) is an *almost symplectic* manifold. If, in addition, $(d\Omega = 0)$, then M is called a symplectic manifold.

Recently pairs of compatible symplectic structures (bi-Poisson geometry) were considered when studying bi-Hamiltonian systems (Bolsinov A.V., Izosimov A.M., Tsonev D.M., Zhang P., 2016-2017). However, the case of almost symplectic structures $(d\Omega \neq 0)$ is also of essential interest, in particular, for an investigation of Hamiltonian vector fields and integrable almost-symplectic Hamiltonian systems (F. Fasso,N. Sansonetto, I. Vaisman)

If (g, J) is an almost Hermitian structure on M, then the tensor field $\Omega(X, Y) = g(X, JY)$ is skew-symmetric, i.e. (M, Ω) is an almost symplectic manifold. Here 2-form Ω is usually called the *fundamental form* (Kähler form) of the structure (g, J) on M. If $d\Omega = 0$, then (g, J) is called *almost Kähler* structure.

Our goal is to construct a collection of invariant almost symplectic structures on homogeneous k-symmetric spaces using canonical almost complex structures on these spaces. Note that the almost symplectic structure on homogeneous k-symmetric space G/H defined by the formula $\Omega_J(X, Y) = g(X, JY)$, where J is a canonical almost complex structure, is said to be *canonical* almost symplectic structure.

Theorem 15. *Let $(G/H, g)$ be a Riemannian homogeneous k-symmetric space, where $-1 \notin$ spec θ and the metric g is invariant with respect to both G and the "symmetries" $\{S_p\}$. Denote by s a number of different pairs k-th roots of unity from spec(θ). Then G/H admits 2^{s-1} different (up to sign) canonical almost symplectic structures, which are invariant with respect to both G and the "symmetries" $\{S_p\}$. In addition, any of these structures Ω_J is invariant with respect to all canonical almost complex structures as well as all canonical almost product structures on G/H.*

Example 1. Riemannian homogeneous 3-symmetric spaces admit the canonical almost symplectic structure Ω_J defined by the canonical almost complex structure $J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$.

Example 2. Riemannian homogeneous 5-symmetric spaces with maximal spectrum of the operator θ admit two canonical almost symplectic structures defined by the canonical almost complex structures J_1 and J_2.

Remark. It's well-known that many flag manifolds, Ledger-Obata spaces, some nilpotent Lie groups have the structure of Riemannian homogeneous k-symmetric spaces. It gives the opportunity to effectively construct canonical almost symplectic structures on these homogeneous manifolds.

We should mention other geometric structures on homogeneous k-symmetric spaces, which are of contemporary interest in geometry and topology:

- symplectic structures on k-symmetric spaces compatible with the corresponding "symmetries" of order k (A.Tralle, M.Bocheński);
- topology of homogeneous k-symmetric spaces, in particular, geometric formality (D. Kotschick, S. Terzić, Jelena Grbić).

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THANK YOU FOR YOUR ATTENTION!