Generalizations of Killing spinors

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(M, g) a (pseudo-) Riemannian spin manifold of dimension n,
∇^g the Levi-Civita and the induced spin connection

Killing spinors

Let $\Psi \in \Gamma(\Sigma M)$, where ΣM is the complex spinor bundle.

$$\nabla^g_X \Psi = a\, X \cdot \Psi \tag{1}$$

- *Killing number* $a \in \mathbb{C}$, either real or purely imaginary
- 1st integrability condition: $\mathcal{R}^g_{X,Y} \Psi = -a^2\, [X\cdot, Y\cdot]\, \Psi$
  $\Rightarrow (M, g)$ is **Einstein** with $\mathrm{Scal}^g = 4\, a^2\, n(n-1)$.
- Ψ is an eigenvector of the *Dirac operator*: $\lambda_0 = -na$
- λ_0 is extremal in the sense of Friedrich's inequality:
  $$\lambda^2 \geq \lambda_0^2 = \tfrac{n}{4(n-1)}\, \mathrm{Scal}^g$$
- **Cone construction** $\Rightarrow$ Holonomy classification for *compact Riemannian manifolds* via Berger's list

Generalized Killing spinors

$$\nabla^g_X \Psi = S(X) \cdot \Psi \tag{2}$$

where $S(X)$ is a section of *symmetric endomorphisms* of $\mathrm{T}M$.

- Originally discovered as the restriction of a parallel spinor to a hypersurface. In this case, $S(X)$ is the *shape operator*.

Special cases by imposing restrictions on $S(X)$:

- *T-Killing spinors* $\equiv$ tr(S) is constant

We are mostly interested in the case when the eigenvalues are constant. The corresponding eigendistributions typically come from an additional special Riemannian structure.

- SU(2)-structures ($n = 5$), SU(3)-structures ($n = 6$), G_2-structures ($n = 7$), Sasakian and 3-Sasakian structures

From this point of view, the equation is **not invariant** with respect to the spin (pseudo-) Riemannian structure alone.

Killing(-Yano) forms

Let $\eta \in \Omega^p(M)$ and $\varphi \in \Omega^{p+1}(M)$.

$$\nabla^g_X \eta = X \lrcorner \varphi \tag{3}$$

- *Projectively invariant* when appropriately weighted.
- Prolongation: $\bigwedge^{p+1} \mathbb{T}^* \cong \bigwedge^p \mathrm{T}^*M \oplus \bigwedge^{p+1} \mathrm{T}^*M$ with the tractor connection modified by the Weyl tensor

Special Killing forms

$$\nabla^g_X \eta = X \lrcorner \varphi, \qquad \nabla^g_X \varphi = -c\, X^\flat \wedge \eta \tag{4}$$

- A prominent example are *Sasakian structures.*
- **Cone construction** $\Rightarrow$ Holonomy classification
- We have $\boxed{\mathrm{Scal} = c\, n(n-1)}$ for compact manifolds.

Q: *Can we deduce* $\mathrm{Scal} = c\, n(n-1)$ *for compact manifolds directly without going through the classification?*

Killing spinor-valued forms

Let $\Phi \in \Omega^p(M, \Sigma M)$ and $\Xi \in \Omega^{p+1}(M, \Sigma M)$.

$$\nabla^g_X \Phi = a\, X \cdot \Phi + X \lrcorner\, \Xi \tag{5}$$

- Briefly appeared in physics in the context of supergravity.
- Tensor products of a Killing spinor and a Killing form.
- The prolongation is similar to the scalar-valued case, just with additional curvature terms on the spinor part.

Special Killing spinor-valued forms

$$\nabla^g_X \Phi = a\, X \cdot \Phi + X \lrcorner\, \Xi, \qquad \nabla^g_X \Xi = a\, X \cdot \Xi - c\, X^\flat \wedge \Phi \tag{6}$$

- Does not imply Einstein in general.
- **Cone construction** again works, but only if $c = 4a^2$.

Q: *Can we deduce $c = 4a^2$ for compact manifolds by clever use of integrability conditions and Stokes theorem?*

Cone construction

The *ε-metric cone* over a (pseudo-) Riemannian manifold (M, g) is the warped product $\overline{M} = M \times \mathbb{R}_+$ with metric

$$\overline{g}_\varepsilon = r^2 \pi_1^*(g) + \varepsilon\, \mathrm{d}r^2, \tag{7}$$

where r is the coordinate function on $\mathbb{R}_+$ and $\varepsilon = \pm 1$.

Analogously to the cases of spinors (Bär 1993, Bohle 2003) and scalar-valued forms (Semmelmann 2003), we have:

Proposition (Somberg, Zima 2016)

A spinor-valued p-form Φ on M is special Killing with constants $a = \pm\frac{1}{2}\sqrt{\varepsilon}$ *and* $c = \varepsilon$ *if and only if the $(p+1)$-form $\Theta_\pm$,*

$$\Theta_\pm = (1 \mp \sqrt{\varepsilon}\partial_r) \cdot (r^p \mathrm{d}r \wedge \pi_1^*(\Phi) + r^{p+1} \pi_1^*(\Xi)) \tag{8}$$

is parallel with respect to the Levi-Civita connection $\overline{\nabla^{g_\varepsilon}}$ on $\overline{M}$.

$\Rightarrow$ $\mathrm{Hol}(\overline{M}, \overline{g}_\varepsilon)$ must fix the spinor-valued $(p+1)$-form $\Theta_\pm$.

2^{nd} order Killing spinors

$\equiv$ Spinor-valued special Killing 0-forms $\Leftrightarrow$ 2^{nd} order equation:

$$\begin{aligned}(\nabla^g)^2_{X,Y}\Psi = -a^2\, X\cdot Y\cdot\Psi\, +& \\ + a\,(Y\cdot(\nabla^g_X\Psi) + X\cdot(\nabla^g_Y\Psi)) - c\, g(X,Y)\,\Psi&\end{aligned} \tag{9}$$

- 1^{st} integrability condition: $\mathcal{R}^g_{X,Y}\Psi = -a^2\,[X\cdot, Y\cdot]\,\Psi$
  $\Rightarrow$ Again (M,g) is **Einstein**.
- Includes Killing spinors with Killing number $a' = -a$.
- Spinorial analog of the equation from Obata's theorem.

Classification for *compact Riemannian manifolds*:

n	$\mathrm{Hol}(\overline{M}, \overline{g}_+)$	Structure on M	Ψ
$2m+1$	$\mathrm{SU}(m+1)$	Sasakian	X
$4k+3$	$\mathrm{Sp}(k+1)$	**3-Sasakian**	✓
7	$\mathrm{Spin}(7)$	G_2-structure	X
6	G_2	Nearly Kähler	?

Sasakian manifolds

Riemannian $(M, g, \varphi, \xi, \eta)$, $n = 2m + 1$, such that:

- *almost contact:* $\boxed{\varphi^2 = -\,\mathrm{Id}_{TM} + \eta \otimes \xi}$, $\boxed{\eta(\xi) = 1}$
- *normal:* Nijenhuis torsion $\boxed{N_\varphi = [\varphi, \varphi] + \mathrm{d}\eta \otimes \xi = 0}$
- *compatible metric:* $\boxed{g(\varphi(X), \varphi(Y)) = g(X, Y) - \eta(X)\eta(Y)}$
- *contact:* $\boxed{\mathrm{d}\eta = 2\Phi}$ where $\Phi(X, Y) = g(X, \varphi(Y))$

Equivalent definitions:

$\Leftrightarrow$ η is a **special Killing 1-form** with $\boxed{c = 1}$ and $\boxed{|\eta| = 1}$.

$\Leftrightarrow$ The cone $(\overline{M}, \overline{g}_+)$ is Kähler.

Theorem (Friedrich, Kath 1990; Bär 2003)

*Let M a complete simply connected Sasaki-Einstein manifold, $m \geq 2$, then M carries **2 Killing spinors** with* $\boxed{a = \pm\frac{1}{2}}$.

3-Sasakian manifolds

Riemannian $(M, g, \varphi_i, \xi_i, \eta_i)$, $n = 4k + 3$, $i = 1, 2, 3$, such that each $(\varphi_i, \xi_i, \eta_i)$ is a Sasakian structure and

$$\varphi_k = \varphi_i\varphi_j - \eta_j \otimes \xi_i = -\varphi_j\varphi_i + \eta_i \otimes \xi_j,$$
$$\xi_k = \varphi_i\xi_j = -\varphi_j\xi_i, \qquad \eta_k = \eta_i\phi_j = -\eta_j\varphi_i.$$

$\Leftrightarrow$ The cone $(\overline{M}, \overline{g}_+)$ is hyper-Kähler.

$\Rightarrow$ Always **Einstein**.

Theorem (Friedrich, Kath 1990; Bär 2003)

*Let M a complete simply connected 3-Sasaki manifold, $k \geq 1$, then M carries **$k + 2$ Killing spinors** with $a = \frac{1}{2}$.*

In dimension 7 the 3 Killing spinors Ψ_i are given by

$$\Psi_i = \xi_i \cdot \Psi_0, \qquad i = 1, 2, 3, \tag{10}$$

where Ψ_0 is a so called **canonical spinor**.

3-(α, δ)-Sasakian manifolds

Split $TM = \mathcal{V} \oplus \mathcal{H}$ to the *vertical* and *horizontal distribution*,

$$\mathcal{V} = \langle \xi_1, \xi_2, \xi_3 \rangle, \qquad \mathcal{H} = \ker \eta_1 \cap \ker \eta_2 \cap \ker \eta_3.$$

Rescale g on $\mathcal{V}$ and $\mathcal{H}$ $\rightsquigarrow$ 2-parameter family of metrics $g_{\alpha,\delta}$
$\rightsquigarrow$ 3-(α, δ)-Sasakian manifolds with $\alpha\delta > 0$

Proposition (Agricola, Dileo 2019)

$(M, g_{\alpha,\delta})$ *is Einstein if and only if* $\delta = \alpha$ *or* $\delta = (2k+3)\,\alpha$.

In dimension 7:

$\delta = \alpha$	$g = g_{1,1}$	the original *3-Sasakian structure*
$\delta = 5\alpha$	$\widetilde{g} = g_{1,5}$	canonical *cocalibrated* G_2*-structure*

G_2-structure on $(M, \widetilde{g})$, *canonical connection* $\widetilde{\nabla^c}$ with torsion
$\rightsquigarrow$ distinguished parallel spinor field Ψ_0,

$$\widetilde{\nabla^c}_X \Psi_0 = 0. \tag{11}$$

Defines the **canonical spinor** of (M, g).

Canonical spinor in dimension 7

Theorem (Agricola, Friedrich 2010)

The canonical spinor Ψ_0 is a ***generalized Killing spinor****,*

$$\begin{aligned} \nabla^g_\xi \Psi_0 &= \tfrac{1}{2}\,\xi \cdot \Psi_0, & \xi &\in \mathcal{V}, \\ \nabla^g_Y \Psi_0 &= -\tfrac{3}{2}\,Y \cdot \Psi_0, & Y &\in \mathcal{H}. \end{aligned} \tag{12}$$

Theorem (Zima *(unpublished)*)

The canonical spinor Ψ_0 is also a ***2^{nd} order Killing spinor*** *with constants $a = -\frac{1}{2}$ and $c = 1$ which is not a Killing spinor.*

- Invariant description of the canonical spinor Ψ_0.
- Discovered on the *Aloff-Wallach space* $N(1,1)$ using CAS and subsequently identified with Ψ_0.
- WIP: Describe the solution Ψ_0 in general for $n = 4k + 3$ directly as an invariant of $\mathrm{Hol}(\overline{M}, \overline{g}_+) = \mathrm{Sp}(k+1)$.

Higher order generalizations

- The spinor-valued skew-symmetric forms are not suitable for higher order generalization, we need to consider **symmetric** (covariant) tensor-spinors of rank $p \geq 2$.
- In order to deduce the appropriate PDE we can start from the cone construction.

Let $\Theta \in \Gamma(\mathrm{Sym}^p(\mathrm{T}^*\overline{M}) \otimes \Sigma\overline{M})$ be *parallel* with respect to $\overline{\nabla^{g_\varepsilon}}$.

$\rightsquigarrow$ Project to **$(p+1)^{\text{th}}$ order Killing spinor** $\Psi_\pm$ on (M, g),

$$\Psi_\pm = (\pi_1)_*\Big(\tfrac{1}{2}(1 \pm \sqrt{\varepsilon}\partial_r) \cdot \Theta(\partial_r, \ldots, \partial_r)\Big). \tag{13}$$

- The prolongation in case of symmetric p-tensors has $p+1$ components and is combinatorially involved.
- Alternatively we can start from the first BGG operator on projective symmetric covariant tractors of rank $p \geq 2$.

References

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THANK YOU FOR YOUR ATTENTION!