

The linear instability of some Einstein metrics

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Dirac operators in differential geometry and global analysis
in memory of Professor Thomas Friedrich

Joint work with Uwe Semmelmann and McKenzie Wang

The normalized total scalar curvature functional

A Riemannian manifold (M^n, g) is **Einstein** if the Ricci curvature Ric_g is constant, i.e.

$$Ric_g = \Lambda g \tag{1}$$

for some constant Λ , and Λ is called **Einstein constant**.

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The **normalized total scalar curvature functional** is defined as

$$\tilde{\mathbf{S}} : \mathcal{M} \rightarrow \mathbb{R}, \quad \tilde{\mathbf{S}}(g) = \frac{1}{V(g)^{\frac{n-2}{n}}} \int_M s_g d\text{vol}_g, \tag{2}$$

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From now on, assume $n \geq 3$.

The first variation formula

Let $g(t)$ for $t \in (-\tau, \tau)$ be a smooth family of metrics on M^n , $g(0) = g$, and $\frac{d}{dt}g(t)|_{t=0} = h \in C^\infty(M, T^*M \odot T^*M)$.

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$$\tilde{\mathbf{S}}'_g \cdot h := \frac{d}{dt} \tilde{\mathbf{S}}(g(t))|_{t=0} = \frac{1}{V(g)^{\frac{n-2}{n}}} \int_M \langle -Ric_g + (\frac{s_g}{2} + \frac{2-n}{2n} \overline{s_g})g, h \rangle d\text{vol}_g, \quad (3)$$

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Theorefore,

$$\begin{aligned} g \text{ is a critical point of } \tilde{\mathbf{S}} &\Leftrightarrow Ric_g = (\frac{s_g}{2} + \frac{2-n}{2n} \overline{s_g})g \\ &\Leftrightarrow s_g \text{ is constant and } g \text{ is Einstein} \\ &\quad (\text{The second Bianchi identity, and } n \geq 3.) \end{aligned}$$

The second variation formula

At a such **Einstein metric** g ,

$$\tilde{\mathbf{S}}_g''(h, h) := \frac{d^2}{dt^2} \tilde{\mathbf{S}}(g(t))|_{t=0} = \frac{1}{V(g)^{\frac{n-2}{n}}} \int_M \langle P_g h, h \rangle d\text{vol}_g, \quad (4)$$

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$$\overline{(\text{tr}_g h)} = \frac{1}{V(g)} \int_M (\text{tr}_g h) d\text{vol}_g, \text{ and } (\mathring{R}h)_{ij} = R_{ikjl} h_{kl}.$$

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Let (M^n, g) be a compact Einstein manifold other than standard spheres. Then

$$C^\infty(M, T^*M \odot T^*M) = \text{Im} \delta_g^* \oplus C^\infty(M) \cdot g \oplus (\ker \text{tr}_g \cap \ker \delta_g). \quad (6)$$

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A decomposition of symmetric 2-tensors on Einstein manifolds

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- An Einstein metric is always a **saddle** point of the normalized total scalar curvature functional.

In the direction orthogonal to diffeomorphism and conformal changes, i.e.
 $h \in \ker \operatorname{tr}_g \cap \ker \delta_g$,

$$\tilde{\mathbf{S}}_g''(h, h) = \frac{-1}{2V(g)^{\frac{n-2}{n}}} \int_M \langle \nabla^* \nabla h - 2\mathring{R}h, h \rangle d\operatorname{vol}_g. \quad (7)$$

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Einstein operator: $\nabla^* \nabla - 2\mathring{R} = \Delta_L - 2\Lambda$.

Definition of linear stability of Einstein metrics

In the direction orthogonal to diffeomorphism and conformal changes, i.e. $h \in \ker \operatorname{tr}_g \cap \ker \delta_g$,

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Einstein operator: $\nabla^* \nabla - 2\mathring{R} = \Delta_L - 2\Lambda$.

Definition: A (compact) Einstein metric (M^n, g) is **$\tilde{\mathbf{S}}$ -linearly stable**, if

$$\langle \nabla^* \nabla h - 2\mathring{R}h, h \rangle_{L^2} \geq 0 \quad (8)$$

for all **TT-tensors** h , i.e. symmetric 2-tensor h satisfying $\operatorname{tr}_g h = 0$ and $\delta_g h = 0$, and otherwise, **$\tilde{\mathbf{S}}$ -linearly unstable**.

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Examples of \tilde{S} -linearly stable Einstein metrics

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- Many compact irreducible symmetric spaces, e.g. S^n , $\mathbb{C}P^n$, $\mathbb{H}P^n$, and all compact simple Lie groups except $\mathrm{Sp}(n)$, $n \geq 2$ and $\mathrm{SU}(n)$, $n \geq 3$ (N. Koiso, 1980).

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- Complete Riemannian manifolds with imaginary Killing spinors (K. Kröncke, 2017, W., 2017).

- $\mathrm{Sp}(n), \mathrm{Sp}(n)/\mathrm{U}(n), n \geq 2$ (N. Koiso, 1980).

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- Warped product Einstein manifolds with dimension $n \leq 6$ are unstable. (W. Batat, S. Hall, T. Murphy, 2017).

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Some other variational characterizations of Einstein metrics

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The second variation formula of ν -entropy at Einstein metrics for $h \in \ker \mathrm{tr}_g \cap \ker \delta_g$ is the same as the second variation formula of the normalized total scalar curvature (up to a positive constant factor).

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However, unlike the case of the $\tilde{\mathbf{S}}$ -functional, the second variation is no longer always positive on conformal change directions. Actually, Einstein metrics could be local **maximum** points of the ν -entropy.

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By H. Cao-C. He's work,

$$\nu\text{-linear stability} \iff \begin{cases} \langle \nabla^* \nabla h - 2\mathring{R}h, h \rangle_{L^2} \geq 0, & \forall h \in \ker \operatorname{tr}_g \cap \ker \delta_g, \\ \lambda_1(M, g) \geq 2\Lambda, \end{cases}$$

where $\lambda_1(M, g)$ is the first non-zero eigenvalue of the Laplace operator on (M, g) .

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\tilde{S} -linearly unstable $\implies \nu$ -linearly unstable $\implies \nu$ -unstable \iff dynamically unstable.

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$$\nabla_X^S \sigma = \mu X \cdot \sigma, \quad (9)$$

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$$\text{Ric}_g = 4\mu^2(n-1)g, \quad (10)$$

i.e. g is an Einstein metric with scalar curvature $4n(n-1)\mu^2$. This implies that μ can only be real or purely imaginary, since scalar curvature is real.

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Professor Thomas Friedrich initiated the mathematical investigation of Killing spinors in 1980.

By the works of T. Friedrich (1981), O. Hijazi (1986), T. Friedrich-I. Kath (1989), R. Grunewald (1990), C. Bär (1993), a complete simply-connected Riemannian manifold of dimension n with a Real Killing spinor and Einstein constant $n - 1$ is isometric one of the following:

- (1) round sphere S^n , if n is even and $n \neq 6$;
- (2) strictly nearly Kähler manifold, if $n = 6$;
- (3) Sasaki-Einstein manifold, if n is odd and $n \neq 7$;
- (4) nearly parallel G_2 manifold (including Sasaki-Einstein and 3-Sasaki), if $n = 7$.

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By the work of J.-B. Butruille (2005), a simply connected homogeneous strictly nearly Kähler 6-manifold is one of the following:

- (1) $S^6 = G_2/SU(3)$,
- (2) $(SU(2) \times SU(2) \times SU(2))/\Delta SU(2)$,
- (3) $\mathbb{CP}^3 = Sp(2)/(Sp(1) \times U(1))$,
- (4) $SU(3)/T^2$,

each equipped with a unique invariant nearly Kähler structure.

Theorem (Simmelmann – W. – Wang, 2019)

A complete strict nearly Kähler 6-manifold with either 2nd or 3rd Betti number nonzero is $\tilde{\mathbf{S}}$ -linearly unstable, and therefore ν -linearly unstable.

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Main ingredients in the proof:

- M. Verbitsky's Hodge decomposition of Harmonic forms on nearly Kähler 6-manifolds (also proved by L. Foscolo in a different way).
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Corollary

If a complete simply connected strict nearly Kähler manifold is $\tilde{\mathbf{S}}$ -linearly stable, then it is a rational homology sphere. In particular, if $H_2(M, \mathbb{Z})$ has no torsion, then it is diffeomorphic to S^6 .

Classification by Friedrich-Kath-Moroianu-Semmelmann (1997):

A compact, simply connected, homogeneous nearly parallel G_2 manifold with only 1 linearly independent Killing spinor is isometric to one of the followings

- (1) S^7 with the Jensen's metric ($\tilde{\mathbf{S}}$ -linearly unstable);
- (2) Aloff-Wallach spaces $N_{k,l} = \mathrm{SU}(3)/U_{k,l}$, where k, l are relatively prime integers with $(k, l) \neq (1, 1)$ and $U_{k,l}$ is the circle $\mathrm{diag}(e^{2\pi i k \theta}, e^{2\pi i l \theta}, e^{-2\pi i (k+l) \theta}) \in \mathrm{SU}(3)$, with invariant Einstein metrics;
- (3) the isotropy irreducible space $\mathrm{Sp}(2)/\mathrm{SU}(2)$, where the embedding of $\mathrm{SU}(2)$ is via the irreducible 4-dimensional symplectic representation.

By the works of M. Wang (1982), Castellani-Romans (1984), Page-Pope (1984), Kawalski-Vlasek (1993), and Nikonorov (2004), each of $N_{k,l}$ admit two $SU(3)$ -invariant Einstein metrics, up to isometry.

By the works of M. Wang (1982), Castellani-Romans (1984), Page-Pope (1984), Kawalski-Vlasek (1993), and Nikonorov (2004), each of $N_{k,l}$ admit two $SU(3)$ -invariant Einstein metrics, up to isometry.

Theorem (W. – Wang, 2018)

The invariant Einstein metrics on $N_{k,l}$ are all $\tilde{\mathbf{S}}$ -linearly unstable, and therefore ν -linearly unstable.

Definition 1 of Sasaki Manifolds: (M^n, g) is a Sasaki manifold if the cone $(\mathbb{R}_+ \times M^n, dr^2 + r^2 g)$ is Kähler, where $\mathbb{R}_+ = (0, +\infty)$, and r is coordinate on \mathbb{R}_+ .

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Definition 2 of Sasaki Manifolds: (M^n, g) is a Sasaki manifold if there exists a Killing vector field ξ of unit length on M^n so that the Riemann curvature satisfies the condition

$$R_{X\xi}Y = -g(\xi, Y)X + g(X, Y)\xi, \quad (11)$$

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From (11), one can easily see that on a Sasaki-Einstein manifold (M^n, g) of dimension n

$$\text{Ric}_g = (n-1)g. \quad (12)$$

Let (B^{2p}, G, J) be a Kähler-Einstein manifold of real dimension $2p$, with the Kähler form $\Omega = G(\cdot, J\cdot)$, and $Ric_G = (2p + 2)G$.

Then let $\pi : M^{2p+1} \rightarrow B^{2p}$ be a principal S^1 -bundle with a connection η with the curvature form $d\eta = 2\pi^*\Omega$.

Let ξ be a vertical vector field on M^{2p+1} , generated by S^1 -action, such that $\eta(\xi) = 1$.

We define a Riemannian metric on M^{2p+1} as

$$g(X, Y) = G(\pi_*X, \pi_*Y) + \eta(X)\eta(Y), \quad (13)$$

for vector field X and Y on M^{2p+1} .

Then (M^{2p+1}, g, ξ) is a regular Sasaki-Einstein manifold.

Proposition (W., 2016)

$$\begin{aligned} & \int_M \langle (\nabla^g)^* \nabla^g \tilde{h} - 2\mathring{R}^g \tilde{h}, \tilde{h} \rangle d\text{vol}_g \\ &= \int_B (\langle (\nabla^G)^* \nabla^G h - 2\mathring{R}^G h, h \rangle + 4\langle h, h \rangle + 4\langle h \circ J, h \rangle) d\text{vol}_G, \end{aligned} \tag{14}$$

for any $h \in C^\infty(B, S^2(B))$, where $\tilde{h} = \pi^* h \in C^\infty(M, S^2(M))$.

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Corollary

The regular Sasaki-Einstein manifold (M^{2p+1}, g) is unstable, if one of the following conditions holds:

- (1) there exists a traceless transverse symmetric 2-tensor h on base manifold B^{2p} such that $\int_B \langle (\nabla^G)^* \nabla^G h - 2\mathring{R}^G h, h \rangle d\text{vol}_G < -8 \int_B \langle h, h \rangle d\text{vol}_G$;

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- (2) the base Kähler-Einstein manifold (B^{2p}, G, J) has $\dim H^{1,1}(B) \geq 2$.

Instability of regular Sasaki-Einstein manifolds with Hermitian symmetric space bases

Theorem (W. – Wang, 2018)

The following simply connected regular Sasaki Einstein manifold are ν -linearly unstable from conformal variations:

- (1) $SO(p+2)/SO(p)$, $p \geq 3$, circle bundle over the complex quadric $SO(p+2)/(SO(p) \times SO(2))$;
- (2) $E_6/Spin(10)$, and E_7/E_6 , which are respectively circle bundles over the hermitian symmetric spaces $E_6/(Spin(10) \cdot U(1))$ and $E_7/(E_6 \cdot U(1))$;
- (3) $SU(p+2)/(SU(p) \times SU(2))$, $p \geq 2$, a circle bundle over the complex Grassmannian $SU(p+2)/S(U(p) \times U(2))$.

Moreover, the Stiefel manifolds in (a) above are also \tilde{S} -linearly unstable, and for $k \geq 4$, $Sp(k)/SU(k)$, which are circle bundles over $Sp(k)/U(k)$, are \tilde{S} -linearly unstable, and so ν -linearly unstable.

THANK YOU!