An intrinsic characterization of projective special Kähler manifolds

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7 October 2019

Overview

Quaternion-Kähler manifolds

Definition

The quaternionic unitary group is

$$\mathrm{Sp}(n) = \{Q \in \mathrm{Mat}(n, \mathbb{H}) | Q^\star Q = I_n\} \leq \mathrm{GL}(n, \mathbb{H}).$$

Define the following left actions on $\mathbb{H}^n$: let $Q \in \mathrm{Sp}(n), q \in \mathrm{Sp}(1)$.

$$\begin{aligned} L_Q \colon \mathbb{H}^n &\longrightarrow \mathbb{H}^n \\ v &\longmapsto Qv \end{aligned} \qquad\qquad \begin{aligned} R_q \colon \mathbb{H}^n &\longrightarrow \mathbb{H}^n \\ v &\longmapsto vq^\star \end{aligned}$$

They provide embeddings

$$L \colon \mathrm{Sp}(n) \longrightarrow \mathrm{GL}(4n, \mathbb{R}), \qquad R \colon \mathrm{Sp}(1) \longrightarrow \mathrm{GL}(4n, \mathbb{R}).$$

Definition

$\mathrm{Sp}(n)\mathrm{Sp}(1)$ *is the subgroup of* $\mathrm{GL}(4n,\mathbb{R})$ *generated by* $\mathrm{Sp}(n)$ *and* $\mathrm{Sp}(1)$.

$$\begin{array}{ccccccc} & & \mathrm{Sp}(n) & & & & \\ & ╱ & & ╲ & & & \\ \mathbb{Z}_2 \cong \{\pm I_{4n}\} & & & & \mathrm{Sp}(n)\mathrm{Sp}(1) & — & \mathrm{GL}(4n,\mathbb{R}) \\ & ╲ & & ╱ & & & \\ & & \mathrm{Sp}(1) & & & & \end{array}$$

$\mathrm{Sp}(n)\mathrm{Sp}(1)$ is isomorphic to $(Sp(n) \times Sp(1))/\mathbb{Z}_2$.

Definition

A quaternion-Kähler manifold of dimension $4n \geq 8$ *is a Riemannian* $4n$*-manifold with holonomy group contained in* $Sp(n)Sp(1)$ *but not in* $Sp(n)$.

Some quaternion Kähler history

- Berger (1955): classification of holonomy groups, QKs appear;
- Wolf (1965): classification of symmetric QK;
- Alekseevsky (1975): classification of QK with simply transitive real solvable group of isometries (first non-sym. examples);
- De Wit, Van Proeyen (1992): obtain an example not appearing in Alekseevsky's classification (c-map);
- Cortés (1996): complete classification;
- Haydys (2008): HK/QK correspondence;
- Hitchin (2009): Mathematical description of the c-map (local);
- Alekseevsky, Cortés, Dyckmanns, Mohaupt (2015): c-map=rigid c-map+HK/QK;
- Macia, Swann (2015): global c-map, equivalence of previous constructions and twist.

C-map

From supergravity and string theory.

$$\begin{tikzcd}
\mathrm{PSK} \arrow[d, dashed] \\
\mathrm{CSK} \arrow[d, dashed] \\
\mathrm{HK} \arrow[d, dashed, bend right, "\mathrm{HK/QK}"'] \arrow[d, dashed, bend left, "\text{twist}"] \\
\mathrm{QK}
\end{tikzcd}$$

rigid c-map

$$\begin{tikzcd}
 & M^{2n} & \\
 & \widetilde{M}^{2n+2} \arrow[u, "\mathbb{C}^*"'] & \\
 & H^{4n+4} = T^*\widetilde{M}^{2n+2} \arrow[u] \arrow[dd, dashed] & \\
Q^{4n+6} \arrow[ur, "\mathbb{C}"'] \arrow[dr, "\mathbb{C}"'] & & P^{4n+5} \arrow[ul, "S^1"] \arrow[dl, "S^1"] \\
 & N^{4n+4} &
\end{tikzcd}$$

Special Kähler manifolds

Definition

A conic special Kähler (CSK) manifold $(M, g, I, \omega, \nabla, \xi)$ *is the data of:*

- *a pseudo-Kähler manifold* (M, g, I, ω)*;*
- *a flat, torsion free, symplectic connection* ∇ *such that* $d^{\nabla} I = 0$*, i.e.* ∇I *is symmetric;*
- *a vector field* ξ *such that*
  - $g(\xi, \xi)$ *is nowhere vanishing;*
  - $\nabla\xi = \nabla^{LC}\xi = \mathrm{id}$*, where* ∇^{LC} *is the Levi-Civita connection;*
  - g *is negative definite on* $\langle \xi, I\xi \rangle$ *and positive definite on its orthogonal complement.*

Let $\mathbb{C}^{n,1}$ be $\mathbb{C}^{n+1}$ endowed with the Hermitian form

$$\langle z, w\rangle = \overline{z_1}w_1 + \cdots + \overline{z_n}w_n - \overline{z_{n+1}}w_{n+1}.$$

Example

The open submanifold

$$\{z \in \mathbb{C}^{n,1} | \langle z, z\rangle < 0\}$$

is conic special Kähler with the induced pseudo Kähler structure, $\nabla = \nabla^{LC}$ *and* ξ *the position vector field.*

Definition

A *projective* *special Kähler (PSK) manifold is a Kähler manifold M endowed with a $\mathbb{C}^*$-bundle $\pi\colon \widetilde{M} \to M$ with $(\widetilde{M}, \widetilde{g}, \widetilde{I}, \widetilde{\omega}, \nabla, \xi)$ conic special Kähler such that ξ and $I\xi$ are the fundamental vector fields associated to $1, i \in \mathbb{C}$ respectively and M is the Kähler quotient with respect to the induced $\mathrm{U}(1)$-action.*

Example

$\mathcal{H}^n_{\mathbb{C}} := \{z \in \mathbb{C}^{n,1} | \langle z, z\rangle < 0\}/\mathbb{C}^*$ is projective special Kähler, called complex hyperbolic n-space.

Problems with definition of a PSK $(\pi : \widetilde{M} \to M, \nabla)$:

1. It depends on objects defined on $\widetilde{M}$;
2. Both algebraic and differential equations to be verified on $\widetilde{M}$ instead of M;
3. Given M it is not so easy to find ∇.

Difference tensor

Let $(\widetilde{M}, \widetilde{g}, \widetilde{I}, \widetilde{\omega}, \nabla, \xi)$ be CSK. We have a section $\widetilde{\eta}$ of $T^*\widetilde{M} \otimes T\widetilde{M} \otimes T^*\widetilde{M}$ such that $\nabla = \nabla^{LC} + \widetilde{\eta}$.
The properties of CSK manifolds imply that if we lower the second index, $\flat_2(\widetilde{\eta})$ has image in $[\![S_{3,0}\widetilde{M}]\!]$, i.e. locally

$$\sum_{i \le j \le k} \left(a_{i,j,k} dz_i dz_j dz_k + \overline{a_{i,j,k}} d\overline{z}_i d\overline{z}_j d\overline{z}_k \right),$$

for $(U, z = (z_1, \ldots, z_{n+1}))$ complex chart, $a_{i,j,k} \in \mathcal{C}^\infty(U, \mathbb{C})$.

Difference tensor

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for $(U, z = (z_1, \ldots, z_{n+1}))$ complex chart, $a_{i,j,k} \in \mathcal{C}^\infty(U, \mathbb{C})$.
It is equivalent to take sections in $S_{3,0}\widetilde{M}$, hence locally of the form

$$\sum_{i \leq j \leq k} a_{i,j,k} dz_i dz_j dz_k.$$

Essentially homogeneous polynomials of degree 3 with coefficients in $\mathcal{C}^\infty(U, \mathbb{C})$ and variables $dz_1, \ldots, dz_{n+1}$.

Consider a PSK $(\pi : \widetilde{M} \to M, \nabla)$.
Given an open subset $U \subseteq M$ and a section $s\colon U \to \widetilde{M}$, it induces a trivialisation

$$(\pi|_{\pi^{-1}(U)}, z)\colon \pi^{-1}(U) \to U \times \mathbb{C}^*.$$

There exists a tensor $\eta \in T_{1,0}U \otimes T^{0,1}U \otimes T_{1,0}U$ such that $\flat_2\eta$ is a tensor in $S_{3,0}U$ and

$$\widetilde{\eta} = z^2\pi^*\eta + \overline{z^2\pi^*\eta} = r^2\cos(2\vartheta)2\operatorname{Re}\pi^*\eta + r^2\sin(2\vartheta)2\operatorname{Im}\pi^*\eta$$

where $z = re^{i\vartheta}$.
We call η the (local) deviance.

Theorem 1

On a Kähler $2n$-manifold (M, g, I, ω), giving a projective special Kähler structure is equivalent to giving an S^1-bundle $\pi_S\colon S \to M$ endowed with a connection form φ and a bundle map $\gamma\colon S \to \sharp_2 S_{3,0} M$ such that:

1. $d\varphi = -2\pi_S^*\omega$;
2. $\gamma(ua) = a^2\gamma(u)$ *for all* $a \in S^1$;
3. *given an open covering* $\{U_\alpha | \alpha \in \mathcal{A}\}$ *of* M *and a family* $\{s_\alpha\colon U_\alpha \to S\}_{\alpha\in\mathcal{A}}$ *of sections, denoting by* η_α *the local 1-form taking values in* $T^{0,1}M \otimes T_{1,0}M$ *determined by* $\gamma \circ s_\alpha$*, for all* $\alpha \in \mathcal{A}$*:*
   - **D1** $\Omega^{LC} + \Omega_{\mathbb{P}^n_{\mathbb{C}}} + [\eta_\alpha \wedge \overline{\eta_\alpha}] = 0$
   - **D2** $d^{LC}\eta_\alpha = 2i s_\alpha^*\varphi \wedge \eta_\alpha$

In this case, 3 is satisfied by every such family of sections.

Proof.

The curvature form of $\nabla = \widetilde{\nabla}^{LC} + \widetilde{\eta}$ is

$$\widetilde{\Omega}^{LC} + [\widetilde{\eta} \wedge \widetilde{\eta}] + d\widetilde{\eta}$$

Flatness of ∇ implies

$$\begin{cases} \widetilde{\Omega}^{LC} + [\widetilde{\eta} \wedge \widetilde{\eta}] = 0 \\ \widetilde{d}^{LC}\widetilde{\eta} = 0 \end{cases} \rightsquigarrow \begin{cases} \Omega^{LC} + \Omega_{\mathbb{P}^n_{\mathbb{C}}} + [\eta_\alpha \wedge \overline{\eta}_\alpha] = 0 \\ d^{LC}\eta_\alpha = 2is^*_\alpha\varphi \wedge \eta_\alpha \end{cases}$$

$\square$

Have we solved the problems?

1. The definition depends on objects defined on $\widetilde{M}$: we still have to find an S^1-bundle (S, φ), but an S^1-bundle is determined (uniquely up to iso) by its Chern class, which is fixed up to torsion by the theorem.
   Problem solved with added conditions (e.g. ω exact);
2. The conditions have to be verified on $\widetilde{M}$: now the conditions **D1**, **D2** are given on M;
3. Not so easy to find ∇: now we only need to find γ, which is locally defined as a polynomial η_α with functional coefficients. It can be done by taking locally a generic η_α, restricting the cases with the algebraic condition **D1** and on the remaining cases verifying the differential condition **D2**.

Classification of projective special Kähler Lie 4-groups

Definition

A projective special Kähler Lie group is a Lie group with a projective special Kähler structure such that the Kähler structure is left-invariant.

Theorem 2

Up to isomorphisms of projective special Kähler manifolds, there are only two connected projective special Kähler Lie groups of dimension 4: $\mathcal{H}_{\sqrt{2}} \times \mathcal{H}_2$ and the complex hyperbolic plane.
Up to isomorphisms that also preserve the Lie group structure, there are four projective special Kähler Lie groups of dimension 4.

#	$\mathfrak{g}$	I	ω	Conditions
1	$\mathfrak{rr}_{3,0}$	$Ie_1 = e_2, Ie_3 = e_4$	$a_1 e^{1,2} + a_2 e^{3,4}$	$a_1, a_2 > 0$
2	$\mathfrak{rr}'_{3,0}$	$Ie_1 = e_4, Ie_2 = e_3$	$a_1 e^{1,4} + a_2 e^{2,3}$	$a_1, a_2 > 0$
3	$\mathfrak{r}_2\mathfrak{r}_2$	$Ie_1 = e_2, Ie_3 = e_4$	$a_1 e^{1,2} + a_2 e^{3,4}$	$a_1, a_2 > 0$
4	$\mathfrak{r}'_{4,0,\delta}$	$Ie_4 = e_1, Ie_2 = e_3$	$a_1 e^{1,4} + a_2 e^{2,3}$	$-a_1, a_2 > 0$
5	$\mathfrak{r}'_{4,0,\delta}$	$Ie_4 = e_1, Ie_2 = -e_3$	$a_1 e^{1,4} + a_2 e^{2,3}$	$a_1, a_2 < 0$
6	$\mathfrak{d}_{4,2}$	$Ie_4 = -2e_1, Ie_2 = e_3$	$a_1 e^{1,4} + a_2 e^{2,3}$	$a_1, a_2 > 0$
7	$\mathfrak{d}_{4,1/2}$	$Ie_4 = e_3, Ie_1 = e_2$	$a_1(e^{1,2} - e^{3,4})$	$a_1 > 0$
8	$\mathfrak{d}'_{4,\delta}$	$Ie_4 = e_3, Ie_1 = e_2$	$a_1(e^{1,2} - \delta e^{3,4})$	$a_1 > 0$
9	$\mathfrak{d}'_{4,\delta}$	$Ie_4 = -e_3, Ie_1 = -e_2$	$a_1(e^{1,2} - \delta e^{3,4})$	$a_1 < 0$

Table: Kähler Lie algebras (Ovando 2004)

#	$\mathfrak{g}$	I	ω	Conditions
3	$\mathfrak{r}_2\mathfrak{r}_2$	$Ie_1 = e_2, Ie_3 = e_4$	$a_1 e^{1,2} + a_2 e^{3,4}$	$a_1 = \frac{1}{2}, a_2 = \frac{1}{4}$
6	$\mathfrak{d}_{4,2}$	$Ie_4 = -2e_1, Ie_2 = e_3$	$a_1 e^{1,4} + a_2 e^{2,3}$	$a_1 = \frac{1}{2}, a_2 > 0$
7	$\mathfrak{d}_{4,1/2}$	$Ie_4 = e_3, Ie_1 = e_2$	$a_1(e^{1,2} - e^{3,4})$	$a_1 = \frac{1}{4}$
8	$\mathfrak{d}'_{4,\delta}$	$Ie_4 = e_3, Ie_1 = e_2$	$a_1(e^{1,2} - \delta e^{3,4})$	$a_1 = \frac{1}{2}$
9	$\mathfrak{d}'_{4,\delta}$	$Ie_4 = -e_3, Ie_1 = -e_2$	$a_1(e^{1,2} - \delta e^{3,4})$	$a_1 = -\frac{\delta}{4}$

Table: Kähler Lie algebras satisfying **D1**

#	$\mathfrak{g}$	I	ω	Conditions
3	$\mathfrak{r}_2\mathfrak{r}_2$	$Ie_1 = e_2, Ie_3 = e_4$	$a_1 e^{1,2} + a_2 e^{3,4}$	$a_1 = \frac{1}{2}, a_2 = \frac{1}{4}$
7	$\mathfrak{d}_{4,1/2}$	$Ie_4 = e_3, Ie_1 = e_2$	$a_1(e^{1,2} - e^{3,4})$	$a_1 = \frac{1}{4}$
8	$\mathfrak{d}'_{4,\delta}$	$Ie_4 = e_3, Ie_1 = e_2$	$a_1(e^{1,2} - \delta e^{3,4})$	$a_1 = \frac{1}{2}$
9	$\mathfrak{d}'_{4,\delta}$	$Ie_4 = -e_3, Ie_1 = -e_2$	$a_1(e^{1,2} - \delta e^{3,4})$	$a_1 = -\frac{\delta}{4}$

Table: PSK Lie algebras

Applying the c-map

$$
\begin{tikzcd}
\mathrm{PSK} \arrow[d, dashed] \\
\mathrm{CSK} \arrow[d, dashed] \\
\mathrm{HK} \arrow[d, dashed, "\text{twist}"'] \\
\mathrm{QK}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{H}^2_{\mathbb{C}} & & \\
\{z \in \mathbb{C}^{2,1} | \langle z, z \rangle < 0\} \arrow[u] & & \\
T^*\{z \in \mathbb{C}^{2,1} | \langle z, z \rangle < 0\} \arrow[u] \arrow[d, dashed] & & P^{13} \arrow[ll] \arrow[dll] \\
\frac{\mathrm{SU}(3,2)}{S(\mathrm{U}(3)\mathrm{U}(2))} & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{H}_{\sqrt{2}} \times \mathcal{H}_2 & & \\
C(\mathcal{H}_{\sqrt{2}} \times \mathcal{H}_2 \times S^1) \arrow[u] & & \\
T^*C(\mathcal{H}_{\sqrt{2}} \times \mathcal{H}_2 \times S^1) \arrow[u] \arrow[d, dashed] & & P'^{13} \arrow[ll] \arrow[dll] \\
\frac{\mathrm{SO}_0(4,3)}{\mathrm{SO}(4)\mathrm{SO}(3)} & &
\end{tikzcd}
$$

Structure constants $\frac{\mathrm{SU}(3,2)}{S(\mathrm{U}(3)\mathrm{U}(2))}$

$$
\begin{aligned}
&du^1 = u^{1,3}; \qquad du^2 = u^{2,3}; \qquad du^3 = 0;\\
&du^4 = -2u^{1,2} - 2u^{3,4}; \qquad du^5 = 0;\\
&du^6 = -2u^{5,6} + 2u^{7,8} + 2u^{9,10} - 2u^{11,12};\\
&du^7 = u^{1,9} + u^{1,11} - u^{2,10} - u^{2,12} - u^{5,7};\\
&du^8 = u^{1,10} + u^{1,12} + u^{2,9} + u^{2,11} - u^{5,8};\\
&du^9 = -u^{1,7} - u^{2,8} + u^{3,11} - u^{4,10} - u^{4,12} - u^{5,9};\\
&du^{10} = -u^{1,8} + u^{2,7} + u^{3,12} + u^{4,9} + u^{4,11} - u^{5,10};\\
&du^{11} = +u^{1,7} + u^{2,8} + u^{3,9} + u^{4,10} + u^{4,12} - u^{5,11};\\
&du^{12} = +u^{1,8} - u^{2,7} + u^{3,10} - u^{4,9} - u^{4,11} - u^{5,12}.
\end{aligned}
$$

Structure constants $\frac{\mathrm{SO}_0(4,3)}{\mathrm{SO}(4)\mathrm{SO}(3)}$

$$du^1 = -\sqrt{2}u^{1,2}; \qquad du^2 = 0; \qquad du^3 = -2u^{3,4};$$
$$du^4 = 0; \qquad du^5 = 0;$$
$$du^6 = -2u^{5,6} - 2u^{7,8} - 2u^{9,10} + 2u^{11,12};$$
$$du^7 = u^{1,9} + u^{1,12} - u^{2,10} + u^{2,11} + u^{3,7} + u^{3,8} - u^{4,8} - u^{5,7};$$
$$du^8 = -u^{1,10} - u^{1,11} - u^{2,9} + u^{2,12} - u^{3,7} - u^{3,8} - u^{4,7} - u^{5,8};$$
$$du^9 = u^{1,7} + \sqrt{2}u^{1,10} - u^{2,8} - u^{3,10} + u^{3,12} + u^{4,11} - u^{5,9};$$
$$du^{10} = -u^{1,8} - \sqrt{2}u^{1,9} - u^{2,7} + u^{3,9} - u^{3,11} + u^{4,12} - u^{5,10};$$
$$du^{11} = -u^{1,8} + \sqrt{2}u^{1,12} + u^{2,7} - u^{3,10} + u^{3,12} + u^{4,9} - u^{5,11};$$
$$du^{12} = u^{1,7} - \sqrt{2}u^{1,11} + u^{2,8} + u^{3,9} - u^{3,11} + u^{4,10} - u^{5,12}.$$

Conclusions

No new examples: if we start from a Lie group we obtain a Lie group (Macia, Swann 2019) $\rightsquigarrow$ Alekseevsky-Cortés' classification.

What if we apply the same procedure to non homogeneous manifolds?

Thanks for your attention!