Dirac operators and representations of Lie groups

“Dirac Operators in Differential Geometry and Global Analysis”

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Topics in Talk

- Bundle–valued Dirac operators
- Representations of compact groups: Highest weights
- Realization on kernels of Dirac operators
- Representations of semisimple Lie groups: characters
- Representations of semisimple Lie groups: discrete series
- Realization on kernels of Dirac operators
- Measurable families of Dirac operators
- Representations of semisimple Lie groups: tempered series
- Realization on kernels of partial Dirac operators

Spinors

- $E = \mathbb{R}^n$ with pos. def. inner product $\langle u, v\rangle$, o.n. basis $\{e_i\}$
- Clifford algebra $C\ell(E)$: gen. e_i, rel. $e_i \cdot e_j + e_j \cdot e_i = \delta_{i,j}$
- $C\ell^{\pm}(E) =$ Span$\{e_{i_1} \dots e_{i_k}\}, i_1 < \cdots < i_k, (-1)^k = \pm 1$
- $x \mapsto \bar{x}$ defined by $e_{i_1} \dots e_{i_k} \mapsto (-1)^k e_{i_k} \dots e_{i_1}$
- $Spin(E) = Spin(n) = \{x \in C\ell^+(E) \mid x \cdot \bar{x} = 1, x \cdot E \cdot \bar{x} = E\}$
- vector rep is $v : Spin(n) \to SO(n)$ by $v(x)e = x \cdot e \cdot \bar{x}$
- left multiplication $\ell(\cdot)$ of $Spin(n)$ on $C\ell(E)_C$
- if $n = 2m + 1$ then $\ell = 2^{m+1}s$, s spin rep, space $S, \dim 2^m$
- if $n = 2m$ then $\ell = 2^m s, s = s^+ \oplus s^-$ spin and half–spin reps, rep spaces $S = S^+ \oplus S^-, \dim S^{\pm} = 2^{m-1}$
- if $n = 2m$ Clifford mult defines maps $m^{\pm} : E_C \otimes S^{\pm} \to S^{\mp}$

Riemannian Spin Bundles

- Y: oriented n–dimensional riemannian manifold
- $\mathbb{F} \to Y$: oriented orthonormal frame bundle
- Assume: Y is spin, i.e. the principal $SO(n)$–bundle $\mathbb{F}$ lifts to a principal $Spin(n)$–bundle $\widetilde{\mathbb{F}} \to Y$.
- Let U be a closed subgroup of $Spin(n)$ and $\mathbb{F}_U \to Y$ the principal U–sub-bundle of $\widetilde{\mathbb{F}} \to Y$ with structure group U.
- μ: finite dim unitary rep of U, representation space V_μ, hermitian vector bundle $\mathbb{V}_\mu \to Y$ associated to $\mathbb{F}_U$ by μ
- cases $\mathbb{S} = \mathbb{V}_s$ and $\mathbb{S}^\pm = \mathbb{V}_{s^\pm}$: spin and half-spin bundles
- cases $\mathbb{S} \otimes \mathbb{V}_\mu \to Y$ and (for n even) $\mathbb{S}^\pm \otimes \mathbb{V}_\mu \to Y$: bundles of $\mathbb{V}_\mu$–valued spinors
- inner products on sections: $\langle \phi, \psi \rangle = \int_Y \langle \phi, \psi \rangle_y dy$

Bundle-valued Dirac Operators

- $L_2(Y;\mathbb{S}\otimes\mathbb{V}_\mu)$: square integrable $\mathbb{V}_\mu$–valued spinors
- Now assume $n=2m$ even: then $L_2(Y;\mathbb{S}\otimes\mathbb{V}_\mu)=L_2(Y;\mathbb{S}^+\otimes\mathbb{V}_\mu)\oplus L_2(Y;\mathbb{S}^-\otimes\mathbb{V}_\mu)$
- $\mathbb{T}\to Y$: complexified tang bundle, $\mathbb{T}^*\to Y$ cotang bundle
- covariant differentials: $\nabla^\pm : C^\infty(Y;\mathbb{S}^\pm\otimes\mathbb{V}_\mu)\to C^\infty(Y;\mathbb{T}^*\otimes\mathbb{S}^\pm\otimes\mathbb{V}_\mu)$ and $\nabla=\nabla^++\nabla^- : C^\infty(Y;\mathbb{S}\otimes\mathbb{V}_\mu)\to C^\infty(Y;\mathbb{T}^*\otimes\mathbb{S}\otimes\mathbb{V}_\mu)$
- Clifford mult: $m^\pm:\mathbb{T}\otimes\mathbb{S}^\pm\to\mathbb{S}^\mp$ and $m:\mathbb{T}\otimes\mathbb{S}\to\mathbb{S}$
- Dirac operators: $D=D^+\oplus D^-$ where $D^\pm=(m^\pm\otimes 1)\circ\nabla^\pm : C^\infty(Y;\mathbb{S}^\pm\otimes\mathbb{V}_\mu)\to C^\infty(Y;\mathbb{S}^\mp\otimes\mathbb{V}_\mu)$
- In a moving orthonormal frame on an open subset of Y:
$$D\phi=\sum_{1\leqq j\leqq n} e_j\cdot\nabla_{e_j}(\phi)$$

Square Integrable Harmonic Spinors

- D is an elliptic operator on $L_2(Y; \mathbb{S} \otimes \mathbb{V}_\mu)$
- D has dense domain $C_c^\infty(Y; \mathbb{S} \otimes \mathbb{V}_\mu)$
- On that domain, D is symmetric
- If Y is complete then D and D^2 are essentially self adjoint
- i.e. closures $\widetilde{D} = D^*$ and $\widetilde{D^2} = (D^2)^*$, so
  - those are the unique self–adjoint extensions
  - and they have well defined spectral decompositions
- their kernel is $H_2(Y; \mathbb{V}_\mu) = \{\phi \in L_2(Y; \mathbb{S} \otimes \mathbb{V}_\mu) \mid D\phi = 0\}$: square integrable $\mathbb{V}_\mu$–valued spinors
- $H_2(Y; \mathbb{V}_\mu)$ closed in $L_2(Y; \mathbb{S} \otimes \mathbb{V}_\mu)$, $\subset C^\infty(Y; \mathbb{S} \otimes \mathbb{V}_\mu)$, is orthogonal sum $H_2(Y; \mathbb{V}_\mu) = H_2^+(Y; \mathbb{V}_\mu) \oplus H_2^-(Y; \mathbb{V}_\mu)$ where $H_2^\pm(Y; \mathbb{V}_\mu) = \{\phi \in L_2(Y; \mathbb{S} \otimes \mathbb{V}_\mu) \mid D^\pm\phi = 0\}$

Homogeneous Spin Bundles

- $Y = G/K$: homogeneous riemannian spin manifold: isotropy representation $K \to SO(n)$ lifts to $K \to Spin(n)$
- μ: unitary representation of K on vector space V_μ
- $\mathbb{V}_\mu \to Y$: corresp G–homog. hermitian vector bundle
- The natural action of G on $L_2(Y; \mathbb{V}_\mu)$ is continuous and defines a unitary representation of G
- The Dirac operator D (and $D^\pm$ if n is even) is invariant
- The natural action of G on $H_2(Y; \mathbb{V}_\mu)$ (resp. $H_2^\pm(Y; \mathbb{V}_\mu)$) is a unitary representation π_μ (resp. $\pi_\mu^\pm$) of G
- Now we will assume $n = 2m$ even, and will look at the structure of $\widehat{G}$ and $L_2(G)$ in terms of the $\pi_\mu^\pm$. The first step uses square integrable representations.

Square Integrable Representations

- G is a separable locally compact group, center Z
- $\pi \in \widehat{G}$ irreducible unitary representation
- if $u, v \in \mathcal{H}_\pi$ then the coefficient $f_{u,v}(g) = \langle u, \pi(g)v \rangle$
- These are equivalent:
  - There exist nonzero $u, v \in \mathcal{H}_\pi$ with $|f_{u,v}| \in L^2(G/Z)$.
  - $|f_{u,v}| \in L^2(G/Z)$ for all $u, v \in \mathcal{H}_\pi$.
  - π is a discrete summand of $\mathrm{Ind}\,_Z^G(\chi_\pi)$.
- π is *square integrable* (mod Z) if $|f_{u,v}| \in L_2(G/Z)$ for some $u \neq 0 \neq v$.
- Then the formal degree $\deg \pi > 0$, is defined by $\int_{G/Z} f_{u,v}(x)\overline{f_{u',v'}(x)}d\mu_{G/Z}(xZ) = \frac{1}{\deg \pi}\langle u, u' \rangle \overline{\langle v, v' \rangle}$

Examples

- If G is compact then every $\pi \in \widehat{G}$ is square integrable
- If G is the universal covering group of $U(n)$ then
  - its center $Z \cong \mathbb{Z}$ and G/Z is compact
  - every $\pi \in \widehat{G}$ is square integrable mod Z
- If $G = SU(1,1)$ acting on $B = \{z \in \mathbb{C} \mid |z| < 1\}$ by linear fractional transformations, and $\mathbb{L} \to B$ is a negative G–homogeneous holomorphic line bundle,
- then the natural action of G on $H_2^0(B; \mathcal{O}(\mathbb{L}))$ is an infinite dimensional square integrable representation
- If G is the Heisenberg group of dimension $2n+1$ then the Fock representations (on Hermite polynomials on $\mathbb{C}^n$) are square integrable (mod the center).

Semisimple Group Characters

- G connected real semisimple (or reductive) Lie group
- $\pi \in \widehat{G}$: has three sorts of characters
  - central character ζ_π: $\pi(gz) = \zeta_\pi(z)\pi(g)$ for $z \in Z_G$
  - infinitesimal character: $d\pi(\Xi) = \chi_\pi(\Xi)$ for $\Xi \in \mathcal{Z}(\mathfrak{g})$
    center of the enveloping algebra of $\mathcal{U}(\mathfrak{g})$
  - distribution character (which specifies π):
    $\Theta_\pi(f) = trace \int_G f(x)\pi(x)dx$ for $f \in C_c^\infty(G)$
- If $\Xi \in \mathcal{Z}(\mathfrak{g})$ then $\Xi(\Theta_\pi) = \chi_\pi(\Xi)\Theta_\pi$
- Elliptic regularity Θ_π is C^∞ on the regular set in $G' \subset G$
- $\operatorname{codim}(G \setminus G') \geqq 2$ and Θ_π has only finite jumps on $G \setminus G'$
- $\Theta_\pi(f) = \int_G f(x)T_\pi(x)dx$ where $T_\pi \in C^\infty(G')$ satisfies
  $\Xi(T_\pi) = \chi_\pi(\Xi)T_\pi$ for all $\Xi \in \mathcal{Z}(\mathfrak{g})$

Discrete Series I

- G: semisimple Lie group w/ compact Cartan subgroup T
- $K \subset G$ max compact subgroup, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, with $K \supset T$
- $\Delta_{\mathfrak{t},\mathfrak{g}}$ roots; $\Delta_{\mathfrak{t},\mathfrak{k}}$ compact roots, $\Delta_{\mathfrak{t},\mathfrak{p}}$ noncompact roots,
- $\Delta^+_{\mathfrak{t},\mathfrak{g}}$, $\Delta^+_{\mathfrak{t},\mathfrak{k}}$, $\Delta^+_{\mathfrak{t},\mathfrak{p}}$ positive roots, $\rho_\bullet = \frac{1}{2}\sum_{\Delta^+_{\mathfrak{t},\bullet}} \nu$, $\rho_\mathfrak{g} = \rho_\mathfrak{k} + \rho_\mathfrak{p}$
- $W = W_{\mathfrak{t},\mathfrak{g}}$ Weyl group , $W^1 = \{w \in W \mid \Delta^+_{\mathfrak{t},\mathfrak{k}} \subset w(\Delta^+_{\mathfrak{t},\mathfrak{g}})\}$
- $\mathcal{F}_0' =$ all $\lambda \in i\mathfrak{t}^*$ such that
  - $e^\lambda \in \widehat{T}$ with $\langle \lambda + \rho_\mathfrak{p}, \alpha\rangle \neq 0$ for $\alpha \in \Delta_{\mathfrak{t},\mathfrak{p}}$ and $q(\lambda) = \#$pos
  - $\lambda + \rho_\mathfrak{p}$ is K–dominant, i.e. $\langle \lambda + \rho_\mathfrak{p}, \alpha\rangle > 0$ for $\alpha \in \Delta^+_{\mathfrak{t},\mathfrak{k}}$
- $\lambda \in \mathcal{F}_0'$: $\pi_{\lambda+\rho_\mathfrak{g}}$ is the unique square integrable rep. of G with distribution character given on $T \cap G'$ by

$$T_{\pi_{\lambda+\rho_\mathfrak{g}}}(x) = (-1)^{q(\lambda+\rho_\mathfrak{g})} \frac{\sum_W \det(w) e^{\lambda+\rho_\mathfrak{g}}(x)}{\prod_{\Delta^+_{\mathfrak{t},\mathfrak{g}}}(e^\alpha - e^{-\alpha})(x)}$$

Discrete Series II

- The *discrete series* $\widehat{G}_{disc}$ consists all equivalence classes of square integrable representations of G
- $\lambda \mapsto \pi_{\lambda+\rho_{\mathfrak{g}}}$ defines a bijection $\mathcal{F}_0' \leftrightarrow \widehat{G}_{disc}$
- Given $\lambda \in \mathcal{F}_0'$: $\mu_{\lambda+\rho_{\mathfrak{p}}}$ rep of K, highest wt $\lambda + \rho_{\mathfrak{p}}$, on $V_{\lambda+\rho_{\mathfrak{p}}}$
  - recall $H_2(G/K; \mathbb{V}_{\lambda+\rho_{\mathfrak{p}}})$: square integrable harmonic spinors with values in $\mathbb{V}_{\lambda+\rho_{\mathfrak{p}}}$ and $\pi_{\mu_{\lambda+\rho_{\mathfrak{p}}}}$ rep of G there
  - $H_2^{\pm}(G/K; \mathbb{V}_{\lambda+\rho_{\mathfrak{p}}})$ and $\pi^{\pm}_{\mu_{\lambda+\rho_{\mathfrak{p}}}}$: half spin summands
- Let $j(\lambda)$ be the sign $\pm$ of $(-1)^{q(\lambda+\rho_{\mathfrak{g}})}$
- Theorem of Parthasrathy (slightly improved):
  If $\lambda \in \mathcal{F}_0'$ then (1) $H_2^j(G/K; \mathbb{V}_{\lambda+\rho_{\mathfrak{p}}}) = 0$ for $j \neq j(\lambda)$ and (2) the representation $\pi^{j(\lambda)}_{\mu_{\lambda+\rho_{\mathfrak{p}}}}$ of G on $H_2^{j(\lambda)}(G/K; \mathbb{V}_{\lambda+\rho_{\mathfrak{p}}})$ is the discrete series representation $\pi^{j(\lambda)}_{\mu_{\lambda+\rho_{\mathfrak{g}}}}$

Harish-Chandra Theory I

- Cayley transform between conj classes of Cartan subalg of $\mathfrak{g}$ defines cascade from max. compact Cartan subalg. to max. noncompact Cartan subalg., defines ordering $\succ$
- $H = T \times A = \theta H$: cuspidal parabolic $P = MAN$ and fibr
$$p : X = G/UAN \to G/MAN = K/U$$
θ Cartan involution of G, $K = G^\theta$ and T CSA in $U = M \cap K$
- H–series: $\{\pi_{\gamma,\sigma} = \mathrm{Ind}\,_P^G(\gamma \otimes e^{i\sigma}) \mid \gamma \in \widehat{M}_{disc}$ and $\sigma \in \mathfrak{a}^*\}$
- G'_H: subset of reg set G' of elements conj Cartan $J \succ H$
- Given $\pi_{\gamma,\sigma}$ in the H–series $\widehat{G}_H$:
  - infinitesimal character $\chi_{\pi_{\gamma,\sigma}} = \chi_{\gamma+\rho_{\mathfrak{m}}}|_{Z_G}$
  - distribution character $\Theta_{\pi_{\gamma,\sigma}}$, concentrated in $\bigcup_{J \succ H} G'_J$, determined by restriction to G'_H, given there by an explicit integral formula

Harish-Chandra Theory II

- $Car(G)$: set of all conj classes of Cartan subgroups of G
- $\widehat{G}_{temp} := \bigcup_{H \in Car(G)} \widehat{G}_H$: all the tempered reps of G
- Plancherel measure on $\widehat{G}$ is concentrated on $\widehat{G}_{temp}$
- There is an explicit Plancherel formula for Schwartz class functions.
- However it is somewhat complicated in its dependence on the detailed structure of G. A relatively short argument and statement can be found in math.berkeley.edu/(tilde)jawolf/publications.pdf/paper_103.pdf
- If G is compact then $\widetilde{G}_{temp} = \widetilde{G}_H = \widetilde{G}$, so the Plancherel Theorem is just the Peter-Weyl Theorem.

Partially Harmonic Spinors

- Recall $H = T \times A$: θ–stable Cartan subgroup of G, $P = MAN$ associated cuspidal parabolic subgroup of G, and $p : X = G/UAN \to G/MAN = K/U$ fibration with riemannian symmetric fiber M/U where $U = M \cap K$.
- On each fiber, apply the realization of discrete series representations by harmonic spinors
- This can be done so that the spaces of bundle valued spinors vary measurably between fibers
- These fit together as spaces of square integrable partially harmonic spinors, $\int_{K/U} H_2^{j(\lambda)}(MA/UA; \mathbb{V}_{\lambda+\rho_{\mathfrak{m}/\mathfrak{u}}})d(kU)$
- Comparing with the construction of induced representations, the H–series representations are realized on those spaces of square integrable partially harmonic spinors

Possible Extension to Nilmanifolds

- N: connected simply connected nilpotent Lie group, center Z
- Suppose that N has representations that are square integrable mod Z
- Then the square integrable reps π_λ are specified by central character $e^{i\lambda}, \lambda \in \mathfrak{z}^*$, and the Plancherel density is a certain polynomial in λ
- In this square integrable case, express π_λ as rep of G on square integrable harmonic spinors
- Starting point: the case of the (generalized) Heisenberg group $\mathrm{Im}\,\mathbb{F} + \mathbb{F}^m$ where $\mathbb{F} = \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$

Thank You for Your Attention