Miscellaneous on Almost Complex Manifolds

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Dirac operators in differential geometry and global analysis
In memory of Thomas Friedrich

October 10, 2019

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This is *not* what this talk is about.

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For $n = 2$: $\mathrm{SO}(4)/\mathrm{U}(2) \cong S^2$, so we have obstructions in $H^3(M;\pi_2(S^2)) = H^3(M;\mathbb{Z})$ and $H^4(M;\pi_3(S^2)) = H^4(M;\mathbb{Z})$.

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Dimension 8: *Müller–Geiges* (2000): There exists an acs on M if and only if

- $W_3 = 0$,
- $\chi(M) \equiv \sigma(M) \mod 4$,
- If there exists a torsion element $c \in H^2(M, \mathbb{Z})$ with $c \equiv \omega_2(M) \mod 2$, then $\chi(M) \equiv 0 \mod 2$, and
- If $b_2(M) = 0$, then $8\chi(M) - 4p_2(M) + p_1^2(M) = 0$.

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- $k\mathbb{C}P^2$ and $k\mathbb{C}P^4$ are almost complex if and only if k is odd.

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4. Use result of Sutherland/Thomas (1965/67): A closed $2d$-dimensional manifold M admits an almost complex structure if and only if it admits a stable almost complex structure whose d-th Chern class equals the Euler class of M.

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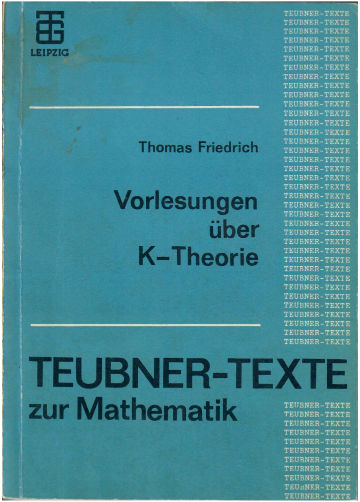

More on connected sums

This result was generalized by Yang:

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For almost complex 4n-dimensional almost complex manifolds $M_1, \ldots, M_k$, the connected sum $(\#_{i=1}^k M_i)\#(k-1)\mathbb{C}P^{2n}$ is almost complex.

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The proof uses results of Kahn (1969) on the obstruction

$$\mathfrak{o}(M, J) \in H^{4n}(M, M \setminus D; \pi_{4n-1}(\mathrm{SO}(4n)/\mathrm{U}(2n)))$$

to extend a given almost complex structure J on $M \setminus D$ to M. He shows that the obstruction vanishes for an explicit stable almost complex structure.

Even more on connected sums

Theorem (Albanese, Milivojević; 2019)

Given r closed almost complex manifolds of dimension n, their connected sum is automatically almost complex if and only if

- $n = 4m$, *and* $r = 1$,
- $n = 8k + 2$, *and* $r \equiv 1 \mod (4k)!$,
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$$\mathfrak{o}(M,J) = \frac{1}{2}(\chi(M) - c_{4k+1}(\tilde{J})) = \frac{1}{2}(\chi(M) - \sum_{i=1}^r \chi(M_i)) = 1 - r.$$

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More examples (Albanese, Milivojević): if $M_1, \ldots, M_r$ are $(4m+2)$-dimensional almost complex manifolds with $H^{2j}(M_i;\mathbb{Q}) = 0$ for $j = 1, \ldots, 2m$, then $M = \#_{i=1}^r M_i$ can only admit an almost complex structure if $r \equiv 1 \mod \frac{1}{2}(2m)!$.

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The almost complex structure on M_i induces a spin^c structure. Its spin^c Dirac operator $\partial\!\!\!/^c_{M_i}$ has index

$$\int_{M_i} \exp(c_1(TM_i)/2)\,\mathrm{ch}(TM_i)\hat{A}(TM_i) = \frac{1}{(2m)!}\chi(M_i) \in \mathbb{Z}$$

If M is almost complex, then analogously

$$\frac{1}{(2m)!}\chi(M) = \frac{1}{(2m)!}\left(\sum_{i=1}^{r}\chi(M_i) - 2(r-1)\right) \in \mathbb{Z}.$$

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- Datta–Subramanian (1990): The only products of two spheres that admit an almost complex structure are $S^2 \times S^2$, $S^6 \times S^6$, $S^2 \times S^6$, and $S^2 \times S^4$.

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- Open for sphere bundles over spheres.

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—, Konstantis, Zoller (2018): found several infinite families of symplectic, in particular almost complex, biquotients.