Quantitative isoperimetric inequalities in geometric settings

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Dirac operators in differential geometry and global analysis
– in memory of Thomas Friedrich (1949–2018)

Będlewo, 08.10.2019.

De Giorgi's optimal isoperimetric inequality

For any set $E \subset \mathbb{R}^n$ of finite perimeter the classical isoperimetric inequality

$$\mathbf{A}_n(E) \le \gamma_n \mathbf{A}_{n-1}(\partial E)^{\frac{n}{n-1}}, \qquad \gamma_n = n^{-\frac{n}{n-1}} \omega_n^{-\frac{1}{n-1}}$$

holds true.

Equality holds if and only if $E = B_\varrho(x_o)$.

Almgren's optimal isoperimetric inequality

In 1986 Almgrem proved in the following higher co-dimension version (in the context of integer multiplicity rectifiable currents):

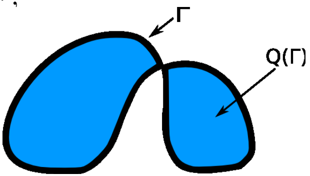
Theorem (Almgren 1986, Indiana Univ. Math.)

For any closed $(n-1)$-dimensional oriented surface T in $\mathbb{R}^{n+k}$ and any area minimizer Q_T with boundary $\partial Q_T = T$ there holds:

$$\mathbf{A}_n(Q_T) \leq \gamma_n \mathbf{A}_{n-1}(T)^{\frac{n}{n-1}},$$

where

$$\gamma_n = n^{-\frac{n}{n-1}} \omega_n^{-\frac{1}{n-1}}.$$



" = " $\Leftrightarrow$ Q_T is a flat n-dimensional disk D.

Flat disks have least boundary area

Corollary

Amongst closed $(n-1)$-dimensional oriented surfaces T in $\mathbb{R}^{n+k}$ spanning the same area (i.e. the area minimizing surfaces with boundary T possess the same area), flat spheres of dimension $(n-1)$ have least area. This means:

Suppose T is a closed $(n-1)$-dimensional oriented surface and Q_T some area minimzer with boundary $\partial Q_T = T$. Then for any disk D with $\mathbf{A}_n(D) = \mathbf{A}_n(Q_T)$ there holds

$$\mathbf{A}_{n-1}(\partial D) \le \mathbf{A}_{n-1}(T). \tag{1}$$

"$=$" $\Leftrightarrow$ *T is the boundary of some flat disk D with volume $\mathbf{A}_n(Q_T)$.*

Stability

A natural geometric question is about the stability of (1) in the following sense:

Suppose that there holds:

$$\mathbf{D}(T) := \frac{\mathbf{A}_{n-1}(T) - \mathbf{A}_{n-1}(\partial D)}{\mathbf{A}_{n-1}(\partial D)} \ll 1 \qquad \mathbf{A}_n(D) = \mathbf{A}_n(Q_T).$$

$\mathbf{D}(T)$ = renormalized isoperimetric gap.

$$\stackrel{?}{\Longrightarrow} \quad \text{Is } T \text{ close to } \partial D?$$

More precisely, we want to prove

$$\mathbf{D}(T) \geq c \operatorname{dist}(T, \partial D)^2$$

for some suitable distance of T to flat spheres.

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Another way to interpret stability

The isoperimetric quotient is defined by

$$\mu(T) := \frac{\mathbf{A}_{n-1}(T)^{\frac{n}{n-1}}}{\mathbf{A}_n(Q_T)}.$$

$\mu(T)$ is invariant under rigid motions.

Almgren proved that $T \mapsto \mu(T)$ attains its unique (i.e. unique up to rigid motions and homotheties) minimum in flat spheres, i.e.

$$\mu(\partial D) = \min\left\{\mu(T) : T \text{ is } (n-1)\text{-dim, oriented, } \subset \mathbb{R}^{n+k},\ \partial T = \varnothing\right\}$$

$$
\begin{aligned}
\boldsymbol{\mu}(T) &= \frac{\mathbf{A}_{n-1}(T)^{\frac{n}{n-1}}}{\mathbf{A}_n(Q_T)} \\
&= \boldsymbol{\mu}(\partial D)\left[1 + \frac{\mathbf{A}_{n-1}(T) - \mathbf{A}_{n-1}(\partial D)}{\mathbf{A}_{n-1}(\partial D)}\right]^{\frac{n}{n-1}} \quad (\mathbf{A}_n(Q_T) = \mathbf{A}_n(D)) \\
&= \boldsymbol{\mu}(\partial D)\big[1 + \mathbf{D}(T)\big]^{\frac{n}{n-1}} \\
&\geq \boldsymbol{\mu}(\partial D)\big[1 + \mathbf{D}(T)\big] \qquad (\mathbf{D}(T) \ll 1)
\end{aligned}
$$

We want to have a bound from below of $\mathbf{D}(T)$ by the square of a quantity which can be interpreted as a suitable distance of T to the $(n-1)$-dimensional flat spheres.

$$
\boldsymbol{\mu}(T) \geq \boldsymbol{\mu}(\partial D) + c\,\mathrm{dist}(T, \partial D)^2.
$$

Classics

- Bernstein (1905), Bonnesen (1924): Planar convex sets.
- Fuglede (1989): Convex sets, nearly spherical sets.
- Hall & Haymann & Weitsman (1991), Hall (1992): general sets in $\mathbb{R}^n$.

Nearly spherical sets

Consider sets $E \subset \mathbb{R}^n$ which are nearly spherical in the sense that

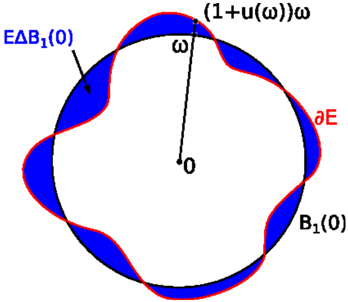
$$\partial E = \left\{(1 + u(\omega))\omega : \omega \in S^{n-1}\right\}$$

for some fuction $u: S^{n-1} \to \mathbb{R}$ satisfying

$$\|u\|_{C^1} \ll 1.$$

S^{n-1} should be the optimal sphere:

- $\mathbf{A}_n(E) = \mathbf{A}_n(B_1)$
- $\mathrm{bar}(E) = 0$



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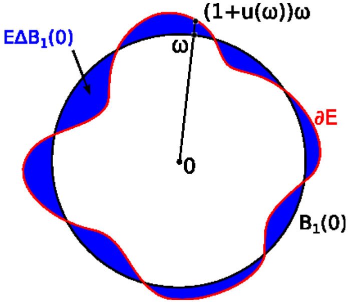

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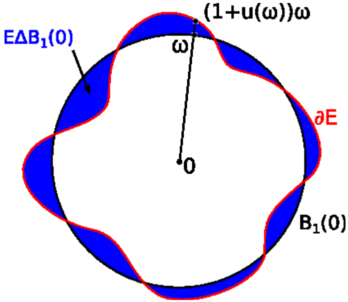

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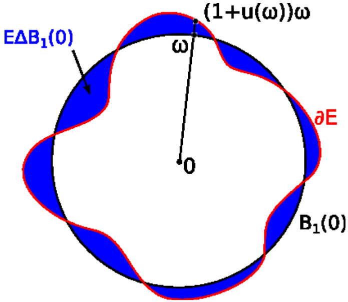

S^{n-1} should be the optimal sphere:

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Fuglede's theorem

Theorem (Fuglede)

There exist constants $\varepsilon_o(n) > 0$ and $c(n) < \infty$ such that there holds: For any nearly spherical set E whose volume is equal to the volume of the unit ball whose barycenter is at the origin and which satisfies $\|u\|_{C^1} \leq \varepsilon_o$ we have

$$\mathbf{A}_{n-1}(\partial E) - \mathbf{A}_{n-1}(\partial B_1) \geq c(n)\|u\|^2_{W^{1,2}(S^{n-1})}.$$

□

In particular, the isoperimetric gap controls the square of the measure of the symmetric difference

$$E \Delta B_1 := (E \setminus B_1) \cup (B_1 \setminus E),$$

since

$$\|u\|^2_{W^{1,2}(S^{n-1})} \geq \|u\|^2_{L^2(S^{n-1})} \geq c\|u\|^2_{L^1(S^{n-1})} \geq c|E \Delta B_1|^2.$$

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Theorem (Fusco-Maggi-Pratelli, Ann. Math. 2008)

For any set E of finite perimeter with $|E| = |B_\varrho|$ the following quantitative isoperimetric inequality

$$\mathbf{D}(E) = \frac{\mathbf{A}_{n-1}(\partial E) - n\omega_n\varrho^{n-1}}{n\omega_n\varrho^{n-1}} \geq c(n)\boldsymbol{\alpha}(E)^2$$

holds true, where

$$\alpha(E) := \min_{x_o} \frac{|E \Delta B_\varrho(x_o)|}{\varrho^n}$$

denotes the Fraenkel asymmetry. □

Different proofs:

- Figalli-Maggi-Pratelli (Invent. Math. 2010): New proof with arguments from optimal mass transport.
- Ciacalese-Leonardi (Arch. Rat. Mech. Anal. 2012): Proof via regularity by a selection principle.

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The isoperimetric gap

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$$\mathbf{D}(T) := \frac{\mathbf{A}_{n-1}(T) - \mathbf{A}_{n-1}(\partial D)}{\mathbf{A}_{n-1}(\partial D)},$$

where D a flat n-dimensional disk with

$$\mathbf{A}_n(D) = \mathbf{A}_n(Q_T) = \inf_{\partial P = T} \mathbf{A}_n(P)$$

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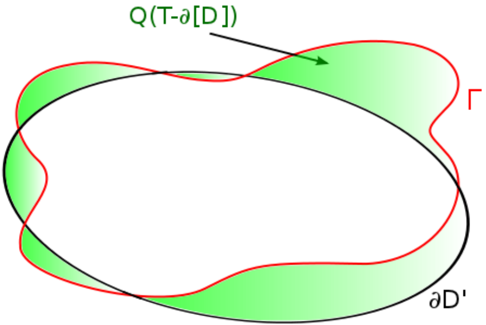
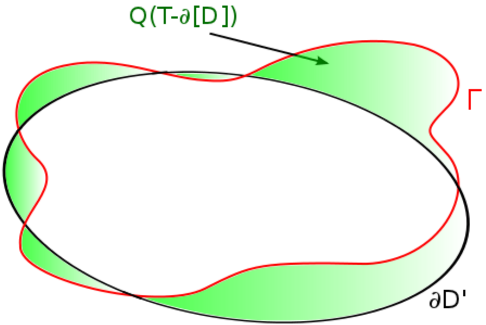


$A_n(Q_{T-\partial D})$ measures how close T and ∂D are.



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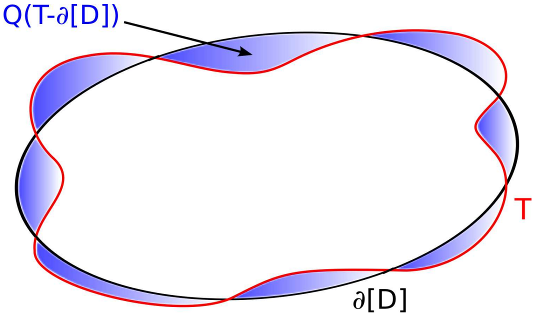

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To measure the deviation of T from round spheres spanning the same mass as T we shall take the infimum over all such spheres.

The asymmetry index is defined by

$$\mathbf{d}(T) := \inf_D \frac{\mathbf{A}_n(Q_{T-\partial D})}{\mathbf{A}_n(D)}$$

whenever T is a closed $(n-1)$-dimensional oriented surface in $\mathbb{R}^{n+k}$. The infimum is taken over all n-dimensional flat disks satisfying $\mathbf{A}_n(D) = \mathbf{A}_n(Q_T)$.

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Theorem (B-Duzaar-Fusco)

There exists a constant $c = c(n,k)$ *such that for any closed* $(n-1)$*-dimensional oriented surface* T *in* $\mathbb{R}^{n+k}$ *the quantitative isoperimetric inequality*

$$\mathbf{D}(T) \geq c\,\mathbf{d}(T)^2 \tag{2}$$

holds true. □

Remark: In the case $k = 0$, (2) reduces to the quantitative isoperimetric inequality of Fusco-Maggi-Pratelli, since in this case

$$\mathbf{d}(T) = \alpha(E) \quad \text{and} \quad \mathbf{D}(T) = \mathbf{D}(E).$$

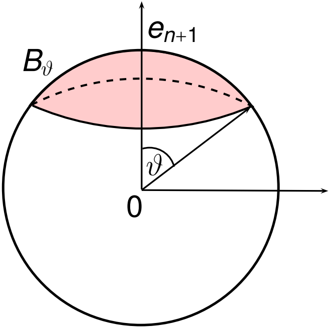
Isoperimetric inequality on the sphere

Theorem (E. Schmidt, Math. Z. 1943/44)

Geodesic balls are the unique isoperimetric sets on S^n, i.e. for any $E \subset S^n$ with $|E| = |B_\vartheta|$ with $0 < \vartheta < \pi$ there holds:

$$\mathbf{A}_{n-1}(\partial B_\vartheta) \leq \mathbf{A}_{n-1}(\partial E).$$

" = " $\Leftrightarrow$ E is a geodesic ball B_ϑ.



Stability

Is there stability for the isoperimetric sets on the sphere?

- Renormalized isoperimetric gap

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- Assymmety index: Fraenkel assymmetry

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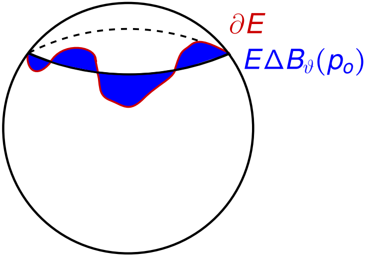

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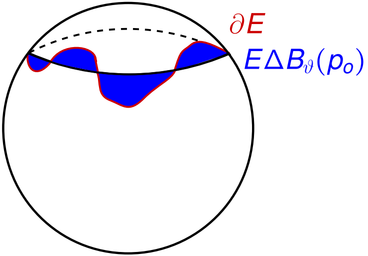

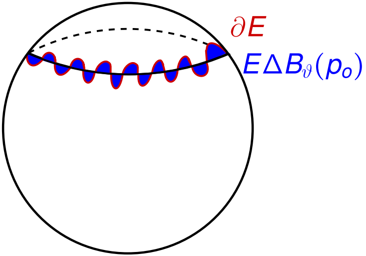

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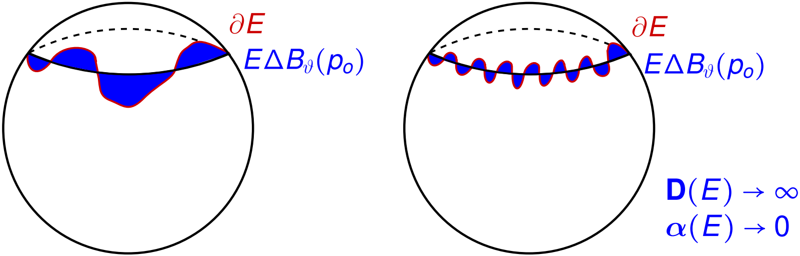


Strong form of the quantitative isop. inequality in $\mathbb{R}^n$

Theorem (Fusco-Julin)

For any set E of finite perimeter with $|E| = |B_\varrho|$ the following strong quantitative isoperimetric inequality

$$\mathbf{D}(E) = \frac{\mathbf{A}_{n-1}(\partial E) - n\omega_n \varrho^{n-1}}{n\omega_n \varrho^{n-1}} \geq c(n)\beta(E)^2$$

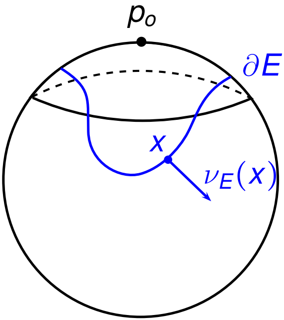
holds true, where

$$\beta(E) := \min_{x_o \in \mathbb{R}^n} \left(\frac{1}{\varrho^{n-1}} \int_{\partial^* E} |\nu_E(x) - \nu_{B_r(x_o)}(\pi_{x_o,\rho}(x))|^2 d\mathcal{H}^{n-1}(x) \right)^{\frac{1}{2}}$$

denotes the L^2-oscillation index. □

Stronger version of the assymmetry index on S^n

L^2-oscillation index



$$\beta(E) := \min_{p_o \in S^n} \left(\frac{1}{\mathcal{H}^{n-1}(B_\vartheta)} \int_{\partial^* E} \left| \nu_E(x) - \nu_{B_{\vartheta(x)}(p_o)}(x) \right|^2 d\mathcal{H}^{n-1}(x) \right)^{\frac{1}{2}}$$

$$\geq \alpha(E),$$

where $\vartheta(x) := \arccos(x \cdot p_o)$.

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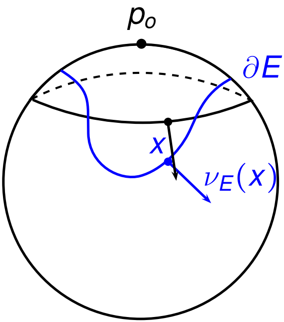

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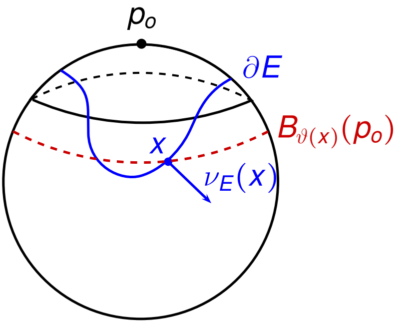

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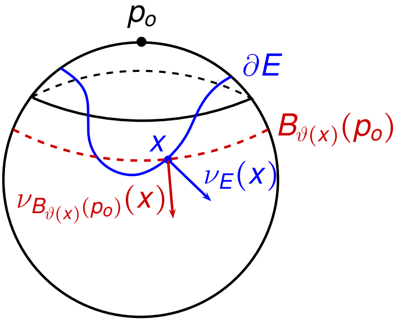

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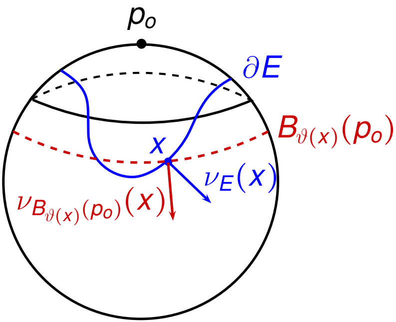

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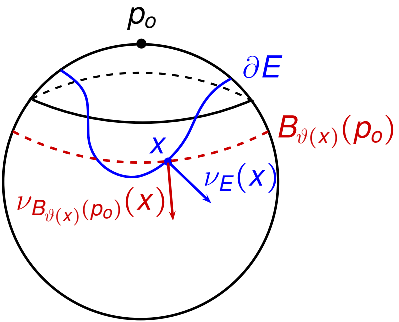

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Strong quantitative isoperimetric inequality on S^n

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Remark: The inequality is sharp in the sense that also the reverse inequality holds:

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Geodesic balls are the unique isoperimetric sets on $\mathbb{H}^n$.

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