Harmonic Spinors on Gravitational Instantons

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The title

- Gravitational instantons: Complete Einstein (or more) Riemannian 4-manifolds. We will consider:

Taub-Bolt	TB	Ricci-flat	Non-compact
Euclidean Schwarzschild	ES	Ricci-flat	Non-compact
Taub-NUT	TN	Hyperkähler	Non-compact

- Harmonic spinors: Solutions of the (massless) Dirac equation

$$\not{D}\psi = 0, \quad \not{D} = \gamma^\mu \nabla_\mu,$$
$$\not{D}_{\mathcal{A}}\psi = 0, \quad \not{D}_{\mathcal{A}} = \gamma^\mu (\nabla_\mu + \mathcal{A}_\mu).$$

- We will construct all the L^2 harmonic spinors on TN, TB, ES.

ALF gravitational instantons

Taub-NUT (TN), Taub-Bolt (TB), Euclidean Schwarzschild (ES).

- Complete, Ricci-flat, non-compact, infinite volume.
- Topology: TN: $\mathbb{C}^2$; ES: $\mathbb{R}^2 \times S^2$; TB: $\mathbb{C}P^2 \setminus \{p\}$.
- Rotationally symmetric ($SU(2)$ or $SO(3)$ isometry group).
- Additional isometric circle action.
- Asymptotically circle bundles over Euclidean $\mathbb{R}^3$ with fibres of finite length (ALF).
  TN, TB twisted (Hopf fibration); ES trivial ($S^2 \times S^1$).
- Fixed point set of the $U(1)$ action (nuts and bolts) topology:
  TN: NUT (single point); TB, ES: bolt (2-sphere).

ALF gravitational instantons (2)

- Metric of bi-axial Bianchi IX type (TN, TB):

$$g = f^2(r)\mathrm{d}r^2 + a^2(r)(\eta_1^2 + \eta_2^2) + c^2(r)\eta_3^2, \quad \text{TN, TB},$$
$$g = f^2(r)\mathrm{d}r^2 + a^2(r)(\eta_1^2 + \eta_2^2) + c^2(r)\mathrm{d}\psi^2, \text{ ES},$$

  η_i left-invariant 1-forms on $SU(2)$, $\eta_1^2 + \eta_2^2 = g_{S^2}$, $\psi \in [0, 2\pi)$.
- TB has the asymptotic topology of TN (twisted circle fibration) and the fixed point set structure of ES (bolt).

A problem and its solution

- TB is not spin.
- TN and ES are spin but by Lichnerowicz's identity

$$\langle \not{D}\psi, \not{D}\psi \rangle = \langle \nabla\psi, \nabla\psi \rangle + \frac{s}{4}\|\psi\|^2$$

  they admit no non-trivial L^2 harmonic spinors.
- Solution: twist the Dirac operator by an Abelian connection $\mathcal{A}$ (introduce a $\mathrm{Spin}^{\mathbb{C}}$ structure).
- It is natural to require the curvature $\mathcal{F} = \mathrm{d}\mathcal{A}$ to be an L^2 harmonic 2-form.

Harmonic cohomology

- By results of Hausel, Hunsicker and Mazzeo

$$L^2\mathcal{H}^2(M) = H^2(X_M), \qquad L^2\mathcal{H}^p(M) = 0 \text{ if } p \neq 2,$$

where $M \in \{\text{TN, TB, ES}\}$ and $X_M = M \cup \Sigma_\infty$ is the compact manifold obtained by collapsing the fibres of the asymptotic fibration which has base Σ_∞. In all three cases $\Sigma_\infty = S^2$.

- Smooth simply connected 4-manifolds are classified up to homeomorphism by their intersection form. Therefore:

M	X_M	$\dim(H^2(X_M))$
TN	$\mathbb{C}P^2$	1
ES	$\mathbb{C}P^1 \times \mathbb{C}P^1$	2
TB	$\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$	2

Harmonic cohomology (2)

- In all 3 cases $L^2\mathcal{H}^2$ is generated by $\{\mathrm{d}\xi^\flat, *\mathrm{d}\xi^\flat\}$, ξ the Killing vector field generating the $U(1)$ isometry. For TN $\mathrm{d}\xi^\flat = *\mathrm{d}\xi^\flat$.
- However it is more convenient to take harmonic representatives of the Poincaré duals (in the compactification) of Σ_∞ and, for ES and TB, of the bolt. For $p, q \in \mathbb{Z}$ we have

$$
\begin{aligned}
\mathcal{F}_{\mathsf{TN}} &= -\pi(2q+1)F_\infty, && \text{self-dual (SD)},\\
\mathcal{F}_{\mathsf{ES}} &= -2\pi\left[q\,F_\infty + p\,F_{\mathrm{bolt}}\right], && \text{SD if } p = q,\\
\mathcal{F}_{\mathsf{TB}} &= -\pi\left[(2q+1)F_\infty - (2p+1)F_{\mathrm{bolt}}\right], && \text{SD if } q = 3p+1.
\end{aligned}
$$

L^2 Harmonic spinors — TB

- We are looking for normalisable solutions of the equation

$$0 = \not{D}_{\mathcal{A}}\psi = \begin{pmatrix} \mathbf{0} & \mathbf{T}_{\mathcal{A}}^{\dagger} \\ \mathbf{T}_{\mathcal{A}} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \Psi \\ \Phi \end{pmatrix}, \qquad \mathrm{d}\mathcal{A} = \mathcal{F},$$

  $\mathbf{T}_{\mathcal{A}} = \phi_1(r, \partial_r)\mathbf{1} + \phi_2(r)\mathbf{P}_{\mathcal{A}}$, with $\mathbf{P}_{\mathcal{A}}$ the twisted Dirac operator on the squashed 3-sphere (TN, TB). L^2 solutions have either $\Psi = 0$ or $\Phi = 0$. Let us focus on $\Phi = 0$.
- Reduce to an ODE by taking $\Psi = h(r)\mathbf{v}$, $\mathbf{v}$ eigenvector of $\mathbf{P}_{\mathcal{A}}$.
- Eigenvectors of $\mathbf{P}_{\mathcal{A}}$ leading to L^2 solutions have the form $(|j, j, m_2\rangle, 0)^T$ or $(0, |j, -j, m_2\rangle)^T$ with $|j, m_1, m_2\rangle := |j, m_1\rangle \otimes |j, m_2\rangle \in V_j \otimes V_j$ with V_j the $\mathfrak{sl}(2, \mathbb{C})$ irreducible representation of dimension $2j + 1$.

L^2 Harmonic spinors — TB (2)

Using coordinates in which the metric has the form

$$g_{\mathrm{TB}} = V\mathrm{d}r^2 + (r^2 - N^2)(\eta_1^2 + \eta_2^2) + 4N^2V^{-1}\eta_3^2,$$
$$V = \frac{r^2 - N^2}{(r-2N)(r-N/2)}, \; N > 0, \; r \in [2N, \infty),$$

the solution of the ODE corresponding to $m_1 = \pm j$ is

$$h = \frac{C}{\sqrt{r+N}}\,\mathrm{e}^{\left(2j+1\mp\left(q+\frac{1}{2}\right)\right)\frac{r}{4N}}.$$
$$(r-2N)^{-\left(j+\frac{3}{4}\mp\frac{1}{2}\left(p+\frac{1}{2}\right)\right)}\,(r-N/2)^{-\left(\frac{j}{4}+\frac{3}{8}\mp\frac{1}{8}\left(p+\frac{1}{2}\right)\right)}.$$

L^2 Harmonic spinors — TB focused (3)

- The $m_1 = \pm j$ TB solution is L^2 if
$$\pm\left(p+\frac{1}{2}\right)+\frac{1}{2} \leq 2j+1 \leq \pm\left(q+\frac{1}{2}\right)-\frac{1}{2}.$$
Recall that $\mathcal{F} = -\pi\left[(2q+1)F_\infty - (2p+1)F_{\mathrm{bolt}}\right]$.
- $|j, \pm j, m_2\rangle$ has multiplicity $2j+1$ as $m_2 \in \{-j, -j+2, \ldots, j\}$, so the number of L^2 harmonic spinors is, as $\mathrm{Ker}(\mathbf{T}^\dagger_{\mathcal{A}}) = 0$,
$$\mathrm{index}(\not{D}_{\mathcal{A}}) = \dim\left(\mathrm{Ker}(\mathbf{T}_{\mathcal{A}})\right) = \left|\frac{q(q+1)}{2} - \frac{p(p+1)}{2}\right|,$$
in agreement with what found using the APS index theorem.

L^2 Harmonic spinors — TN and ES

For both L^2 harmonic spinors $(\Psi, \Phi)^T$ have either $\Phi = 0$ or $\Psi = 0$.

- TN is completely analogous. $\Psi = k(r)(|j, j, m_2\rangle, 0)^T$ or $k(r)(0, |j, -j, m_2\rangle)^T$. The ODE for k is different; the allowed values of j for $m_2 = \pm j$ are $1 \leq 2j + 1 \leq \pm(q + 1/2) - 1/2$.
- ES needs a slightly different treatment since the $SO(3)$ orbits are 2-dimensional. L^2 harmonic spinors have the form

$$\Psi = h_\pm(r)\mathrm{e}^{\pm i(n+1/2)\psi}\mathbf{v}_\pm, \quad n \in \mathbb{Z},\ 0 \leq n \leq \pm q - 1,$$

where $\mathbf{v}_\pm$ is a zero mode of the Dirac operator on S^2 twisted by the line bundle with first Chern number p. $\mathbf{v}_\pm$ belongs to the $\mathfrak{sl}(2, \mathbb{C})$ irrep of dimension $|p|$. Upper (lower) sign for $p \geq 1$ ($p \leq -1$).

Harmonic spinors and topology

$L^2\mathcal{H}^2$ is related to the structure of the $U(1)$ action fixed point set $\mathcal{S}$. Its dimension is equal to the number of bolts.

L^2 harmonic spinors belong to $\mathfrak{sl}(2,\mathbb{C})$ irreps. The allowed dimension $2j+1$ of the irrep is related to the value of $\mathcal{A}$ over $\mathcal{S}$: Let $\omega = \frac{1}{2}\eta_3$ be the connection on the $U(1)$ bundle over S^2 with first Chern number 1. Then $(m_1 = +j)$

M	$\mathcal{A}\vert_{\text{nut/bolt}}$	$\mathcal{A}\vert_{\Sigma_\infty}$	range of j	index
TB	$\left(p+\frac{1}{2}\right)\omega$	$\left(q+\frac{1}{2}\right)\omega$	$\frac{p}{2},\ldots,\frac{q-1}{2}$	$\frac{1}{2}\vert q(q+1)-p(p+1)\vert$
TN	0	$\left(q+\frac{1}{2}\right)\omega$	$0,\ldots,\frac{q-1}{2}$	$\frac{1}{2}\vert q(q+1)\vert$
ES	$p\,\omega$	$p\,\omega$	$j=\frac{p-1}{2}$	$\vert pq\vert$

In all cases the result agrees with the APS index theorem.

Thank you very much for your attention!