Quaternion-Symplectic Structures

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Almost Quaternionic Structures

An Almost quaternionic structure on a manifold M is a smooth algebra sub-bundle $Q \subset \mathrm{End}(TM)$ modelled on the associative algebra of quaternions

$$\mathbb{H} = \langle 1, I, J, K \rangle$$

which is described by

$$I^2 = J^2 = K^2 = IJK = -1$$

Definition

An algebra bundle trivialization of an almost quaternionic structure is called an almost hypercomplex structure. Not every almost quaternionic manifold is trivializable.

An almost quaternionic manifold is of real dimension $4n$, where n is called the quaternionic dimension.

Almost Quaternionic and Subordinated Structures

There has been a lot of development of the theory of quaternionic and almost quaternionic structures and their reductions. The subject can be treated in several ways, for a classical treatment see the 1996 paper by Alekseevsky and Marchiafava. In that work, they considered not only almost quaternionic structures but also reductions:

- Unimodular quaternionic structures (Q, vol)
- Almost hypercomplex structures (I, J, K)
- quaternion pseudo-Hermitian structures (Q, g)
- and more ...

A lot of inspiration for our current project came from this work, as well as works by Swann and Cabrera on almost quaternion-Hermitian structures. A lot of other names should be mentioned.

Motivation

In this talk, we will describe an additional Irreducible reduction which was not previously considered.

Consider a quaternionic right vector space V of quaternionic dimension n. With respect to the structure group $Sp(1)GL(n, \mathbb{H})$, there is a decomposition of the space of (real-valued) two-forms:

$$\Lambda^2(V) = \Lambda^2_{\mathrm{scal}}(V) \oplus \Lambda^2_{\mathrm{Im}\mathbb{H}}(V)$$

- We have $\Lambda^2_{\mathrm{scal}}(V) = \Lambda^2(V)^{Sp(1)}$, called forms of scalar type
- $\Lambda^2_{\mathrm{Im}\mathbb{H}}(V)$ are called forms of imaginary type. Transforms like quaternion-valued forms.

The (local) Kähler forms from quaternion-Hermitian and quaternion-Kähler structures take values in the sections of the forms of imaginary type. In this talk we will discuss the geometries which arise from the scalar type two-forms.

Let V be a quaternionic right vector space.

Definition

A linear quaternion-symplectic structure is a non-degenerate scalar type two-form $\omega \in \Lambda^2_{\text{scal}} V^*$.

Definition

A linear quaternion-symplectic structure is a non-degenerate two-form $\omega \in \Lambda^2 V^*$ *such that the contractions between* I, J, K *and* ω *in one index are symmetric (denoted* g_I, g_J, g_K*).*

Definition

A linear quaternion-symplectic structure is a quaternion-valued inner product $h : V \times V \to \mathbb{H}$ *which is skew-symmetric in the real part and symmetric in the imaginary part. (Set* $\omega = Re(h)$*.)*

Proposition

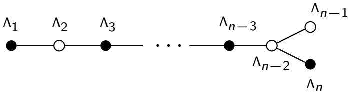
These definitions are equivalent.

From now on, we will use the first definition $\omega \in \Lambda^2_{\mathrm{scal}} V^*$. The structure group of a linear quaternionic-symplectic structure on $\mathbb{H}^n$ is $Sp(1)SO^*(2n) \subset \mathrm{Aut}(Q) = Sp(1)GL(n, \mathbb{H})$. Subgroup diagram:

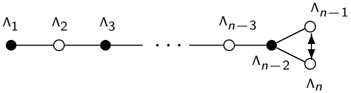
$$\begin{tikzcd}
 & Sp(1)GL(n,\mathbb{H}) & \\
 & Sp(1)SL(n,\mathbb{H}) \arrow[u] & \\
Sp(1)Sp(p,q) \arrow[ur] & & Sp(1)SO^*(2n) \arrow[ul]
\end{tikzcd}$$

The group $G = SO^*(2n)$ is different for even and odd quaternionic dimension. This is reflected in the Satake diagrams of their Lie algebras.

$n = 2k$:



$n = 2k + 1$:



Therefore we will give separate descriptions of the structures by means of finding special subgroups in $SO^*(2n)$ that allow us to write the quaternion-symplectic forms as a product of more well known tensors.

For even $n = 2k$, consider the irreducible subgroup $Sp(1)Sp(2k, \mathbb{R})$. Note that this $Sp(1)$ is not the same as the ideal in $Sp(1)SO^*(2n)$! With respect to this reduction, we can write $\mathbb{H}^n \simeq \mathbb{H} \otimes \mathbb{R}^{2k}$, with the structure

$$\omega = g_{\mathbb{H}} \otimes \omega_0$$

This realizes the linear quaternion-symplectic structure as a quaternionification of a linear symplectic structure. A tensor product basis here yields a quaternionic "Darboux basis".

$$\mathbb{H}^n = \langle v_1^e, v_1^i, v_1^j, v_1^k, v_2^e, \cdots, v_n^e, v_n^i, v_n^j, v_n^k \rangle$$

such that

$$\omega = \theta_1^e \wedge \theta_2^e + \theta_1^i \wedge \theta_2^i + \theta_1^j \wedge \theta_2^j + \theta_1^k \wedge \theta_2^k + \theta_3^e \wedge \theta_4^e + \cdots + \theta_{n-1}^k \wedge \theta_n^k$$

i.e. $\mathbb{H}^n$ decomposes into a sum of (quaternionic) one-dimensional quaternionic subspaces which are isotropic.

For odd $n = 2k+1$, we consider the subgroup $SO(2k+1, \mathbb{C})$. Then we can write $\mathbb{H}^n \simeq \mathbb{C}^{2k+1} \otimes_{\mathbb{R}} \mathbb{C} = (\mathbb{C}^n)^{\mathbb{C}}$.

$$\mathrm{Re}(g) \otimes \omega_0$$

This is the quaternionification of the complex-bilinear form g on a complex vector space.

Remark

Although the odd and even subgroups are quite different, they were found by the same method. The technique is to look for "diagonal" subgroups, i.e. subgroups for which the standard module $\mathbb{H}^n$ branches into a sum of several equivalent modules. In the even case this yields a diagonal $Sp(1) \subset SO^*(2n)$. In the odd case however, a diagonal $Sp(1)$ does not exist.

Definition

An almost quaternion-symplectic structure on an almost quaternionic manifold (M, Q) is a non-degenerate two form $\omega \in \Omega^2_{\mathrm{scal}}(M)$ of scalar type.

Definition

An almost hypercomplex-symplectic structure on an almost hypercomplex manifold (M, I, J, K) is a non-degenerate two form $\omega \in \Omega^2_{\mathrm{scal}}(M)$ of scalar type.

These manifolds are G-structures for $G = Sp(1)SO^*(2n)$ and $G = SO^*(2n)$, respectively.

Remark

There is a fundamental tensor $\Phi \in S^4 T^* M$.

$$\Phi = g_I \odot g_I + g_J \odot g_J + g_K \odot g_K$$

Quaternionified symplectic structures

Let (M, ω_0) be an almost symplectic manifold. Define

$$M^{\mathbb{H}} = M \times M_i \times M_j \times M_k$$

where each factor is a diffeomorphic copy of M. For arbitrary local coordinates $\{x^I\}$ on M, define coordinates $\{x^I, Ix^I, Jx^I Kx^I\}$ on $M^{\mathbb{H}}$. Then, for $A \in \{I, J, K\}$, set

$$A(\partial_{qx^I}) = \partial_{(Aq)x^I}$$

modulo $\partial_{-qx^I} = -\partial_{qx^I}$. Now we have

$$T_q M^H \simeq T_x M \otimes \langle 1, I, J, K \rangle \simeq T_x M \otimes \mathbb{H}$$

which allows us to use the formula

$$\omega = \omega_0 \otimes g_{\mathbb{H}}$$

to realize $(M^{\mathbb{H}}, I, J, K, \omega)$ as an even dimensional almost hypercomplex-symplectic structure.

Cotangent bundles

Let (N^{4n}, Q, ∇) be an affine quaternionic manifold, ie. equipped with a torsion-free quaternionic connection. Set $M = T^*N$. Then, we have the splitting into horizontal and vertical subspace

$$T(T^*N) = \mathcal{H} \oplus \mathcal{V} \simeq TN \oplus T^*N$$

and the map

$$\rho : \mathrm{End}(TN) \to \mathrm{End}(TN) \oplus \mathrm{End}(T^*N) \xrightarrow{\nabla} \mathrm{End}(TM)$$

Thus we can induce a quaternionic structure $\rho(Q)$ on M. Then, the canonical symplectic form $\omega = -d\theta$ is of scalar-type, and in fact this happens if and only if ∇ is torsion-free. Thus we have constructed a quaternion-symplectic structure on $M = T^*N$.

Remark

Removing the dependence on ∇...

The almost hypercomplex-symplectic structure group can be regarded as a subgroup of $SO(2n, 2n)$. Therefore, we can in principle treat it as a pseudo-Riemannian reduction. However, the three metrics g_I, g_J, g_K are on equal footing, and this method requires a choice.
The almost quaternion symplectic structure group, on the other hand, admits no invariant pseudo-Riemannian metric.

Intrinsic torsion of general G-structures

We want to consider the integrability properties of almost quaternion-symplectic structures. To do this in general we need to use adapted connections. For general G-structures, such connections are not necessarily unique or canonical. In this situation we may use the Spencer cohomology,

$$\mathfrak{g}^{(1)} \to \mathfrak{g} \otimes V^* \xrightarrow{\delta} V \otimes \Lambda^2 V^*$$

This measures how the torsion of an adapted connection can be calibrated by adding gauge terms with values in the structure algebra. Thus we have a notion of minimal intrinsic torsion.

Theorem

For the Lie algebra $\mathfrak{g} = \mathfrak{sp}(1) \oplus \mathfrak{so}^*(2n)$, we have the Spencer δ-cohomology

$$H^{0,2}(\mathfrak{g}) = H(E + K + S^3_0 E) + S^3 H(\Lambda_3 E + K).$$

This is the module where intrinsic torsion lives. The prolongation $\mathfrak{g}^{(1)} = T^* \otimes \mathfrak{g} \cap S^2 T^* \otimes T$ give torsion-preserving gauges.

Berger criterion

There is a necessary condition for a Lie algebra $\mathfrak{g}$ to appear as a holonomy algebra of a torsion-free connection. This is called the Berger criterion. This criterion is not satisfied by $\mathfrak{so}^*(2n)$, Hence torsion-free hypercomplex-symplectic structures are necessarily locally equivalent, meaning we have in a sense quaternionic Darboux coordinates. However, $\mathfrak{sp}(1) \oplus \mathfrak{so}^*(2n)$ does satisfy the Berger criterion, and can appear as a holonomy subalgebra of a torsion-free connection. This means that fully 1-integrable quaternion-symplectic structures admit local geometry. (This is not due to us, see Berger, Bryant and Schwachhöfer.)

The torsion-free case

Torsion-free quaternion symplectic structures include, in particular,

- a symplectic structure ω,
- a torsion-free connection ∇^* with $\nabla^*\omega = 0$,
- with reduced holonomy $Hol(\nabla^*) = G = Sp(1)SO^*(2n) \subset Sp(4n, 2\mathbb{R})$.

This situation can therefore be considered from the symplectic point of view, and falls under the umbrella of special symplectic holonomy. This was explored by Schwachhöfer and Cahen in 2009, which yielded a local classification. We specialize their results to our structure.

Theorem (Schwachhöfer, Cahen)

A torsion-free quaternion-symplectic structure (Q, ω) is locally equivalent to a symplectic reduction of the parabolic contact geometry $SO^*(2n+2)/P_2$ by a symmetry vector field. Hence the moduli space of such structures is finite dimensional.

Proposition

We have the first prolongations $\mathfrak{g}^{(1)} = \{0\}$ *for* $\mathfrak{g} = \mathfrak{so}^*(2n)$ *and* $\mathfrak{g} = \mathfrak{sp}(1) \oplus \mathfrak{so}^*(2n)$

Corollary

Let (M, Q, ω) *be an almost quaternion-symplectic structure. Then there exists a unique minimal connection* ∇^* *with torsion such that*

- *The torsion component in* $S^3H(\Lambda^3E + K)$ *coincides with the intrinsic torsion of the almost-quaternionic structure* Q.
- *The torsion component in* $H(E + K)$ *is the projections of the intrinsic torsion of the almost-symplectic structure* ω*, branched.*
- *The torsion component in* $H\,S^3_0E$ *is the "compatibility" torsion of* (Q, ω).

There is a corresponding statement for almost hypercomplex-symplectic structures, but with many more torsion components.

Remark

Almost quaternionic structures or almost-symplectic structures admit no canonical choice of connection.

First, notice that we can take the tensor power

$$\mathrm{vol} = \omega^{2n},$$

which is gives us a reduction to a unimodular quaternionic structure (Q, vol). Following Alekseevsky, Marchiafava we may start with an arbitrary Oproiu connection ∇ then obtain a unique quaternionic connection preserving vol:

$$\nabla^{(Q,\mathrm{vol})} = \nabla + \frac{1}{4(n+1)} S^{\theta}$$

Where θ is defined by $\nabla \mathrm{vol} = \theta \otimes \mathrm{vol}$ and S^{θ} is an equivariant map from T^*M to $TM \otimes S^2 T^*M$. Then we may take the adaptation

$$\nabla^* = \nabla^{(Q,\mathrm{vol})} + A$$

where A depends on $\nabla^{(Q,\mathrm{vol})}\omega$: $\omega(A(X,Y),Z) = \frac{1}{2}(\nabla_X^{(Q,\mathrm{vol})}\omega)(Y,Z)$.

Non-tensorial representations of Lie algebras

Let $\mathfrak{g}$ be a real simple Lie algebra, and let V be a real $\mathfrak{g}$-module of lowest possible dimension. Call a $\mathfrak{g}$-module tensorial if it appears as a submodule in a tensor power $\bigotimes^k V$ for some k.

Proposition

All finite-dimensional simple $\mathfrak{g}$-modules are tensorial unless $\mathfrak{g}$ is one of

- $\mathfrak{so}(n)$
- $\mathfrak{so}(p,q)$
- $\mathfrak{so}^*(2n)$

The existence of a non-tensorial $\mathfrak{g}$-module implies the existence of a non-trivial finite covering $G' \to G = \exp(\mathfrak{g})^2 \subset \mathrm{End}(V)$.

The group $SO^*(2n)$ has maximal compact subgroup $U(n)$, hence $\pi_1(SO^*(2n)) = \mathbb{Z}$.

Definition

Let $Spin^*(2n) = \exp(\mathfrak{g})^2 \subset End(W)$ be the Lie group acting faithfully on $W = \bigoplus_\alpha V_\alpha$, where W is the direct sum of all the fundamental representations of $\mathfrak{so}^*(2n)$.

The cover $Spin^*(2n) \to SO^*(2n)$ can be given by restricting the representation to $V = V_{\pi_1}$, the first fundamental representation.

Definition

Let (M, Q, ω) be a quaternion-symplectic manifold. We may attempt to define an analogue of the Riemannian spin-structures and spinor-bundles by considering principal $Sp(1)Spin^*(2n)$-bundles compatible with the structure group, and bundles associated to non-tensorial $\mathfrak{sp}(1) \oplus \mathfrak{so}^*(2n)$-modules.

Due to the canonical connection ∇^*, we may then attempt to mimic the construction of Dirac operators.