Spinorial description of almost contact metric manifolds

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Dirac operators in differential geometry and global analysis

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Classical results [Friedrich, Kath, Moroianu, Semmelmann]

Riemannian Killing equation

$$\nabla^g_X \psi = \mu X \cdot \psi, \qquad \mu \in \mathbb{C}$$

1. Describes equality case in Friedrich's inequality for the smallest eigenvalue λ of D on closed spin manifolds

$$\lambda^2 \geq \frac{n}{4(n-1)} \mathrm{Scal}_{\mathrm{min}}$$

2. Automatically Einstein
3. In odd dimensions $n \neq 7$: $\exists$ 1 RKS $\Leftrightarrow$ $\exists$ 2 RKS $\Leftrightarrow$ E-Sasaki
4. $n = 7$: $\exists$ 1 RKS $\Leftrightarrow$ nearly parallel G_2;
   $\exists$ 2 RKS $\Leftrightarrow$ E-Sasaki;
   $\exists$ 3 RKS $\Leftrightarrow$ 3-Sasaki

$SU(3)$-manifolds in $n = 6$ [Agricola, Chiossi,Friedrich, Höll]

A 6-manifold admits an $SU(3)$-structure iff it has a spinor ψ of length 1. It satisfies the equation

$$\nabla^g_X \psi = S(X) \cdot \psi + \eta(X) \cdot j(\psi)$$

for some $S\colon TM \to TM$ and a 1-form η (and $j := e_1 e_2 e_3 e_4 e_5 e_6$).

Class	Description	dim
χ_1	$S = \lambda J$, $\eta = 0$	1
$\chi_{\bar{1}}$ (nK)	$S = \mu\,\mathrm{Id}$, $\eta = 0$	1
χ_2	$S \in \Lambda^2\mathbb{R}^6$, $SJ = JS$, $\mathrm{tr}(JS) = 0$, $\eta = 0$	8
$\chi_{\bar{2}}$	$S \in S^2_0\mathbb{R}^6$, $SJ = JS$, $\eta = 0$	8
χ_3	$S \in S^2_0\mathbb{R}^6$, $SJ = -JS$, $\eta = 0$	12
χ_4	$S \in \Lambda^2\mathbb{R}^6$, $SJ = -JS$, $\eta = 0$	6
χ_5	$S = 0$, $\eta \neq 0$	6

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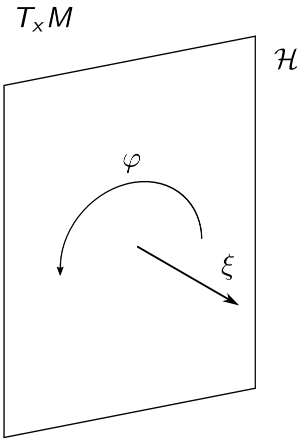
Almost contact metric manifolds

$(M^{2k+1}, g, \varphi, \xi, \eta)$ is an almost contact metric manifold if

1. η 1-form dual to ξ
2. $\mathcal{H} := \langle \xi \rangle^{\perp}$ admits an almost complex structure φ compatible with g
3. In formulae:
   $\varphi^2 = -I + \eta \otimes \xi$, $\eta \circ \varphi = 0$
   $\varphi(\xi) = 0$ and
   $g(\varphi X, \varphi Y) =$
   $g(X, Y) - \eta(X)\eta(Y)$
4. Fundamental 2-Form:

$$\Phi(X, Y) := g(X, \varphi(Y))$$



Definition

Almost contact metric structure (φ, ξ) is *$\mathcal{H}$-parallel* if

$$(\nabla^g_X \Phi)(Y, Z) = 0 \qquad \forall X \in TM,\; Y, Z \in \mathcal{H},$$

or equivalently $\nabla^g \Phi \in TM \otimes \mathcal{H} \wedge \mathcal{V}$.

Remark

All information about $\nabla^g \Phi$ is contained in $(\nabla^g_X \varphi)(\xi)$.

Definition

The *intrinsic endomorphism* of the a.c.m. structure is

$$S\colon TM \to \mathcal{H}, \qquad S(X) := -(\nabla^g_X \varphi)(\xi) = \varphi(\nabla^g_X \xi)$$

Example

If α-Sasaki then $\nabla^g_X \xi = -\alpha\varphi(X)$, hence $S = \alpha\, \mathrm{Id}_{\mathcal{H}}$.

Chinea-Gonzalez' classification of a.c.m. manifolds

Class	Defining equation
$\mathcal{C}_1$	$(\nabla^g_X \Phi)(X,Y) = 0, \nabla^g \eta = 0$
$\mathcal{C}_2$	$d\Phi = \nabla^g \eta = 0$
$\mathcal{C}_3$	$(\nabla^g_X \Phi)(Y,Z) - (\nabla^g_{\varphi(X)} \Phi)(\varphi(Y),Z) = 0, \delta\Phi = 0$
$\mathcal{C}_4$	$(\nabla^g_X \Phi)(Y,Z) = \frac{-1}{2(k-1)}\left[g(\varphi(X),\varphi(Y))\delta\Phi(Z) - g(\varphi(X),\varphi(Z))\delta\Phi(Y) + \ldots\right.$
$\mathcal{C}_5$	$(\nabla^g_X \Phi)(Y,Z) = \frac{1}{2k}\left[\eta(Y)\Phi(X,Z) - \eta(Z)\Phi(X,Y)\right]\delta\eta$
$\mathcal{C}_6$	$(\nabla^g_X \Phi)(Y,Z) = \frac{1}{2k}\left[\eta(Y)g(X,Z) - \eta(Z)g(X,Y)\right]\delta\Phi(\xi)$
$\mathcal{C}_7$	$(\nabla^g_X \Phi)(Y,Z) = \eta(Z)(\nabla^g_Y \eta)(\varphi(X)) + \eta(Y)(\nabla^g_{\varphi(X)}\eta)(Z), \delta\Phi = 0$
$\mathcal{C}_8$	$(\nabla^g_X \Phi)(Y,Z) = -\eta(Z)(\nabla^g_Y \eta)(\varphi(X)) + \eta(Y)(\nabla^g_{\varphi(X)}\eta)(Z), \delta\eta = 0$
$\mathcal{C}_9$	$(\nabla^g_X \Phi)(Y,Z) = \eta(Z)(\nabla^g_Y \eta)(\varphi(X)) - \eta(Y)(\nabla^g_{\varphi(X)}\eta)(Z)$
$\mathcal{C}_{10}$	$(\nabla^g_X \Phi)(Y,Z) = -\eta(Z)(\nabla^g_Y \eta)(\varphi(X)) - \eta(Y)(\nabla^g_{\varphi(X)}\eta)(Z)$
$\mathcal{C}_{11}$	$(\nabla^g_X \Phi)(Y,Z) = -\eta(X)(\nabla^g_\xi \Phi)(\varphi(Y),\varphi(Z))$
$\mathcal{C}_{12}$	$(\nabla^g_X \Phi)(Y,Z) = \eta(X)\eta(Z)(\nabla^g_\xi \eta)(\varphi(Y)) - \eta(X)\eta(Y)(\nabla^g_\xi \eta)(\varphi(Z))$

Description of $\mathcal{H}$-parallel a.c.m. structures in terms of S

Remark: For pure classes: $\mathcal{H}$-parallel $\Leftrightarrow$ $S \neq 0$.

Class	Description	dim
$\mathcal{C}_5$	$S = \alpha\varphi$, $(\alpha = -\frac{1}{2k}\delta\eta)$	1
$\mathcal{C}_6$	$S = \alpha\,\mathrm{Id}_{\mathcal{H}}$, $(\alpha = \frac{1}{2k}\delta\Phi(\xi))$	1
$\mathcal{C}_7$	$S \in S^2_0\mathcal{H}^*$, $S\varphi = \varphi S$	$k^2 - 1$
$\mathcal{C}_8$	$S \in \Lambda^2\mathcal{H}^*$, $S\varphi = \varphi S$, $\mathrm{tr}(\varphi S) = 0$	$k^2 - 1$
$\mathcal{C}_9$	$S \in S^2_0\mathcal{H}^*$, $S\varphi = -\varphi S$	$k^2 + k$
$\mathcal{C}_{10}$	$S \in \Lambda^2\mathcal{H}^*$, $S\varphi = -\varphi S$	$k^2 - k$
$\mathcal{C}_{12}$	$S\|_{\mathcal{H}} = 0$	$2k$

- $\mathcal{C}_5$: α-Kenmotsu, $\mathcal{C}_6$: α-Sasaki, $\mathcal{C}_5 \oplus \mathcal{C}_6$: trans-Sasaki
- $\mathcal{C}_6 \oplus \mathcal{C}_7$: quasi-Sasaki
- Normal $(N = [\varphi, \varphi] + d\eta \otimes \xi = 0)$: $\mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$
- ξ Killing: $\mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_{10}$

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The spinor bundle

(M^n, g) Riemannian spin manifold. The spinor bundle $\Sigma^{\mathbb{C}} M$

- Complex bundle over M of rank $2^{\lfloor n/2 \rfloor}$
- Clifford product: $X \cdot \psi$, $X \in T_p M$, $\psi \in \Sigma^{\mathbb{C}}_p M$, s.t.

$$X \cdot Y \cdot \psi + Y \cdot X \cdot \psi = -2g(X, Y)\psi$$

  in particular if $X \perp Y$ then $X \cdot Y \cdot \psi = -Y \cdot X \cdot \psi$, and $X \cdot X \cdot \psi = -g(X, X) \cdot \psi$.
- Scalar product $(\cdot, \cdot)$ s.t. $(X \cdot \psi, Y \cdot \psi) = g(X, Y)(\psi, \psi)$
- Lift of Levi-Civita connection is again metric and satisfies

$$\nabla^g_X (Y \cdot \psi) = (\nabla^g_X Y) \cdot \psi + Y \cdot \nabla^g_X \psi$$

The spinor bundle of an a.c.m. manifold

(M^{2k+1}, g) spin, $\Sigma^{\mathbb{C}} M$ its spinor bundle

$$L_{1,2} := \{\psi \in \Gamma(\Sigma M) : (\varphi(X) \pm iX) \cdot \psi = 0 \ \forall X \in \mathcal{H}\}$$

are two 1-dimensional subbundles and on L_1: $\xi \cdot \psi = i\psi$; on L_2: $\xi \cdot \psi = (-1)^k i\psi$. [Becker-Bender]

Definition

Family of connections on for $j = 1, 2$ and $s \in \mathbb{R}$:

$$\nabla^{j,s}_X \psi = \nabla^g_X \psi - \epsilon_j \, [\frac{1}{2} S(X) \cdot \psi + s\eta(X)\xi \cdot \psi]$$

where $\epsilon_1 = 1$ and $\epsilon_2 = (-1)^k$.

Proposition

$\nabla^{j,s}$ is a connection on L_j if and only if the a.c.m. structure is $\mathcal{H}$-parallel.

Remark

By a result of Cabrera and Gonzalez-Davila, $\nabla^g_X - \frac{1}{2}S(X)$ coincides with the lift of the minimal $U(n)$-connection $\nabla^g + \Gamma$ to the bundle L_j, where Γ is the intrinsic torsion.

For simplicity we now consider only L_1.

For $\psi \in \Gamma(L_1)$, $\|\psi\| = 1$, we know $\mathrm{Re}(\nabla^g_X \psi, \psi) = 0$ and $\nabla^g_X \psi - \frac{1}{2}S(X) \cdot \psi \in L_1$, therefore

$$\nabla^g_X \psi - \frac{1}{2}S(X) \cdot \psi = i\,\mu(X)\psi$$

for a real 1-form μ.

Question: What are the interesting/special spinors in L_j?

Example (Einstein-Sasaki)

L_j each contains a Killing spinor. We know $S = \mathrm{Id}_{\mathcal{H}}$. If $\nabla^g_X \psi = \lambda X \cdot \psi$ then

$$\nabla^g_X \psi - \lambda\, \mathrm{Id}_{\mathcal{H}}(X) \cdot \psi = \lambda\, \eta(X) \xi \cdot \psi$$

Example (η-Einstein [Friedrich, Kim])

Sasaki mfd s.t. $\mathrm{Ric} = \nu g + \kappa \eta \otimes \eta$: *generalized Killing spinor* in L_1 satisfies

$$\nabla^g_X \psi = aX \cdot \psi + b\eta(X) \cdot \xi \cdot \psi$$

i.e.

$$\nabla^g_X \psi - a\, \mathrm{Id}_{\mathcal{H}}(X) \cdot \psi = (a+b)\, \eta(X) \xi \cdot \psi.$$

Remark

A rescaling of $S \propto \mathrm{Id}_{\mathcal{H}}$ and $\mu \propto \eta$ can be obtained by a suitable homothetic deformation.

Conclusion:

Look for parallel sections w.r.t the connection
$\nabla^1_X\psi = \nabla^g_X\psi - \frac{1}{2}S(X)\cdot\psi - s\,\eta(X)\xi\cdot\psi$!

Unfortunately...

$$R^s(X,Y)\psi = R^g(X,Y)\psi + [S(X),S(Y)]\cdot\psi - \frac{1}{2}\varphi(R^g(X,Y)\xi)\cdot\psi \\ + \Phi(\nabla^g_X\xi,\nabla^g_Y\xi) - s\,d\eta(X,Y)\xi\cdot\psi$$

Remark

Observe that dependence on s is only given by $s\,d\eta$. Indeed, the hyperbolic space $\mathbb{H}^7$ can be equipped with a Kenmotsu structure $(\mathcal{C}_5)$, and parallel spinors for $\nabla^{1,s}$ can be constructed for any $s\in\mathbb{R}$.

Generalized Killing spinors can occur in the following classes, provided that $\mu \propto \eta$:

$$\nabla^g_X \psi = \frac{1}{2} S(X) \cdot \psi + \eta(X) \xi \cdot \psi.$$

Class	Description	dim
$\mathcal{C}_6$	$S = \alpha\, \mathrm{Id}_{\mathcal{H}}$, $(\alpha = \frac{1}{2k} \delta\Phi(\xi))$	1
$\mathcal{C}_7$	$S \in S^2_0 \mathcal{H}^*$, $S\varphi = \varphi S$	$k^2 - 1$
$\mathcal{C}_9$	$S \in S^2_0 \mathcal{H}^*$, $S\varphi = -\varphi S$	$k^2 + k$

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Converse construction in dimension 7

Let (M^7, g) a Riemannian spin manifold.

- $\Sigma^{\mathbb{C}} M$ is complexification of a real 8-dimensional bundle $\Sigma^{\mathbb{R}} M$
- If M is closed or non-closed with $\omega_6(M) = 0$, $\exists$ one (and then two) nowhere vanishing vector fields

Hence Σ will then admit a trivial rank-2 subbundle $E = \mathrm{span}(\psi, V \cdot \psi)$ which will play the role of $L_1 \oplus L_2$.

Given $\psi \in \Sigma^{\mathbb{R}}_p M$, $\|\psi\| = 1$ we have $(X \cdot \psi, \psi) = 0$ and therefore

$$\Sigma^{\mathbb{R}}_p M = \langle \psi \rangle \oplus T_p M \cdot \psi$$

and the map $X \mapsto X \cdot \psi$ is an isometry between $T_p M$ and $\psi^{\perp}$.

Lemma

Given $E \subset \Sigma M$ a trivial rank 2 subbundle of ΣM then

1. *There exists $\xi \in \mathfrak{X}(M)$, $\|\xi\| = 1$, unique up to sign s.t.* $\xi \cdot E = E$ *and* $\xi \cdot E^{\perp} = E^{\perp}$.
2. *An almost contact metric structure on M can be defined by*

$$\varphi(X) \cdot \psi := \xi \cdot X \cdot \psi \qquad \forall X \perp \xi, \qquad \varphi(\xi) := 0.$$

independently of ψ.

Proof.

Take $\psi_1, \psi_2 \in E$ orthogonal, $\|\psi_i\| = 1$, then $\psi_2 \in \Gamma(\psi_1^{\perp}) \cong \Gamma(TM \cdot \psi_1)$ i.e. $\psi_2 = \xi \cdot \psi_1$.

Since $\xi \cdot \xi \cdot \psi = -\psi$, multiplication with ξ defines an almost complex structure in $E^{\perp}$ and hence an a.c.m. structure in TM. $\square$

Given an a.c.m. structure on (M^7, g) the bundle E can be retreived via

$$E = \{\psi \in \Gamma(\Sigma^{\mathbb{R}} M) : (\varphi(X) - \xi X) \cdot \psi = 0 \ \forall X \perp \xi\}.$$

Then: $E_{\mathbb{C}} \cong L_1 \oplus L_2$.

Corollary

$$\{\textit{a.c.m. structures}^*\} \overset{\textit{1:1}}{\longleftrightarrow} \{E^2 \subset \Sigma^{\mathbb{R}} M \ \textit{trivial}\}$$

For $\psi \in \Gamma(E)$ with $\|\psi\| = 1$ we now have that $(\nabla^g_X \psi, \psi) = 0$, and since $TM \cong \langle\psi\rangle^\perp$ there exist $S\colon TM \to \mathcal{H}$ and $\mu \in T^*M$ s.t.

$$\nabla^g_X \psi = \frac{1}{2} S(X) \cdot \psi + \mu(X) \xi \cdot \psi$$

Question: In which cases are S and μ independent of ψ?

Proposition

Let S', μ' s.t. $\nabla^g_X(\xi \cdot \psi) = \frac{1}{2} S'(X) \cdot (\xi \cdot \psi) + \mu'(X) \xi \cdot (\xi \cdot \psi)$ then

$$S(X) + S'(X) = 2\varphi(\nabla^g_X \xi)$$

and $\mu' = \mu$. In particular, S does not depend on ψ if and only if $S(X) = \varphi(\nabla^g_X \xi)$ and the structure is $\mathcal{H}$-parallel.