Lagrangian submanifolds of twistor spaces and superminimal surfaces

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7 October 2019

Dirac operators in differential geometry and global analysis – in memory of Thomas Friedrich (1949-2018)

Motivation

My interest comes from two particular cases:

$$\mathbb{C}P^3 \to S^4 \quad \text{and} \quad \mathbb{F}_{1,2}(\mathbb{C}^3) \to \mathbb{C}P^2.$$

These twistor spaces admit both a Kähler and a nearly Kähler structure. Lagrangian submanifolds of the remaining two homogeneous nearly Kähler spaces $S^3 \times S^3$ and S^6 have been studied a lot; see [Eji81, DVW18, BMVV19, DVV90, Lot11, GIP03].

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Goal

Find Lagrangian submanifolds of twistor spaces.

A start is made for $\mathbb{F}_{1,2}(\mathbb{C}^3)$ in [Sto19].

Twistor space

Let (M^4, g) be an oriented Riemannian 4-manifold. For a complex structure $I \in \mathrm{End}(T_xM)$ we say I is compatible with the orientation if $(e_1, I(e_1), e_3, I(e_3))$ is an oriented basis and we denote this by $I \gg 0$. Consider the bundle $\pi : Z \to M^4$ whose fibre over a point $x \in M^4$ is given by

$$\pi^{-1}(x) = \{I \in \mathrm{End}(T_xM) : I^*g = g,\ I^2 = -1 \text{ and } I \gg 0\}.$$

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$$Z \cong \mathcal{F}_{SO}(M^4) \times_{SO(4)} SO(4)/U(2).$$

A point $[u, I] \in \mathcal{F}_{SO}(M^4) \times_{SO(4)} SO(4)/U(2)$ is an equivalence class of a pair consisting of a frame $u : \mathbb{R}^4 \to T_x M$ and a complex structure I on $\mathbb{R}^4$ compatible with the metric and orientation.

The Levi-Civita connection induces a splitting:

$$
\begin{tikzcd}
T_I Z = T_I^v Z \oplus T_I^h Z \arrow[d, "\pi_*", "\cong"'] \\
T_{\pi(I)} M.
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Theorem ([AHS78])

J^+ is integrable if and only if (M^4, g) is anti-self-dual.

The almost complex structure J^- is never integrable; see [Sal85].

In addition to the almost complex structures we fix a Riemannian metric

$$g_\lambda = \lambda g_{\mathbb{C}P^1} \oplus \pi^* g, \quad \lambda > 0.$$

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Definition

A submanifold $i : L \to (Z, J^\pm, g_\lambda)$ is *Lagrangian* when $\dim(L) = 3$ and

$$J^\pm(T_x L) \perp T_x L$$

for all $x \in L$.

Superminimal surfaces

For an oriented surface $f : \Sigma^2 \to M$ there is a lift $F_0 : \Sigma^2 \to Z$ given by

$$F_0(x) = \left\{\text{a rotation by } \frac{\pi}{2} \text{ in } T_x\Sigma\right\} \oplus \left\{\text{a rotation by } \frac{\pi}{2} \text{ in } T_x\Sigma^\perp\right\}.$$

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Let $u = (e_1, e_2, e_3, e_4)$ be an oriented o.n. local frame of f^*TM, such that (e_1, e_2) and (e_3, e_4) are oriented bases of $T\Sigma$ and $T\Sigma^\perp$, respectively. The complex structure F_0 is in the frame u expressed as

$$\begin{tikzcd} & Z \arrow[d, "\pi"] \\ \Sigma^2 \arrow[ur, "F_0"] \arrow[r, "f"'] & M \end{tikzcd}$$

$$J_0 := u^{-1} \circ F_0 \circ u = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \text{ where } i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

In terms of the associated bundle we have

$$F_0(x) = [u(x), J_0].$$

Definition

An immersion f is *superminimal* if the vertical component $(\mathrm{d}F_0)^v$ of $\mathrm{d}F_0$ vanishes, i.e. if F_0 is horizontal.

Remark

If f is superminimal, then f is minimal. If f is minimal and F_0 is holomorphic, then f is superminimal; see [Fri84].

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Lemma

For an oriented surface $f : \Sigma \to M$ the following are equivalent

1. *the surface Σ is superminimal,*
2. *$F_0 \in \mathrm{End}(f^*TM)$ is parallel, i.e. $\mathrm{Hol}(f^*\nabla) \subseteq U(2)$.*

Goal

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Let $\Sigma \subset M$ be an oriented surface. Define $L_{\Sigma} \subset Z$ by

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We will prove the following:

Theorem

If $\Sigma \subset M$ is a superminimal surface, then $L_\Sigma \subset Z$ is Lagrangian for both J^+ and J^-. Conversely, if $L \subset Z$ is Lagrangian for both J^+ and J^-, then $\pi(L) \subset M$ is a superminimal surface.

The fibre of L_Σ is in $u = (e_1, e_2, e_3, e_4)$ parametrized by $\theta \in S^1$ as

$$J_\theta(e_1) = \cos(\theta)e_3 + \sin(\theta)e_4, \quad J_\theta(e_2) = \sin(\theta)e_3 - \cos(\theta)e_4.$$

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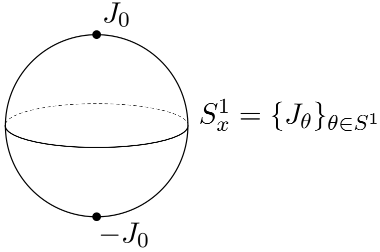

Figure: Complex structures of $T_x M^4$ at a point $x \in \Sigma$.

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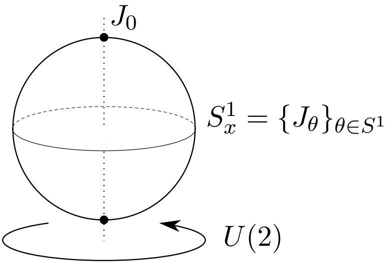

Remark

The group $U(2)$ also preserves S^1_x.

Proof of the theorem $\Rightarrow$

Let $\Sigma \subset M$ be superminimal. A point in L_Σ can be expressed as J_θ. Note that $T_{J_\theta} S^1_x \subset T_{J_\theta} L_\Sigma$. Consider the map

$$F_\theta(x) = [u(x), J_\theta] : \Sigma \to L_\Sigma.$$

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$$F_\theta(x) = [u(x), J_\theta] : \Sigma \to L_\Sigma.$$

By the lemma from before $\downarrow$ we find that $(\mathrm{d}F_\theta)^v \subset T_{J_\theta} S^1_x$. Consequently,

$$\mathrm{im}(\mathrm{d}F_\theta) \oplus T_{J_\theta} S^1_x = T_{J_\theta} L_\Sigma.$$

Lemma

For an oriented surface $\Sigma \subset M$ the following are equivalent

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By the lemma from before we find that $(\mathrm{d}F_\theta)^v \subset T_{J_\theta} S^1_x$. Consequently,

$$\mathrm{im}(\mathrm{d}F_\theta) \oplus T_{J_\theta} S^1_x = T_{J_\theta} L_\Sigma.$$

Thus we can express a tangent vector $X \in T_{J_\theta} L_\Sigma$ as

$$X = X^v + X^h.$$

For $Y \in T_{J_\theta} L_\Sigma$ we have

$$g_\lambda(J^\pm(X), Y) = \pm g_\lambda(J_{\mathbb{C}P^1}(X^v), Y^v) + g_\lambda(J_\theta(X^h), Y^h) = 0.$$

Thus L_Σ is a Lagrangian submanifold.

Lemma

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1. *the surface Σ is superminimal,*
2. $F_0 \in \mathrm{End}(f^*TM)$ *is parallel, i.e.* $\mathrm{Hol}(f^*\nabla) \subseteq U(2)$.

Proof.

Let $\gamma : I \to \Sigma$ be a curve.

$$\begin{aligned}(\mathrm{d}_{\gamma'(t)} F_0)^v &= \left(\frac{d}{dt}[u(\gamma(t)), J_0]\right)^v = \left(\frac{d}{dt}[u^h(\gamma(t)) \cdot g(t), J_0]\right)^v \\ &= \left(\frac{d}{dt}[u^h(\gamma(t)), g(t)^{-1} J_0]\right)^v = \frac{d}{dt} g(t)^{-1} J_0,\end{aligned}$$

Thus $g(t) \in \mathrm{stab}(J_0) \cong U(2)$ if and only if $(\mathrm{d}_{\gamma'(t)} F_0)^v = 0$. $\square$

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