Elliptic operators on homogeneous bundles over compact manifolds

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Dirac operators in Differential geometry and Global analysis

In memory of professor Thomas Friedrich

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Symplectic Dirac operators

Defined by Katharina Habermann [KH, AGAG 1995],[KH, CMP 1997]

$Mp(2n, \mathbb{R}) \to Sp(2n, \mathbb{R})$ 2 : 1 cover

Does not have any faithful finite dimensional representation.

It has unitary faithful representation on $S = L^2(\mathbb{R}^n)$

symplectic spinors

$\cdot : \mathbb{R}^{2n} \times S \to S$ symplectic Clifford multiplication, defined on a dense subset of S

metaplectic structures defined similarly ("same diagram")

If a symplectic manifold admits a metapl. structure, symplectic Dirac operator is defined locally as $\mathfrak{D}^S = \sum_{i=1}^{2n} e_i \cdot \nabla^S_{e_i} s$, for a lift ∇^S to S of a symplectic connection ∇^ω, $(e_i)_i$ a local symplectic basis

Further development (A. Klein, Brasch, Wyss, Krysl, Gutt, Rawnsley...)

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In case of lack of C^k-differentiability, principal bundle pictures – spaces $C^\infty(\mathcal{G}, \mathbb{H})^H$ – should be used. For $k = 0$, homeomorphism (wrt. to comp.-open topologies)

Classical and other known situations

- **Borel–Weil**: G semisimple Lie group, H Borel subgroup (connected solvable Lie subgroup of G), $\dim_c \mathbb{H} = 1$, such that the abelian part A of H acts by $A \ni X \mapsto \exp^{(\log(X),\lambda)}$ where λ is in a positive Weyl chamber and $(,)$ is the Killing form on $\mathfrak{g}^c$. Then $\Gamma^{\text{hol}}(G/H, \mathcal{H})$ is irreducible G-module with highest weight λ.

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- **Bott–Borel–Weil**: G semisimple, H Borel, $\mathbb{H}$ also one dimensional: Representations characterized by cohomology of sheaves resolving holomorphic sections (twisted Dolbeault operators). $H^{l(w)}(G/H, \mathcal{H})$ irrep if $w(\lambda + \rho)$ is integral and in the positive Weyl chamber.

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Generalize BBW in direction of rank

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Images shall be closed and ortho-complemented.

Theorem A[Krysl, AGAG 2014, 2015]: Let A be a C^*-algebra and $(\mathcal{H}^i \to M)_i$ be a sequence of finitely generated projective sufficiently differentiable Hilbert A-bundles over a compact manifold M. Suppose that **images of Laplacians** of an elliptic A-equivariant complex in $\Gamma(\mathcal{H}^i)$ **are closed**. Then the cohomology groups of the complex are finitely generated projective Hilbert A-modules.

Used analytic structures

C^*-algebras $= (A, *, || \, ||)$, such that $(A, *)$ is involutive antiautomorphism, $(A, || \, ||)$ is Banach algebra (ie., complete, norm submultiplicative), and C^*-identity $||aa^*|| = ||a||^2$ holds

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Examples: End($\mathbb{H}$) – linear operators on finite dimensional scalar product space, $B(\mathbb{H})$ – bounded operators on a Hilbert space, $C(\mathbb{H})$ – compact operators on a Hilbert space, $*$ is the adjoint, norm is the operator supremum norm; $C^0(X)$, X compact, $f^*(x) = \overline{f(x)}$, $||f|| = \sup_{x \in X} |f(x)|$ (norm).

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Not example: $L^1(G)$ – for G a Lie group of positive dimension and a Haar measure on G.

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Hilbert A-module = right A-module with product $(,)$ – A-valued, sesquilinear (in the second input), hermitian, A-invariant; the norm $||v|| = \sqrt{||(v,v)||_A}$ makes it a Banach space. Generalization of Hilbert space.
Note that $\mathbb{H}$ is generated just by one vector over $B(\mathbb{H})$ or $C(\mathbb{H})$.

Non-projective Hilbert C^*-module (Manuilov)
$A = C([0,1])$, $M = \ell_2(A)$ with ON-basis $(e_i)_i$

$$\phi_i = \begin{cases} 0, & \text{on } [0,1/i] \cup [1/i+1,1] \\ 1, & \text{for } a_i = 1/2(1/i + 1/i + 1) \\ \text{linear}, & \text{otherwise} \end{cases}$$

For $T(e_i) = \phi_i e_1$, the subspace 'graph of T' is not ortho-complementable in $(M \times \{0\}) \oplus (M \times \{1\})$.

Hodge theory for A-bundles, second version

Theorem B [Krysl, JGP]: If A is a C^*-algebra of compact operators (any C^*-subalgebra of $C(\mathbb{H})$) and $(\mathcal{H}^i \to N)_i$ is a sequence of finitely generated projective Hilbert A-bundles over a compact manifold N, then cohomology groups of an elliptic A-equivariant complex in $\Gamma(\mathcal{H}^i)$ are finitely generated projective Hilbert A-modules.

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- If $U(\mathbb{H})$ is equipped with the strong ($=$ uniform operator) operator topology, the Čech (i.e., sheaf) cohomology group is trivial $\Longrightarrow \exists\ J : \mathcal{H} \to M \times \mathbb{H}$ a fixed trivialization, defining a bundle atlas on $\mathcal{H}$.

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- Let $C(\mathbb{H})$ be the algebra of compact operators on $\mathbb{H}$. Considering a normalisation (unitary part of the polarization) of this trivialization, actions of $C(\mathbb{H})$ on $\mathcal{H}$ and on $\Gamma(\mathcal{H})$, and a $C(\mathbb{H})$-valued product on $\Gamma(\mathcal{H})$ may be defined making $\mathcal{H}$ a finitely generated projective Hilbert bundle

Cohomology of bundles induced from principal bundles

Theorem C [Krysl, CMP 2019]: Let M be a compact manifold, $\mathcal{G} \to M$ a principal G-bundle, and $\mathcal{H} \to M$ be a finitely generated projective Hilbert $C(\mathbb{H})$-bundle associated to $\mathcal{G}$ via a unitary representation of G on infinite dimensional $\mathbb{H}$. Let $D_J^\bullet$ be the deRham complex twisted by the trivial connection induced by (the normalization of) a trivialization J. Then the cohomology groups of this complex are projective Hilbert $C(\mathbb{H})$-modules with $C(\mathbb{H})$-rank equal to the Betti numbers of M.

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Proof. [Krysl, CMP 2019]

- [BG] Bakić, D., Guljaš, B., Operators on Hilbert H^*-modules, J. Oper. Theory 46 (2001), no. 1, 123–137.
- [DM] Dubois-Violette, M., Michor, P., Connections on central bimodules in noncommutative differential geometry, J. Geom. Phys. 20 (1996) Issue 2–3, 218–232.
- [FM] Fomenko, A., Mishchenko (Miščenko), A., The index of elliptic operators over C^*-algebras, Izv. Akad. Nauk SSSR, Ser. Mat. 43 (1979) No. 4, 831–859, 967, translation of the Russian orig.
- [GS] Griffiths, P., Schmid, W., Locally homogeneous complex manifolds. Acta Math. 123 (1969), 253–302.
- [KH, 1995] Habermann, K., The Dirac operator on symplectic spinors, Ann. Global Anal. Geom. 13 (1995), no. 2, 155–168.
- [KH, 1997] Habermann, K., Basic properties of symplectic Dirac operators, Comm. Math. Phys. 184 (1997), no. 3, 629–652.

- [HH] Habermann, K., Habermann, L., Introduction to symplectic Dirac operators, Lecture Notes in Mathematics 1887. Springer-Verlag, 2006.
- [Krysl, AGAG, 2015] Krýsl, S., Hodge theory for self-adjoint parametrix possessing complexes over C^*-algebras, Annals Glob. Anal. Geom. 47 (2015) No. 4, 359–372.
- [Krysl, JGP 2016] Krýsl, S., Elliptic complexes over C^*-algebras of compact operators, Journal of Geometry and Physics 101 (2016), 27–37.
- [Krysl, CMP 2019] Krýsl, S., Induced C^*-complexes in metaplectic geometry, Comm. Math. Phys. 365 (2019) No. 1, 61–91.
- [S] Schick, T., L^2-index theorems, KK-theory, and connections, New York J. Math. 11 (2005), 387–443.
- [ST] Solovyov (Solov'ëv), Y., Troitsky (Troickij), E., C^*-algebras and elliptic operators in differential topology.

Transl. of Mathem. Monographs, 192, AMS, Providence, Rhode-Island, 2001.

[W] Wong, H.-Wai, Cohomological induction in various categories and the maximal globalization conjecture, Duke Math. J. 96 (1999) No. 1, 1–27.