Generalizations of 3-Sasakian manifolds and connections with skew torsion

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Goals & Motivation

Define and investigate new classes of such manifolds:

- the Levi-Civita connection is not well-adapted to the structure
- look for 'good' metric connections with skew torsion

In particular,

- introduce notion of φ-compatible connections,
- make them unique by a certain extra condition $\rightarrow$ canonical connection,
- define the new class of canonical almost 3-contact metric manifolds
- define and study 3-(α, δ)-Sasaki manifolds
- compute torsion, holonomy, curvature of the canonical connection,
- provide lots of examples, classify the homogeneous ones, further applications (metric cone, existence of generalized Killing spinors. . .)

Almost contact metric structures

$(M^{2n+1}, \varphi, \xi, \eta, g)$ almost contact metric manifold if

- ξ is a vector field ξ, called the *Reeb vector field*,
- $\eta = g(\xi, \cdot)$,
- φ is a (1,1)-tensor field such that $\varphi\xi = 0$ and

$$\varphi^2 = -I, \quad g(\varphi X, \varphi Y) = g(X, Y) \text{ on } \langle\xi\rangle^\perp = \ker\eta.$$

Equivalently, the structural group is reducible to $U(n) \times \{1\}$.

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Then,

- the fundamental 2-form is defined by

$$\Phi(X, Y) = g(X, \varphi Y),$$

- it is called normal if $N_\varphi := [\varphi, \varphi] + d\eta \otimes \xi \equiv 0$,
- α-Sasakian, $\alpha \in \mathbb{R}^*$, if $d\eta = 2\alpha\Phi, \quad N_\varphi \equiv 0 \ (\Rightarrow \xi$ Killing)
- Sasakian if 1-Sasakian.

A metric connection ∇ on (M, g) has totally skew-symmetric torsion (skew torsion for brief) if the $(0,3)$-tensor field

$$T(X, Y, Z) := g(\nabla_X Y - \nabla_Y X - [X, Y], Z)$$

is a 3-form ($\Leftrightarrow$ same geodesics as ∇^g)

$$\textcolor{red}{\Rightarrow} \quad g(\nabla_X Y, Z) \;=\; g(\nabla_X^g Y, Z) + \tfrac{1}{2}\, T(X, Y, Z)$$

Given a G-structure ($G \subset SO(n)$) on (M, g), if there exists a unique ∇ with skew torsion preserving the structure, ∇ is called characteristic connection.

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Given a G-structure ($G \subset SO(n)$) on (M, g), if there exists a unique ∇ with skew torsion preserving the structure, ∇ is called **characteristic connection**.

Theorem (Friedrich-Ivanov, 2002)

An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ admits a unique **metric connection ∇ with totally skew symmetric torsion**, *and such that $\nabla\eta = \nabla\xi = \nabla\varphi = 0$, if and only if*

1. *the tensor $N_\varphi := [\varphi, \varphi] + d\eta \otimes \xi$ is totally skew-symmetric,*

2. *ξ is a Killing vector field.*

In particular, it exists for α-Sasaki manifolds and its torsion $T = \eta \wedge d\eta$ is parallel.

Almost 3-contact metric manifolds

$(M^{4n+3}, \varphi_i, \xi_i, \eta_i, g)$, $i = 1, 2, 3$ is almost 3-contact metric manifold if

- each structure $(\varphi_i, \xi_i, \eta_i, g)$ is almost contact metric
- on the *vertical distribution* $\mathcal{V} := \langle \xi_1, \xi_2, \xi_3 \rangle$:

$$\varphi_i \xi_j = \xi_k = -\varphi_j \xi_i \quad (\Rightarrow \xi_1, \xi_2, \xi_3 \text{ are orthogonal})$$

- on the *horizontal distribution* $\mathcal{H} := \mathcal{V}^\perp = \bigcap_{i=1}^{3} \ker \eta_i$:

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for every even permutation (i, j, k) of $(1, 2, 3)$.

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$$\varphi_i \varphi_j = \varphi_k = -\varphi_j \varphi_i$$

for every even permutation (i, j, k) of $(1, 2, 3)$. Then,

- structure group reducible to $\mathrm{Sp}(n) \times \{1_3\}$
- M is said to be hypernormal if $N_{\varphi_i} \equiv 0$, $i = 1, 2, 3$.
- 3-α-Sasakian if each structure is α-Sasakian
- 3-Sasakian if each structure is Sasakian $\quad \Rightarrow$ Einstein!

Theorem (Kashiwada, 2001)

If $d\eta_i = 2\Phi_i$, $i = 1, 2, 3$, then M is hypernormal (and thus 3-Sasakian).

The associated sphere of structures

An almost 3-contact metric manifold $(M, \varphi_i, \xi_i, \eta_i, g)$ carries a sphere $\Sigma_M \cong S^2$ of almost contact metric structures.

For every $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ such that $a_1^2 + a_2^2 + a_3^2 = 1$, put

$$\varphi_a = \sum_{i=1}^3 a_i \varphi_i, \quad \xi_a = \sum_{i=1}^3 a_i \xi_i, \quad \eta_a = \sum_{i=1}^3 a_i \eta_i.$$

Then $(\varphi_a, \xi_a, \eta_a, g)$ is an almost contact metric structure.

Theorem (Cappelletti Montano - De Nicola - Yudin, 2016)

If $N_{\varphi_i} = 0$ for all $i = 1, 2, 3$, then $N_\varphi = 0$ for all $\varphi \in \Sigma_M$.

Theorem

If each N_{φ_i} is skew symmetric on $\mathcal{H}$ (respectively on TM), then for all $\varphi \in \Sigma_M$, N_φ is skew symmetric on $\mathcal{H}$ (respectively on TM).

Proposition

Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a almost 3-contact metric manifold. If each $(\varphi_i, \xi_i, \eta_i, g)$, $i = 1, 2, 3$ admits a characteristic connection, the same holds for every structure in the sphere.

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Do these connections coincide?

Is it possible to find a metric connection with skew torsion parallelizing ALL the structure tensor fields?

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! For a 3-Sasakian manifold the characteristic connection of the structure $(\varphi_i, \xi_i, \eta_i, g)$ is

$$\nabla^i = \nabla^g + \frac{1}{2} T_i, \qquad T_i = \eta_i \wedge d\eta_i.$$

For $i \neq j$, $T_i \neq T_j$ and thus $\nabla^i \neq \nabla^j$.

We need to relax the requirement that the structure should be parallel

Canonical connection for 7-dimensional 3-Sasaki manifolds
(Agricola-Friedrich, 2010)

Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a 7-dimensional 3-Sasakian manifold.
The 3-form

$$\omega := \frac{1}{2} \sum_i \eta_i \wedge d\eta_i + 4\,\eta_{123} \qquad \eta_{123} := \eta_1 \wedge \eta_2 \wedge \eta_3$$

defines a *cocalibrated G_2-structure* and hence admits a characteristic connection ∇; its torsion is

$$T = \sum_{i=1}^{3} \eta_i \wedge d\eta_i$$

∇ is called the canonical connection, and verifies the following:

- it preserves $\mathcal{H}$ and $\mathcal{V}$,
- $\nabla T = 0$,
- ∇ admits a parallel spinor ψ, called *canonical spinor*, such that the Clifford products $\xi_i \cdot \psi$ are exactly the 3 Riemannian Killing spinors.

Canonical connection for quaternionic Heisenberg groups

N_p connected, simply connected 2-step nilpotent Lie group with Lie algebra

$$\mathfrak{n}_p = \mathrm{span}(\xi_1, \xi_2, \xi_3, \tau_r, \tau_{p+r}, \tau_{2p+r}, \tau_{3p+r}), \quad r = 1, \ldots, p,$$

and non-vanishing commutators ($\lambda > 0$):

$$[\tau_r, \tau_{p+r}] = \lambda \xi_1 \qquad [\tau_r, \tau_{2p+r}] = \lambda \xi_2 \qquad [\tau_r, \tau_{3p+r}] = \lambda \xi_3$$
$$[\tau_{2p+r}, \tau_{3p+r}] = \lambda \xi_1 \qquad [\tau_{3p+r}, \tau_{p+r}] = \lambda \xi_2 \qquad [\tau_{p+r}, \tau_{2p+r}] = \lambda \xi_3.$$

N_p admits an almost 3-contact metric structure $(\varphi_i, \xi_i, \eta_i, g_\lambda)$:

η_i dual 1-form of ξ_i

g_λ Riemannian metric such that $\{\xi_i, \tau_l\}$ is orthonormal

$$\begin{aligned}
\varphi_i \;=\; & \eta_j \otimes \xi_k - \eta_k \otimes \xi_j + \sum_{r=1}^{p} [\theta_r \otimes \tau_{ip+r} - \theta_{ip+r} \otimes \tau_r \\
& + \theta_{jp+r} \otimes \tau_{kp+r} - \theta_{kp+r} \otimes \tau_{jp+r}]
\end{aligned}$$

$(\theta_l, l = 1, \ldots, 4p, \text{ dual } 1- \text{ form of } \tau_l)$

The structure is hypernormal with $d\Phi_i \neq 0$.

The canonical connection (Agricola-Ferreira-Storm, 2015) is the metric connection ∇ with skew torsion

$$T = \sum_{i=1}^{3} \eta_i \wedge d\eta_i - 4\,\lambda\,\eta_{123}$$

It satisfies:

- $\nabla T = \nabla R = 0 \rightsquigarrow$ naturally reductive homogeneous space,
- $\mathfrak{hol}(\nabla) \simeq \mathfrak{su}(2)$, acting irreducibly on $\mathcal{V}$ and $\mathcal{H}$.

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In the 7-dimensional case, ∇ is the *characteristic connection* of the cocalibrated G_2 structure

$$\omega = -\eta_1 \wedge (\theta_{12} + \theta_{34}) - \eta_2 \wedge (\theta_{13} + \theta_{42}) - \eta_3 \wedge (\theta_{14} + \theta_{23}) + \eta_{123}.$$

Then, it admits a parallel spinor field ψ and the spinor fields $\psi_i := \xi_i \cdot \psi$, $i = 1, 2, 3$, are generalised Killing spinors:

$$\nabla^g_{\xi_i} \psi_i = \frac{\lambda}{2}\,\xi_i \cdot \psi_i, \quad \nabla^g_{\xi_j} \psi_i = -\frac{\lambda}{2}\,\xi_j \cdot \psi_i \ (i \neq j), \quad \nabla^g_X \psi_i = \frac{5\lambda}{4}\,X \cdot \psi_i, X \in \mathcal{H}$$

Given an almost 3-contact metric manifold $(M, \varphi_i, \xi_i, \eta_i, g)$, on the **metric cone**

$$(\bar{M}, \bar{g}) = (M \times \mathbb{R}^+, a^2 r^2 g + dr^2), \quad a > 0,$$

one can define an almost hyperHermitian structure $(\bar{g}, J_1, J_2, J_3)$.

Well-known:

- the metric cone of a 3-Sasakian manifold is hyper-Kähler
- the metric cone of the quaternionic Heisenberg group is a hyper-Kähler manifold with torsion ('HKT manifold')

Agricola-Höll, 2015: Criterion when the metric cone (for suitable $a > 0$) is a HKT manifold (but unclear what a 'good' large class of manifolds satisfying the criterion could be)

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> Is it possible to find a larger class of almost 3-contact metric manifolds with similar properties?

3-(α, δ)-Sasaki manifolds

Definition

A 3-(α, δ)-Sasaki manifold is an almost 3-contact metric manifold $(M, \varphi_i, \xi_i, \eta_i, g)$ such that

$$d\eta_i = 2\alpha\Phi_i + 2(\alpha - \delta)\eta_j \wedge \eta_k,$$

$\alpha \in \mathbb{R}^*, \delta \in \mathbb{R}, \ (i, j, k)$ even permutation of $(1, 2, 3)$.

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- 3-α-Sasakian manifolds: $d\eta_i = 2\alpha\Phi_i \rightsquigarrow \alpha = \delta$
- quat. Heisenberg groups: $d\eta_i = \lambda(\Phi_i + \eta_j \wedge \eta_k) \rightsquigarrow 2\alpha = \lambda, \delta = 0$

We call the structure degenerate if $\delta = 0$ and nondegenerate otherwise.

$3\text{-}(\alpha, \delta)$-Sasaki manifolds

Definition

A $3\text{-}(\alpha, \delta)$-*Sasaki manifold* is an almost 3-*contact metric manifold* $(M, \varphi_i, \xi_i, \eta_i, g)$ *such that*

$$d\eta_i = 2\alpha\Phi_i + 2(\alpha - \delta)\eta_j \wedge \eta_k,$$

$\alpha \in \mathbb{R}^*, \delta \in \mathbb{R}, (i, j, k)$ *even permutation of* $(1, 2, 3)$.

- $3\text{-}\alpha$-Sasakian manifolds: $d\eta_i = 2\alpha\Phi_i \rightsquigarrow \alpha = \delta$
- quat. Heisenberg groups: $d\eta_i = \lambda(\Phi_i + \eta_j \wedge \eta_k) \rightsquigarrow 2\alpha = \lambda, \delta = 0$

We call the structure degenerate if $\delta = 0$ and nondegenerate otherwise.

Theorem

- *The structure is hypernormal (generalization of Kashiwada's thm, case* $\alpha = \delta$*).*
- *The distribution* $\mathcal{V}$ *is integrable with totally geodesic leaves.*
- *Each* ξ_i *is a Killing vector field, and* $[\xi_i, \xi_j] = 2\delta\xi_k$.

Definition

An $\mathcal{H}$-homothetic deformation of an almost 3-contact metric strucure
($\varphi_i, \xi_i, \eta_i, g$) is given by

$$\eta_i' = c\eta_i, \quad \xi_i' = \frac{1}{c}\xi_i, \qquad \varphi_i' = \varphi_i, \qquad g' = ag + b\sum_{i=1}^{3}\eta_i \otimes \eta_i,$$

$a, b, c \in \mathbb{R}$, $a > 0$, $c^2 = a + b > 0$.

If $(\varphi_i, \xi_i, \eta_i, g)$ is 3-(α, δ)-Sasaki, then $(\varphi_i', \xi_i', \eta_i', g')$ is 3-(α', δ')-Sasaki
with
$$\alpha' = \alpha\frac{c}{a}, \qquad \delta' = \frac{\delta}{c}.$$

- the class of degenerate 3-(α, δ)-Sasaki structures is preserved
- in the non-degenerate case, the sign of $\alpha\delta$ is preserved.

Definition

We say that a 3-(α, δ)-*Sasaki manifold is* positive *(resp.* negative*) if*
$\alpha\delta > 0$ *(resp* $\alpha\delta < 0$*).*

Proposition

$\alpha\delta > 0 \iff M$ *is $\mathcal{H}$-homothetic to a 3-Sasakian manifold ($\alpha = \delta = 1$)*
$\alpha\delta < 0 \iff M$ *is $\mathcal{H}$-homothetic to one with $\alpha = -1$, $\delta = 1$.*

Do there exist 3-(α, δ)-Sasaki manifolds with $\alpha\delta < 0$?

YES - here is the construction:

Definition

A negative 3-Sasakian manifold is a normal almost 3-contact manifold $(M^{4n+3}, \varphi_i, \xi_i, \eta_i)$ endowed with a compatible semi-Riemannian metric $\tilde{g}$ of has signature $(3, 4n)$ such that $d\eta_i(X, Y) = 2\tilde{g}(X, \varphi_i Y)$.

Proposition

If $(M, \varphi_i, \xi_i, \eta_i, \tilde{g})$ is a negative 3-Sasakian manifold, take

$$g = -\tilde{g} + 2\sum_{i=1}^{3} \eta_i \otimes \eta_i.$$

Then $(\varphi_i, \xi_i, \eta_i, g)$ is a 3-(α, δ)-Sasaki structure with $\alpha = -1$ and $\delta = 1$.

It is known that quaternionic Kähler (not hyperKähler) manifolds with negative scalar curvature admit a canonically associated principal $\mathrm{SO}(3)$-bundle $P(M)$ which is endowed with a negative 3-Sasakian structure (Konishi, 1975 - Tanno, 1996).

φ-compatible connections

Definition

Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be an almost 3-contact metric manifold, (φ, ξ, η, g) a structure in the associated sphere Σ_M. Let ∇ be a metric connection with skew torsion on M. We say that ∇ is a φ-compatible connection if

1) ∇ *preserves the splitting* $TM = \mathcal{H} \oplus \mathcal{V}$,

2) $(\nabla_X \varphi)Y = 0 \qquad \forall X, Y \in \Gamma(\mathcal{H})$.

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Theorem

M *admits a φ-compatible connection if and only if*

1) N_φ *is skew-symmetric on $\mathcal{H}$;*

2) $(\mathcal{L}_{\xi_i} g)(X, Y) = 0$ *for every* $X, Y \in \Gamma(\mathcal{H})$ *and* $i = 1, 2, 3$*;*

3) $(\mathcal{L}_X g)(\xi_i, \xi_j) = 0$ *for every* $X \in \Gamma(\mathcal{H})$ *and* $i, j = 1, 2, 3$*.*

Remark If each ξ_i is Killing, 2) and 3) hold.

! φ-compatible connections are not uniquely determined

they are parametrized by their parameter function

$$\gamma := T(\xi_1, \xi_2, \xi_3) \in C^\infty(M).$$

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> $\nabla \varphi_i \equiv 0$ is too strong
>
> φ-compatibility is too weak

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> $\nabla\varphi_i \equiv 0$ is too strong
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> φ-compatibility is too weak

$\rightsquigarrow$ suppose ∇ preserves the 3-dimensional distribution in $\mathrm{End}(TM)$ spanned by φ_i as do quaternionic connections (qK case):

$$\nabla_X\varphi_i = \beta(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k) \quad \forall X \in \mathfrak{X}(M)$$

for every (i, j, k) even permutation of $(1, 2, 3)$.

The canonical connection: general existence

Theorem

An almost 3-contact metric manifold $(M, \varphi_i, \xi_i, \eta_i, g)$ admits a metric connection ∇ with skew torsion such that for some smooth function β,

$$\nabla_X \varphi_i = \beta(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k) \quad \forall X \in \mathfrak{X}(M)$$

for every even permutation (i, j, k) of $(1, 2, 3)$, if and only if

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for every even permutation (i, j, k) of $(1, 2, 3)$, if and only if

   1) *each ξ_i is a Killing vector field,*

   2) *each N_{φ_i} is totally skew-symmetric on $\mathcal{H}$,*

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1) *each ξ_i is a Killing vector field,*

2) *each N_{φ_i} is totally skew-symmetric on $\mathcal{H}$,*

3) *for any $X, Y, Z \in \Gamma(\mathcal{H})$ and any $i, j = 1, 2, 3$,*

$$N_{\varphi_i}(X, Y, Z) - d\Phi_i(\varphi_i X, \varphi_i Y, \varphi_i Z) = N_{\varphi_j}(X, Y, Z) - d\Phi_j(\varphi_j X, \varphi_j Y, \varphi_j Z),$$

4)

The canonical connection: general existence

Theorem

An almost 3-*contact metric manifold* $(M, \varphi_i, \xi_i, \eta_i, g)$ *admits a metric connection* ∇ *with skew torsion such that for some smooth function* β,

$$\nabla_X \varphi_i = \beta(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k) \quad \forall X \in \mathfrak{X}(M)$$

for every even permutation (i, j, k) *of* $(1, 2, 3)$, *if and only if*

1) *each* ξ_i *is a Killing vector field,*

2) *each* N_{φ_i} *is totally skew-symmetric on* $\mathcal{H}$,

3) *for any* $X, Y, Z \in \Gamma(\mathcal{H})$ *and any* $i, j = 1, 2, 3$,

$$N_{\varphi_i}(X, Y, Z) - d\Phi_i(\varphi_i X, \varphi_i Y, \varphi_i Z) = N_{\varphi_j}(X, Y, Z) - d\Phi_j(\varphi_j X, \varphi_j Y, \varphi_j Z),$$

4) β *is a* Reeb Killing function, *that is*

$$A_{ii}(X, Y) = 0, \qquad A_{ij}(X, Y) = -A_{ji}(X, Y) = \beta\Phi_k(X, Y)$$

$$A_{ij}(X, Y) := g((\mathcal{L}_{\xi_j}\varphi_i)X, Y) + d\eta_j(X, \varphi_i Y) + d\eta_j(\varphi_i X, Y)$$

for every $X, Y \in \Gamma(\mathcal{H})$ *and even permutation* (i, j, k) *of* $(1, 2, 3)$.

If such a connection ∇ exists, it is unique and φ-compatible for every almost contact metric structure φ in the associated sphere Σ_M.

∇ is called the canonical connection of M. It satisfies

$$\nabla_X \varphi_i = \beta(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k),$$
$$\nabla_X \xi_i = \beta(\eta_k(X)\xi_j - \eta_j(X)\xi_k),$$
$$\nabla_X \eta_i = \beta(\eta_k(X)\eta_j - \eta_j(X)\eta_k).$$

If $\beta = 0$, then $\nabla \varphi_i = \nabla \xi_i = \nabla \eta_i = 0$.

Definition

We say that an almost 3-contact metric manifold is canonical if it admits a (unique) canonical connection.

If $\beta = 0$ ($\Leftrightarrow A_{ij} = 0 \ \forall i, j = 1, 2, 3$) M will be called parallel canonical.

The canonical connection ∇ satisfies

$$\nabla\Psi = 0, \qquad \nabla\eta_{123} = 0,$$

$\Psi := \Phi_1 \wedge \Phi_1 + \Phi_2 \wedge \Phi_2 + \Phi_3 \wedge \Phi_3$, fundamental 4-form. In particular

$$\mathfrak{hol}(\nabla) \subset (\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)) \oplus \mathfrak{so}(3) \subset \mathfrak{so}(4n) \oplus \mathfrak{so}(3).$$

For parallel canonical manifolds ($\beta = 0$): $\mathfrak{hol}(\nabla) \subset \mathfrak{sp}(n)$

The canonical connection ∇ satisfies

$$\nabla\Psi = 0, \qquad \nabla\eta_{123} = 0,$$

$\Psi := \Phi_1 \wedge \Phi_1 + \Phi_2 \wedge \Phi_2 + \Phi_3 \wedge \Phi_3$, fundamental 4-form. In particular

$$\mathfrak{hol}(\nabla) \subset (\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)) \oplus \mathfrak{so}(3) \subset \mathfrak{so}(4n) \oplus \mathfrak{so}(3).$$

For parallel canonical manifolds ($\beta = 0$): $\mathfrak{hol}(\nabla) \subset \mathfrak{sp}(n)$

Theorem

For a canonical manifold, each structure $(\varphi_i, \xi_i, \eta_i, g)$ (and thus each $\varphi \in \Sigma_M$) admits a characteristic connection ∇^i, which is related to ∇ by

$$\nabla = \nabla^i - \frac{\beta}{2}(\eta_j \wedge \Phi_j + \eta_k \wedge \Phi_k)$$

(i, j, k) even permutation of $(1, 2, 3)$.

For $\beta = 0$: $\nabla^1 = \nabla^2 = \nabla^3 = \nabla$. [first known examples where this happens!]

Theorem

Every $3\text{-}(\alpha, \delta)$*-Sasaki manifold is canonical with* $\beta = 2(\delta - 2\alpha)$.

It is parallel canonical iff $\delta = 2\alpha$ *(*$\Rightarrow \alpha\delta > 0$*).*

The canonical connection of a $3\text{-}(\alpha, \delta)$*-Sasaki manifold has torsion*

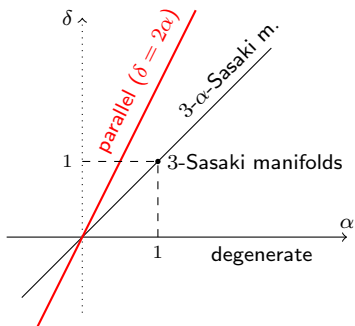$$T = \sum_{i=1}^{3} \eta_i \wedge d\eta_i + 8(\delta - \alpha)\, \eta_{123}$$

and satisfies $\nabla T = 0$.

- $3\text{-}\alpha$-Sasaki manifolds ($\alpha = \delta$): $T = \sum_i \eta_i \wedge d\eta_i$
- quat. Heisenberg groups ($\delta = 0, 2\alpha = \lambda$): $T = \sum_i \eta_i \wedge d\eta_i - 4\lambda\eta_{123}$
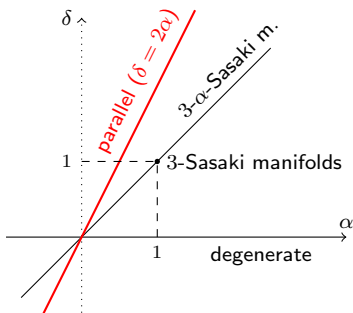
The geometry of 3-(α, δ)-Sasaki manifolds

Using the canonical connection ∇ and applying Agricola-Höll criterion:

Theorem

Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a 3-(α, δ)-Sasaki manifold. Then the metric cone
$$(\bar{M}, \bar{g}) = (M \times \mathbb{R}^+, a^2 r^2 g + dr^2), \quad a = -\frac{\beta}{2},$$
is HKT manifold.

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is HKT manifold.

Moreover, every 3-(α, δ)-Sasakian manifold admits an underlying quaternionic contact structure, and the canonical connection turns out to be a quaternionic contact connection. In fact, it is qc-Einstein (Ivanov - Minchev - Vassilev, 2016) and this allows to determine the Riemannian Ricci curvature:

Theorem

The Riemannian Ricci curvature of a 3-(α, δ)-Sasaki manifold is

$$\mathrm{Ric}^g = 2\alpha\big(2\delta(n+2) - 3\alpha\big)g + 2(\alpha - \delta)\big((2n+3)\alpha - \delta\big)\sum_{i=1}^{3}\eta_i \otimes \eta_i$$

The ∇-Ricci curvature is

$$\mathrm{Ric} = 4\alpha\{\delta(n+2) - 3\alpha\}\,g + 4\alpha\{\delta(2-n) - 5\alpha\}\sum_{i=1}^{3}\eta_i \otimes \eta_i.$$

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$$\mathrm{Ric} = 4\alpha\{\delta(n+2) - 3\alpha\}\, g + 4\alpha\{\delta(2-n) - 5\alpha\}\sum_{i=1}^{3}\eta_i \otimes \eta_i.$$

The property of being symmetric follows for Ric from $\nabla T = 0$.

- M is Riemannian Einstein iff $\alpha = \delta$ or $\delta = (2n+3)\alpha$.
- The manifold is ∇-Einstein iff $\delta(2-n) = 5\alpha$.
- The manifold is both Riemannian Einstein and ∇-Einstein if and only if $\dim M = 7$ and $\delta = 5\alpha$ (happens for example for 'compatible' nearly parallel G_2-str., see next).

7-dimensional 3-(α, δ)-Sasaki manifolds

Theorem

Any 7-dimensional 3-(α, δ)-Sasaki manifold admits a a cocalibrated G_2-structure (Fernandez-Gray type $W_1 \oplus W_3$) given by the 3-form

$$\omega \; := \; \sum_{i=1}^{3} \eta_i \wedge \Phi_i^{\mathcal{H}} + \eta_{123}.$$

Its characteristic connection ∇ coincides with the canonical connection.

This G_2-structure defines a unique canonical spinor field ψ_0 such that

$$\nabla \psi_0 \; = \; 0, \quad \omega \cdot \psi_0 = -7\psi_0, \quad |\psi_0| = 1.$$

Theorem

1. The canonical spinor field ψ_0 is a generalized Killing spinor:

$$\nabla^g_X \psi_0 = -\frac{3\alpha}{2} X \cdot \psi_0 \ \text{ for } X \in \mathcal{H}, \quad \nabla^g_Y \psi_0 = \frac{2\alpha - \delta}{2} Y \cdot \psi_0 \ \text{ for } Y \in \mathcal{V}.$$

   The two generalized Killing numbers coincide iff $\delta = 5\alpha$, corresponding to a *nearly parallel G_2-structure.* (Gray-Fernandez type W_1)
   [$\delta = 5\alpha$ is the only case where M is Einstein and ∇-Einstein].

2. The Clifford products $\psi_i := \xi_i \cdot \psi_0$, $i = 1, 2, 3$, are generalized Killing spinors:

$$\nabla^g_{\xi_i} \psi_i = \frac{2\alpha - \delta}{2} \xi_i \cdot \psi_i, \quad \nabla^g_{\xi_j} \psi_i = -\frac{3\delta - 2\alpha}{2} \xi_j \cdot \psi_i \ (i \neq j),$$

$$\nabla^g_X \psi_i = \frac{\alpha}{2} X \cdot \psi_i \ \text{ for } X \in \mathcal{H}.$$

   Any two of the generalized Killing numbers coincide iff $\alpha = \delta$, i. e. if M^7 is *3-α-Sasakian.*

Homogeneous 3-Sasakian manifolds

Theorem (Boyer, Galicki, Mann, 1994)

Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a homogeneous 3-Sasakian manifold. Then M is one of the following homogeneous spaces:

$$\frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)}, \qquad \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n) \times \mathbb{Z}_2}, \qquad \frac{\mathrm{SU}(m+2)}{S(\mathrm{U}(m) \times \mathrm{U}(1))}, \qquad \frac{\mathrm{SO}(k+4)}{\mathrm{SO}(k) \times \mathrm{Sp}(1)},$$

$$\frac{\mathrm{G}_2}{\mathrm{Sp}(1)}, \qquad \frac{\mathrm{F}_4}{\mathrm{Sp}(3)}, \qquad \frac{\mathrm{E}_6}{\mathrm{SU}(6)}, \qquad \frac{\mathrm{E}_7}{\mathrm{Spin}(12)}, \qquad \frac{\mathrm{E}_8}{\mathrm{E}_7}.$$

Here $n \geq 0$, $m \geq 1$ and $k \geq 3$.

- They are all simply connected except for $\mathbb{R}P^{4n+3} \simeq \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n) \times \mathbb{Z}_2}$

- 1-1 correspondence between simply connected 3-Sasakian homogeneous manifolds and compact simple Lie algebras

Definition

A 3-Sasakian data is a triple (G, G_0, H) of Lie groups such that

- G *is a compact, simple Lie Group*
- $H \subset G_0 \subset G$ *connected Lie subgroups*

Uniform description of homogeneous 3-Sasakian manifolds

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Remark In total the Lie algebra decomposes as

$$\mathfrak{g} = \overbrace{\mathfrak{h} \oplus \underbrace{\mathfrak{sp}(1)}_{\mathfrak{m}} \oplus \mathfrak{g}_1}^{\mathfrak{g}_0}$$

($\mathfrak{m}$ is a reductive complement for $M = G/H$)

Homogeneous 3-Sasakian model

Theorem (Draper, Ortega, Palomo, 2018)

Let (G, G_0, H) be 3-Sasakian data. On $M = G/H$ consider the G-invariant structure defined by the $\mathrm{Ad}(H)$-invariant tensors on $\mathfrak{m}$:

- *the inner product g*

$$g\big|_{\mathfrak{sp}(1)} = \frac{-\kappa}{4(n+2)}, \qquad g\big|_{\mathfrak{g}_1} = \frac{-\kappa}{8(n+2)}, \qquad g\big|_{\mathfrak{sp}(1) \times \mathfrak{g}_1} = 0$$

  κ the Killing form on G.

- *$\xi_i = \sigma_i$, $i = 1, 2, 3$, σ_i standard basis of $\mathfrak{sp}(1) = \mathcal{V} \subset \mathfrak{g}_0$, $\eta_i = g(\xi_i, \cdot)$*

- *the endomorphisms φ_i as*

$$\varphi_i\big|_{\mathfrak{sp}(1)} = \frac{1}{2}\,\mathrm{ad}(\xi_i), \qquad \varphi_i\big|_{\mathfrak{g}_1} = \mathrm{ad}(\xi_i).$$

Then $(M, \varphi_i, \xi_i, \eta_i, g)$ defines a homogeneous 3-Sasakian manifold. Conversely every homogeneous 3-Sasakian manifold $M \neq \mathbb{R}P^{4n+3}$ is obtained by this construction.

Remark: M fibers over the quaternion Kähler symmetric space G/G_0.

Homogeneous positive 3-(α, δ)-Sasakian model

Idea: Use $\mathcal{H}$-homothetic deformation to obtain 3-(α, δ)-Sasakian mnfds for $\alpha\delta > 0$

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Theorem

Let (G, G_0, H) be 3-Sasakian data, $\alpha\delta > 0$. On $M = G/H$ consider the G-invariant structure by the $\mathrm{Ad}(H)$-invariant tensors on $\mathfrak{m}$:

$$g\big|_{\mathfrak{sp}(1)} = \frac{-\kappa}{4\delta^2(n+2)}, \qquad g\big|_{\mathfrak{g}_1} = \frac{-\kappa}{8\alpha\delta(n+2)}, \qquad g\big|_{\mathfrak{sp}(1)\times\mathfrak{g}_1} = 0$$

$$\xi_i = \delta\sigma_i, \qquad \eta_i = g(\xi_i, \cdot)$$

$$\varphi_i\big|_{\mathfrak{sp}(1)} = \frac{1}{2\delta}\,\mathrm{ad}(\xi_i), \qquad \varphi_i\big|_{\mathfrak{g}_1} = \frac{1}{\delta}\,\mathrm{ad}(\xi_i).$$

Then $(M, \varphi_i, \xi_i, \eta_i, g)$ defines a homogeneous 3-(α, δ)-Sasakian mnfd.
Conversely every homogeneous 3-(α, δ)-Sasakian manifold $M \neq \mathbb{R}P^{4n+3}$ with $\alpha\delta > 0$ is obtained by this construction.

Remark: $(G/H, g)$ is naturally reductive $\Leftrightarrow \delta = 2\alpha \Leftrightarrow$ parallel 3-(α, δ).

Generalized setup

Definition

A generalized 3-Sasakian data is a triple (G, G_0, H) of Lie groups such that

- G *is a real simple Lie Group*
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and the Lie algebras $\mathfrak{h} \subset \mathfrak{g}_0 \subset \mathfrak{g}$ satisfy:

- $\mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{sp}(1)$ *with $\mathfrak{sp}(1)$ and $\mathfrak{h}$ commuting subalgebras,*
- $(\mathfrak{g}, \mathfrak{g}_0)$ *form a symmetric pair, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$,*
- *the complexification $\mathfrak{g}_1^{\mathbb{C}} = \mathbb{C}^2 \otimes_{\mathbb{C}} W$ for some $\mathfrak{h}^{\mathbb{C}}$-module of $\dim_{\mathbb{C}} W = 2n$,*
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If $(\mathfrak{g}, \mathfrak{g}_0)$ is a compact symmetric pair such that (G, G_0, H) is 3-Sasakian data, then (G^*, G_0, H) is generalized 3-Sasakian data, where $(\mathfrak{g}^*, \mathfrak{g}_0)$ is the dual non-compact symmetric pair.

Negative homogeneous $3\text{-}(\alpha, \delta)$-Sasakian manifolds

Theorem

Let (G^*, G_0, H) be non-compact generalized 3-Sasakian data, $\alpha\delta < 0$.

On $M = G^*/H$ consider the G^*-invariant structure defined by the $\mathrm{Ad}(H)$-invariant tensors on $\mathfrak{m}$

$$g\big|_{\mathfrak{sp}(1)} = \frac{-\kappa}{4\delta^2(n+2)}, \qquad g\big|_{\mathfrak{g}_1} = \frac{-\kappa}{8\alpha\delta(n+2)}, \qquad g\big|_{\mathfrak{sp}(1)\times\mathfrak{g}_1} = 0,$$

$$\xi_i = \delta\sigma_i, \qquad \eta_i = g(\xi_i, \cdot),$$

$$\varphi_i\big|_{\mathfrak{sp}(1)} = \frac{1}{2\delta}\,\mathrm{ad}(\xi_i), \qquad \varphi_i\big|_{\mathfrak{g}_1} = \frac{1}{\delta}\,\mathrm{ad}(\xi_i),$$

κ the Killing form on G^*, σ_i standard basis $\mathfrak{sp}(1) = \mathcal{V} \subset \mathfrak{g}_0$.

Then $(M, g, \xi_i, \eta_i, \varphi_i)$ defines a homogeneous $3\text{-}(\alpha, \delta)$-Sasakian manifold.

Question: Does this model cover all homogenous negative $3\text{-}(\alpha, \delta)$-Sasaki manifolds?

In total we obtain homogeneous $3\text{-}(\alpha,\delta)$-Sasakian structures on the following list of homogeneous spaces (G/H compact, G^*/H non-compact):

G	G^*	H	G_0	dim
$\mathrm{Sp}(n+1)$	$\mathrm{Sp}(n,1)$	$\mathrm{Sp}(n)$	$\mathrm{Sp}(n)\mathrm{Sp}(1)$	$4n+3$
$\mathrm{SU}(n+2)$	$\mathrm{SU}(n,2)$	$S(\mathrm{U}(n)\times\mathrm{U}(1))$	$S(\mathrm{U}(n)\mathrm{U}(2))$	$4n+3$
$\mathrm{SO}(n+4)$	$\mathrm{SO}(n,4)$	$\mathrm{SO}(n)\times\mathrm{Sp}(1)$	$\mathrm{SO}(n)\mathrm{SO}(4)$	$4n+3$
G_2	G_2^2	$\mathrm{Sp}(1)$	$\mathrm{SO}(4)$	11
F_4	F_4^{-20}	$\mathrm{Sp}(3)$	$\mathrm{Sp}(3)\mathrm{Sp}(1)$	31
E_6	E_6^2	$\mathrm{SU}(6)$	$\mathrm{SU}(6)\mathrm{Sp}(1)$	43
E_7	E_7^{-5}	$\mathrm{Spin}(12)$	$\mathrm{Spin}(12)\mathrm{Sp}(1)$	67
E_8	E_8^{-24}	E_7	$\mathrm{E}_7\mathrm{Sp}(1)$	115

.

Remark: $\mathbb{R}P^{4n+3} = \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)\times\mathbb{Z}_2}$ and non compact dual $\frac{\mathrm{Sp}(n,1)}{\mathrm{Sp}(n)\times\mathbb{Z}_2}$ also admit $3\text{-}(\alpha,\delta)$-Sasaki structures, as the quotient of $S^{4n+3} = \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)}$, resp. $\frac{\mathrm{Sp}(n,1)}{\mathrm{Sp}(n)}$ by $\mathbb{Z}_2$ inside the fiber.

Definiteness of curvature operators

Consider the Riemannian curvature as a symmetric operator

$$\mathcal{R}^g : \Lambda^2 M \to \Lambda^2 M \qquad \langle \mathcal{R}^g(X \wedge Y), Z \wedge V \rangle = -g(R^g(X,Y)Z, V).$$

Definition

A Riemannian manifold (M, g) is said to have strongly positive curvature *if there exists a 4-form ω such that $\mathcal{R}^g + \omega$ is* positive-definite *at every point $x \in M$ (Thorpe, 1971).*

For every 2-plane σ, being $\langle \omega(\sigma), \sigma \rangle = 0$, one has

$$\sec(\sigma) = \langle \mathcal{R}^g(\sigma), \sigma \rangle = \langle (\mathcal{R}^g + \omega)(\sigma), \sigma \rangle.$$

Then,

$\mathcal{R}^g > 0 \implies$ strongly positive curvature $\implies$ positive sectional curvature

$\mathcal{R}^g \geq 0 \implies$ strongly non-negative curvature $\implies$ non-negative sec. curv.

On a 3-(α, δ)-Sasakian manifold the symmetric operators defined by the Riemannian curvature and the curvature of the canonical connection:

$$\mathcal{R}^g : \Lambda^2 M \to \Lambda^2 M \qquad \mathcal{R} : \Lambda^2 M \to \Lambda^2 M$$

are related by

$$\mathcal{R}^g - \frac{1}{4}\sigma_T = \mathcal{R} + \frac{1}{4}\mathcal{G}_T$$

with

$$\langle \mathcal{G}_T(X \wedge Y), Z \wedge V \rangle := g(T(X,Y), T(Z,V)),$$

$$\langle \sigma_T(X \wedge Y), Z \wedge V \rangle := \frac{1}{2} dT(X,Y,Z,V).$$

(M, g) is strongly non-negative with 4-form $-\frac{1}{4}\sigma_T$ if and only if

$$\mathcal{R} + \frac{1}{4}\mathcal{G}_T \geq 0.$$

Being $\mathcal{G}_T \geq 0$, if $\mathcal{R} \geq 0$ we directly have strong non-negativity.

Theorem

Let M be a homogeneous 3-(α, δ)-Sasakian manifold obtained from a generalized 3-Sasakian data.

- *If $\alpha\delta < 0$ then $\mathcal{R} \leq 0$.*
- *If $\alpha\delta > 0$ then*

$$\mathcal{R} \geq 0 \text{ if and only if } \alpha\beta \geq 0$$

.

Then, on a positive homogeneous 3-(α, δ)-Sasaki manifold with $\alpha\beta \geq 0$:

$$\mathcal{R}^g - \frac{1}{4}\sigma_T = \mathcal{R} + \frac{1}{4}\mathcal{G}_T \geq 0.$$

The converse also holds, i.e.

Theorem

A positive homogeneous 3-(α, δ)-Sasaki manifold is strongly non-negative with 4-form $-\frac{1}{4}\sigma_T$ if and only if $\alpha\beta \geq 0$.

Strong positivity is much more restrictive than strong non-negativity.

Strong positivity implies strict positive sectional curvature.

Homogeneous manifolds with strictly positive sectional curvature have been classified (Wallach 1972, Bérard Bergery 1976).

Only the 7-dimensional Aloff-Wallach-space $W^{1,1}$, the spheres S^{4n+3} and real projective spaces $\mathbb{R}P^{4n+3}$ admit homogeneous 3-(α, δ)-Sasaki structures.

Theorem

The 3-(α, δ)-Sasakian spaces

- $W^{1,1} = \mathrm{SU}(3)/S^1$ *with 4-form* $-(\frac{1}{4} + \varepsilon)\sigma_T$ *for small* $\varepsilon > 0$,
- S^{4n+3}, $\mathbb{R}P^{4n+3}$, $n \geq 1$, *with 4-form* $\frac{\delta}{8\alpha}\sigma_T|_{\Lambda^4\mathcal{H}} - (\frac{1}{4} + \varepsilon)\sigma_T$ *for small* $\varepsilon > 0$

are strongly positive if and only if $\alpha\beta > 0$.

Short bibliography

I. Agricola, *The Srní lectures on non-integrable geometries with torsion*, Arch. Math.(Brno) **42** (2006), suppl., 5–84.

I. Agricola, G. Dileo, *Generalizations of 3-Sasakian manifolds and skew torsion*, Adv. Geom. (2019).

I. Agricola, G. Dileo, L. Stecker, *Curvature and homogeneity properties of non-degenerate* $3\text{-}(\alpha, \delta)$*- Sasaki manifolds*, in preparation.

I. Agricola, T. Friedrich, 3*-Sasakian manifolds in dimension seven, their spinors and* G_2*-structures*, J. Geom. Phys. (2010).

I. Agricola, J. Höll, *Cones of* G *manifolds and Killing spinors with skew torsion*, Ann. Mat. Pura Appl. (2015).

C.P. Boyer, K. Galicki, B. M. Mann, *The geometry and topology of 3-Sasakian manifolds* J. Reine Angew. Math. (1994).

C. Draper, M. Ortega, F.J. Palomo, *Affine Connections on 3-Sasakian Homogeneous Manifolds*, Math. Z. (2019).

T. Friedrich, S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian J. Math. (2002).

S. Ivanov, I. Minchev, D. Vassilev, *Quaternionic contact Einstein structures and the quaternionic contact Yamabe problem*, Mem. AMS (2014).

T. Kashiwada, *On a contact* 3*-structure*, Math. Z. (2001).