Second-order deformations of associative submanifolds

Kotaro Kawai

Gakushuin University (Japan)

A 7-dimensional Riemannian manifold (M^7, g) said to have a nearly parallel G_2-structure if its cone $(C(M^7), \overline{g}) = (\mathbb{R}_{>0} \times M^7, dr^2 + r^2 g)$ has holonomy contained in $\mathrm{Spin}(7)$.

In this talk, we assume that $M^7 = S^7$ for simplicity.

A nearly parallel G_2-structure $\varphi \in \Omega^3(S^7)$ on S^7, which satisfies $d\varphi = 4 * \varphi$, is defined by the following.

$$\Phi = \frac{1}{2}\omega^2 + \mathrm{Re}\,\Omega, \qquad \varphi = i(\partial_r)\Phi|_{S^7}.$$

where $\omega = \frac{1}{2}\sum_{j=1}^{4} dz^i \wedge d\bar{z}^i, \Omega = dz^1 \wedge \cdots \wedge dz^4$ are the standard Kähler form and the holomorphic volume form on $\mathbb{C}^4 \cong \mathbb{R}^8$.

Define $\chi \in \Omega^3(S^7, TS^7)$ by

$$g(\chi(x, y, z), w) = *\varphi(x, y, z, w).$$

A 3-submanifold $L^3 \subset S^7$ is an associative submanifold if

$$\varphi|_{TL} = \mathrm{vol}_L \iff \Phi|_{TC(L)} = \mathrm{vol}_{C(L)} \iff \chi|_{TL} = 0.$$

Recall:

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$$\varphi|_{TL} = \text{vol}_L \Longleftrightarrow \Phi|_{TC(L)} = \text{vol}_{C(L)}.$$

Example (Examples of associative submanifolds)

- Some totally geodesic $S^3 \subset S^7$,
- Special Legendrian submanifolds,
- The pull back of holomorphic curves in $\mathbb{C}P^3$ via $S^7 \to \mathbb{C}P^3$.

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- ([Lotay, Mashimo]) Classification of homogeneous associative submanifolds
- Lotay constructed an example (A_3) not arising from examples above.
  - $A_3 \cong \mathrm{SU}(2)$ is a orbit of the irreducible $\mathrm{SU}(2)$ representation on $S^3\mathbb{C}^2 \cong \mathbb{C}^4 \cong \mathbb{R}^8$.

- Such examples (up to the $\mathrm{Spin}(7)$-action) other than A_3 are not known.
- Can we get new examples having the same property by deforming A_3?

Deformation of associative submanifolds

Define the moduli space of associative submanifolds by

$$\mathcal{M} = \{L' \subset S^7 : \text{compact associative submanifolds}\}.$$

Fix $L^3 \in \mathcal{M}$. Let $\nu \to L$ be a normal bundle. Since $\exp : \nu \to S^7$ is a diffeomorphism around the zero section, we have

$$\exp : \Gamma(L, \mathcal{U}) := \{\text{small sections of } \nu\} \cong \{\text{submanifolds near } L\}.$$

Define $F : \Gamma(L, \mathcal{U}) \to \Omega^3(L, \nu) \cong \Gamma(L, \nu)$ by

$$F(V) = "\phi_V"(exp_V^* \chi).$$

(Here, $\exp_V : L \to M, \quad \exp_V(x) = \exp_x(V_x)$.)
$(\phi_V : \nu_V \to \nu$ is a bundle isomorphism. We need this so that F takes values in $\Gamma(L, \nu)$.)

$$\exp_V(L) \text{ :associative} \Leftrightarrow F(V) = 0 \quad \text{(:1st order nonlinear PDE).}$$

$$\mathcal{M} \overset{locally}{\cong} F^{-1}(0).$$

Proposition (Mclean, Akbulut-Salur, Gayet)

Then a linearization of F at 0 is given by

$$D = (dF)_0 : \Gamma(L, \nu) \to \Omega^3(L, \nu) \cong \Gamma(L, \nu),$$

$$DV = \sum_{i=1}^{3} e_i \times \nabla^{\perp}_{e_i} V + V,$$

where $\{e_1, e_2, e_3\}$ is a local oriented orthonormal frame of TL s.t. $e_i = e_{i+1} \times e_{i+2} = \varphi(e_{i+1}, e_{i+2}, \cdot)^{\sharp}$ for $i \in \mathbb{Z}/3$, $\nabla^{\perp}$ is the connection on ν induced by the Levi-Civita connection ∇ of S^7.

- $\exists$ a rank 4 vector bundle $E \to M$ s.t. $\nu \cong \mathbb{S} \otimes_{\mathbb{H}} E$, where $\mathbb{S} \to M$ is a spinor bundle. Then $\sum_{i=1}^{3} e_i \times \nabla^{\perp}_{e_i} V$ is a twisted Dirac operator.
- The space of infinitesimal associative deformations of L ($=$ ker D) is a (-1)-eingenspace of a twisted Dirac operator.

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- D is elliptic.
- D is self-adjoint $\Rightarrow \ker D = \mathrm{Coker} D$.

The standard technique to prove the smoothness of a moduli space is an implicit function theorem (cf. [McLean]).
But in our case, D is surjective iff $\dim \ker D = 0$.

Strategy

So that $F^{-1}(0)$ is smooth, the following must hold.

$\forall V \in \ker D, \exists \{V(t)\}_{t \in (-\epsilon, \epsilon)} \subset \Gamma(L, \nu)$ s.t.

$$F(V(t)) = 0 \qquad \text{and} \qquad V(0) = 0, \quad \frac{d}{dt} V(t)\bigg|_{t=0} = V.$$

$V \in \ker D$: unobstructed $\overset{def}{\Leftrightarrow} \exists \{V(t)\}$ as above.

How to check : Consider a formal power series expansion w.r.t. t

$$V(t) = \sum_{k=1}^{\infty} V_k t^k / k! \qquad (V_k \in \Gamma(L, \nu)).$$

Then decide V_k so that $F(V(t)) = 0$.

$$0 = \frac{d^2}{dt^2} F(V(t))\bigg|_{t=0} = \frac{d}{dt}\left((dF)_{V(t)}\left(\frac{dV(t)}{dt}\right)\right)\bigg|_{t=0}$$
$$= (d^2 F)_0(V_1, V_1) + D(V_2).$$

Thus if $(d^2 F)_0(V_1, V_1) \in \mathrm{Im} D$, $V = V_1$ is not unobstructed. We can repeat this process. If it stops for some k, V is not unobstructed.

This idea is used for deformations of many geometric problems.

Example

- All infinitesimal deformations of the following are not unobstructed to second order.
    - Einstein metrics on $\mathbb{C}P^{2l} \times S^2 (l \geq 2)$ ([Koiso]),
    - nearly Kähler structures on $\mathrm{SU}(3)/T^2$ ([Foscolo]).
- All infinitesimal deformations of harmonic maps $T^2 \to S^3$ are not unobstructed to third order ([Mukai]).
- All infinitesimal deformations of complex structures on a Calabi-Yau manifold are unobstructed ([Tian,Todorov]).

Remark

- When we cannot use the implicit function theorem, it is rare that all infinitesimal deformations are unobstructed.
  $\rightsquigarrow$ The result of Tian, Todorov is surprising.
  $\rightsquigarrow$ This was generalized by Goto （"The geometric structures defined by closed differential forms"）.

- The infinitesimal deformations of the normal homogeneous nearly parallel G_2-manifolds are studied by [Alexandrov-Semmelmann]. Nearly parallel G_2-manifolds are all rigid except for the Aloff-Wallach manifold $SU(3)/U(1)$, which admits an 8-dimensional infinitesimal deformations.
  It is an open problem whether these infinitesimal deformations are unobstructed or not.

- The more general deformation theory in terms of DGLA(Differential graded Lie algebra) （or L_∞ algebra） :
  Tian,Todorov $\rightsquigarrow$ Manetti,   Goto $\rightsquigarrow$ Papayanov.

Now we go back to the associative case.
For any $V = V_1 \in \ker D$, we want to find $V_2 \in \Gamma(L, \nu)$ s.t.

$$(d^2 F)_0(V_1, V_1) + D(V_2) = 0.$$

Since D is elliptic, self-adjoint and L is compact, we have

$$\Gamma(L, \nu) = \operatorname{Im} D \oplus \ker D \qquad : L^2 \text{ orthogonal decomposition.}$$

Hence

Lemma

$V_1 \in \ker D$ is unobstructed to second order (i.e. $\exists V_2$ as above.) $\Leftrightarrow$ $(d^2 F)_0(V_1, V_1) = \frac{d^2}{dt^2} F(tV_1)\Big|_{t=0} \perp_{L^2} \ker D$.

- $\frac{d^2}{dt^2} F(tV_1)\Big|_{t=0}$ is computed explicitly.

Proposition

For $V \in \ker D$, we have

$$\frac{d^2}{dt^2}F(tV)\bigg|_{t=0} = 2\sum_{i,j=1}^{3} g(V, II(e_i, e_j))e_i \times \nabla^{\perp}_{e_j}V.$$

where II is the second fundamental form of L in M.

- More generally, we can describe $\frac{d^2}{dt^2}F(tV)\big|_{t=0}$ for any associative submanifolds of a manifold with a G_2-structure.

In my previous paper, I proved that all infinitesimal deformations of homogeneous associative submanifolds (not contained in a totally geodesic S^6) except A_3 are unobstructed.

<u>The properties of $A_3 \cong \mathrm{SU}(2)$</u>

- $\ker D \cong S^6 \mathbb{C}^2 \oplus S^4 \mathbb{C}^2 \oplus S^4 \mathbb{C}^2$: $\dim \ker D = 34$.
- The automorphism group $\mathrm{Spin}(7)$ induces 17-dim infinitesimal deformations:

$$\mathfrak{spin}(7)/(\mathfrak{su}(2) \oplus \mathbb{R}j) \hookrightarrow \ker D.$$

  ($j : S^3 \mathbb{C}^2 \to S^3 \mathbb{C}^2$:a structure map, which is a $\mathbb{C}$-antilinear $\mathrm{SU}(2)$-equivariant map satisfying $j^2 = -1$.)

- We don't know whether the other infinitesimal deformations of a $34 - 17 = 17$-dim subspace in $\ker D$ are unobstructed or not. (Since A_3 does not arise from other known geometries, we can not explain the deformations in terms of other geometries.)
  $\Rightarrow$ Then we study the second order deformations of A_3.

Theorem (K.)

All infinitesimal deformations of A_3 are unobstructed to second order.

(Sketch of the proof:) Define an $SU(2)$-invariant map by
$T : S^2(\ker D) \otimes \ker D \to \mathbb{R}$

$$T(V \odot V, W) = \left\langle \left. \frac{d^2}{dt^2} F(tV) \right|_{t=0}, W \right\rangle_{L^2}.$$

$\ker D \cong S^6\mathbb{C}^2 \oplus S^4\mathbb{C}^2 \oplus S^4\mathbb{C}^2$ & Clebsch-Gordan decomposition
$\Rightarrow T \equiv 0.$

Remark

- *This theorem is not so strong because we do not decide whether infinitesimal deformations of A_3 are unobstructed or not.*

Higher order deformations?

Since

$$
\begin{aligned}
\frac{d^2}{dt^2}F(V(t)) =& \frac{d}{dt}\left((dF)_{V(t)}\left(\frac{dV(t)}{dt}\right)\right) \\
=& (d^2F)_{V(t)}\left(\frac{dV(t)}{dt}, \frac{dV(t)}{dt}\right) + (dF)_{V(t)}\left(\frac{d^2V(t)}{dt^2}\right),
\end{aligned}
$$

we have to find V_3 s.t.

$$
\left.\frac{d^3}{dt^3}F(V(t))\right|_{t=0} = (d^3F)_0(V_1, V_1, V_1) + 3(d^2F)_0(V_1, V_2) + D(V_3) = 0.
$$

$$(d^3F)_0(V_1, V_1, V_1) + 3(d^2F)_0(V_1, V_2) + D(V_3) = 0.$$

- The computation of 3rd order deformations seems to be very hard.
  - $\dim \ker D = 34 \Rightarrow \dim S^3(\ker D) = \begin{pmatrix} 34 + 3 - 1 \\ 3 \end{pmatrix} = 7140.$
  - For $V = V_1 \in \ker D$, the choice of V_2 is not unique.
- For an associative submanifold L^3 in a torsion-free G_2-manifold, $\Omega^*(L^3, \nu)$ admits a structure of an L_∞ algebra ([Fiorenza-Lê-Schwachhöfer-Vitagliano]).
  - Can we apply this method to the case of A_3?