The theory of Lipschitz structures

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Objective

The main goal of this talk is to discuss recent developments in the theory of *Lipschitz structures*, which consist on certain generalized spinorial structures that can be naturally associated to a large class of bundles of real or complex Clifford modules over a pseudo-Riemannian manifold (M, g) of any dimension d and signature (p, q). The theory of Lipschitz structures was initiated by T. Friedrich and A. Trautman in:

- Friedrich, T. and Trautman, A. Annals of Global Analysis and Geometry (2000) 18: 221.

for bundles of faithful complex Clifford modules in Riemannian signature, and further continued in:

- CSS and C. Lazaroiu. Arxiv:1606.07894, to appear in Asian Journal of Mathematics.
- CSS and C. Lazaroiu. Differential Geometry and its Applications, Vol. 61, 2018.
- CSS and C. Lazaroiu. Arxiv:1809.09084, preprint.

for bundles of real/complex weakly faithful Clifford modules over (M, g).

Let (V, h) be a regular quadratic vector space of signature (p, q) and dimension d, with associated Clifford algebra $\mathrm{Cl}(V, h)$.

Definition

A Clifford representation is a morphism of unital algebras $\eta : \mathrm{Cl}(V, h) \to \mathrm{End}(\Sigma)$, where Σ is a finite-dimensional vector space (over $\mathbb{R}$ or $\mathbb{C}$).

Definition

Let $\eta : \mathrm{Cl}(V, h) \to \mathrm{End}(\Sigma)$ and $\eta' : \mathrm{Cl}(V', h') \to \mathrm{End}(\Sigma')$ be two Clifford representations. A morphism from η to η' is a pair (f_0, f) such that:

1. $f_0 : V \to V'$ is an isometry from (V, h) to (V', h')
2. $f : \Sigma \to \Sigma'$ is an linear map
3. $\eta'(\mathrm{Cl}(f_0)(x)) \circ f = f \circ \eta(x)$ for all $x \in \mathrm{Cl}(V, h)$.

Definition

A Clifford representation $\eta : \mathrm{Cl}(V, h) \to \mathrm{End}(\Sigma)$ is called weakly faithful if the restriction $\eta|_V : V \to \mathrm{End}(\Sigma)$ is an injective map.

Statement of the problem II.

- Let (M, g) denote a pseudo-Riemannian manifold of signature (p, q) and dimension d, with bundle of Clifford algebras $\mathrm{Cl}(M, g)$.
- Let ClRep denote the category of Clifford modules with the notion of morphism defined above; ClRep_w of weakly faithful Clifford modules.
- Every Clifford representation $\eta \in \mathrm{Ob}(\mathrm{ClRep})$ defines an equivalence class $[\eta] \in \mathrm{ClRep}^{\times}$ in the groupoid defined by ClRep.

Definition

A *(s)pinor bundle of type* $[\eta]$ over a pseudo-Riemannian manifold (M, g) is a pair (S, Γ) consisting on a vector bundle $S \to M$ and a morphism of bundles of unital and associative algebras $\Gamma : \mathrm{Cl}(M, g) \to \mathrm{End}(S)$ such that $(S_m, \Gamma_m) \in [\eta]\ \forall\, m \in M$. When η is fiberwise-irreducible, (S, η) is called a bundle of irreducible spinors.

Dirac operators can be defined on S if and only if S admits a globally well-defined Clifford multiplication $TM \otimes S \to S$, which amounts to the condition that S is a spinor bundle as defined above.

Remark

Spinor bundles are abundant. For instance: every pseudo-Riemannian spin/spin^c manifold admits an irreducible real/spinor bundle. Such spinorial structures are well-known to be obstructed (obstruction expressed in terms of Stiefel-Whitney classes). On a given (M, g) spinor bundles may not be unique: spin/spin^c structures are torsor over $H^1(M, \mathbb{Z}_2)/H^2(M, \mathbb{Z})$. General problem open.

Natural questions:

1. Are spinor bundles (S, Γ) obstructed? If so, what are the obstructions? Alternatively, what is the topological obstruction to the existence of a Dirac operator on S?
2. Are spinor bundles (S, Γ) always associated to a *spinorial structure*, such as a spin or a spin^c structure? If so, can these spinorial structures be classified?

In the following we will approach questions (1) and (2) through the use of the theory of *Lipschitz structures*, introduced by T. Friedrich and A. Trautman (2000) for the particular case of bundles of complex faithful Clifford modules.

Motivation

- Study of connections and Dirac operators on spinor bundles over non-spin manifolds.
- Obtaining formulas for the index theorem of Dirac operators on (S, Γ).
- Sytematic study of holonomy and stabilizers of non-vanishing spinors.
- Study of generalized Killing spinors.
- Computation of anomalies in QFT and string theory.
- Recent developments in the mathematical theory of supergravity require considering generalized spinorial structures for the study of its supersymmetric solutions and supersymmetric moduli spaces. There is a priori no guiding principle to choose such spinorial structure: the theory of Lipschitz structures provides such natural choice (compatible with R-symmetry) and therefore fits naturally within the theory of supergravity.

Let $\eta : \mathrm{Cl}(V, h) \to \mathrm{End}(\Sigma)$ be a weakly faithful Clifford representation. The restriction of η gives a linear isomorphism $\eta|_V : V \xrightarrow{\sim} W(\eta)$ which transports h to a symmetric non-degenerate pairing $g_\eta : W(\eta) \times W(\eta) \to \mathbb{R}$. Thus $(W(\eta), g_\eta)$ is a quadratic space and $\eta|_V : (V, h) \xrightarrow{\sim} (W(\eta), g_\eta)$ is an invertible isometry.

Definition

The group $\mathrm{L}(\eta) \stackrel{\mathrm{def.}}{=} \mathrm{Aut}_{\mathrm{ClRep}}(\eta) = \{a \in \mathrm{Aut}_{\mathbb{R}}(S) | \mathrm{Ad}(a)(W) = W\}$ is called the Lipschitz group of η.

Definition

The group morphism $\lambda_\eta : \mathrm{L}(\eta) \to \mathrm{O}(V, h)$ given by:

$$\lambda_\eta(a) \stackrel{\mathrm{def.}}{=} \eta^{-1}|_V \circ \mathrm{Ad}(a)|_W \circ \eta|_V \,,$$

for every $a \in \mathrm{L}(\eta)$ is called the adjoint or vector representation of $\mathrm{L}(\eta)$.

Definition

A real pinor bundle (S, Γ) of type $[\eta]$ is called weakly faithful if η is weakly faithful.

Let $\mathrm{ClB}(M, g)$ denote the category of real pinor bundles over (M, g) and based spinor bundle morphisms and let $\mathrm{ClB}_w(M, g)$ denote the full sub-category whose objects are weakly faithful pinor bundles.

Definition

The type of a real pinor bundle (S, Γ) is the isomorphism class $[\eta]$ of its fiber-wise Clifford representation $\Gamma_p : \mathrm{Cl}(T_p^* M, g_p^*) \to \mathrm{End}(S_p)$.

Let $\mathrm{ClB}_w^\eta(M, g)$ denote the full sub-category of $\mathrm{ClB}_w(M, g)$ consisting of all real pinor bundles of type η and $\mathrm{Cl}_w^\eta(M, g)^\times$ the corresponding groupoid.

Lipschitz structures II

Definition

Let $F(M,g)$ denote the orthonormal frame bundle of (M,g) and $\eta \in \mathrm{Ob}(\mathrm{ClRep}_w)$. A Lipschitz structure of type η on (M,g) is a pair (Q,Λ) consisting of a principal $\mathrm{L}(\eta)$ bundle over M and a λ_η-reduction $\Lambda\colon Q \to F(M,g)$.

In other words, a Lipschitz structure (Q,Λ) is a principal $\mathrm{L}(\eta)$ bundle Q fitting in the following commutative diagram:

$$\begin{tikzcd} Q \times \mathrm{L}(\eta) \arrow[r] \arrow[d, "\Lambda\times\lambda_\eta"'] & Q \arrow[d, "\Lambda"] \\ F(M,g) \times \mathrm{O}(V,h) \arrow[r] & F(M,g) \end{tikzcd}$$

where the horizontal arrows denote the right action of the group on the corresponding bundle.

Definition

Let $\Lambda : Q \to F(M,g)$ and $\Lambda' : Q' \to F(M,g)$ be two Lipschitz structures relative to $[\eta]$. An *isomorphism of Lipschitz structures* from Λ to Λ' is a based isomorphism of principal L_η-bundles $f : Q \to Q'$ such that $\Lambda' \circ f = \Lambda$.

$$\begin{tikzcd} Q \arrow[r, "f"] \arrow[d, "\Lambda"'] & Q' \arrow[d, "\Lambda'"] \\ F(M,g) \arrow[r, "\mathrm{id}"] & F(M,g) \end{tikzcd} \tag{1}$$

We denote by $L_\eta(M,g)$ the groupoid of Lipschitz structures of (M,g) relative to η.

Theorem

The groupoids $\mathrm{ClB}^{\eta}_{w}(M,g)^{\times}$ *and* $\mathrm{L}_{\eta}(M,g)$ *are equivalent. That is, there exist functors* $\mathcal{Q}_{\eta}\colon \mathrm{ClB}^{\eta}_{w}(M,g)^{\times} \to \mathrm{L}_{\eta}(M,g)$ *and* $\mathcal{S}_{\eta}\colon \mathrm{L}_{\eta}(M,g) \to \mathrm{ClB}^{\eta}_{w}(M,g)^{\times}$ *such that* $\mathcal{Q}_{\eta}\circ\mathcal{S}_{\eta} \simeq \mathrm{Id}_{\mathrm{L}_{\eta}(M,g)}$ *and* $\mathcal{S}_{\eta}\circ\mathcal{Q}_{\eta} \simeq \mathrm{Id}_{\mathrm{ClB}^{\eta}_{w}(M,g)^{\times}}$ *(in the sense of natural transformations).*

Remark

This theorem shows that the classification of weakly faithful real spinor bundles of type $[\eta]$ over (M,g) (up to based isomorphism of real spinor bundles) is equivalent with that of Lipschitz structures of type $[\eta]$ (up to isomorphism of Lipschitz structures). The first classification problem asks one to determine the set of isomorphism classes of objects in the category $\mathrm{ClB}^{\eta}_{w}(M,g)^{\times}$, while the latter asks for the set of isomorphism classes of objects in the category $\mathrm{L}_{\eta}(M,g)$. The theorem implies that there exists a canonically defined bijection between these two sets

Scketch of the proof I

We construct functors $\mathcal{Q}_\eta \colon \mathrm{ClB}^\eta_w(M,g)^\times \to \mathrm{L}_\eta(M,g)$ and $\mathcal{S}_\eta \colon \mathrm{L}_\eta(M,g) \to \mathrm{ClB}^\eta_w(M,g)^\times$ as follows:

- **A.** To (S,Γ) we associate the principal bundle $Q \stackrel{\mathrm{def.}}{=} \mathcal{Q}_\eta(S,\Gamma)$:

$$Q \stackrel{\mathrm{def.}}{=} \sqcup_{p\in M} \mathrm{Hom}_{\mathrm{ClRep}^\times_w}(\eta, \Gamma_p)\,,$$

projection given by $\pi(q) = p$ for $q \in Q_p = \mathrm{Hom}_{\mathrm{ClRep}^\times_w}(\eta,\eta_p)$ and:

$$\Lambda_p(q) \stackrel{\mathrm{def.}}{=} q_0 \in \mathrm{Hom}_{\mathrm{Quad}^\times}((V,h),(T^\ast_p M, g^\ast_p)) = F(M,g)_p\,.$$

- **B.** To (Q,Γ) we associate $S \stackrel{\mathrm{def.}}{=} \mathcal{S}_\eta(Q,\Lambda) \stackrel{\mathrm{def.}}{=} Q\times_{\rho_\eta}\Sigma$ through the tautological representation $\rho_\eta \colon \mathrm{L}(\eta)\to \mathrm{Aut}(\Sigma)$ of L_η. Clifford structure morphism $\eta \stackrel{\mathrm{def.}}{=} \eta(Q,\Lambda)\colon \mathrm{Cl}(M,g)\to\mathrm{End}(S)$ defined as follows:

$$\Gamma_p(x)([q,s]) \stackrel{\mathrm{def.}}{=} [q, \eta(\mathrm{Cl}(\Lambda_p(q)^{-1})(x))(s)]\,, \quad \forall\, x\in \mathrm{Cl}(T^\ast_p M, g^\ast_p)\,,$$

for all $q\in Q_p$ and $s\in\Sigma$.

Scketch of the proof II

The following must be now verified:

- $\mathcal{Q}_\eta(S,\Gamma)$ is indeed a principal bundle with structure group $\mathrm{L}(\eta)$ and there exists a *reduction* $\Lambda\colon Q \to F(M,g)$ along $\lambda_\eta\colon \mathrm{L}(\eta) \to \mathrm{O}(V,h)$.
- There exist isomorphisms:

$$\mathcal{S}_\eta \circ \mathcal{Q}_\eta(S,\Gamma) \simeq (S,\Gamma)\,, \qquad \mathcal{Q}_\eta \circ \mathcal{S}_\eta(Q,\Lambda)\,.$$

Corollary

A pseudo-Riemannian manifold (M,g) admits a spinor bundle (S,Γ) of type η if and only if it admits a Lipschitz structure (Q,Λ) of type η.

Since every real irreducible real Clifford representation is weakly irreducible (real spinors are more important for supergravity and string theory applications), we can apply the previous results to irreducible spinor bundles. Goal: compute obstructions and classify associated spinorial structures (Lipschtiz structures).

Classification of real Clifford algebras

Real Clifford algebras are isomorphic to matrix algebras over either $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. They are classified in eight different classes specified by $(p-q)\,\mathrm{mod}(8)$.

- If $(p-q)\,\mathrm{mod}(8) = 0, 2$ then: $\mathrm{Cl}(V,h) = \mathrm{Mat}(2^{\frac{d}{2}}, \mathbb{R})$.
- If $(p-q)\,\mathrm{mod}(8) = 1$ then:
  $\mathrm{Cl}(V,h) = \mathrm{Mat}(2^{\frac{d-1}{2}}, \mathbb{R}) \oplus \mathrm{Mat}(2^{\frac{d-1}{2}}, \mathbb{R})$.
- If $(p-q)\,\mathrm{mod}(8) = 3, 7$ then: $\mathrm{Cl}(V,h) = \mathrm{Mat}(2^{\frac{d-1}{2}}, \mathbb{C})$.
- If $(p-q)\,\mathrm{mod}(8) = 4, 6$ then: $\mathrm{Cl}(V,h) = \mathrm{Mat}(2^{\frac{d-2}{2}}, \mathbb{H})$.
- If $(p-q)\,\mathrm{mod}(8) = 5$ then:
  $\mathrm{Cl}(V,h) = \mathrm{Mat}(2^{\frac{d-3}{2}}, \mathbb{H}) \oplus \mathrm{Mat}(2^{\frac{d-3}{2}}, \mathbb{H})$.

A Clifford irrep $\eta : \mathrm{Cl}(V, h) \to \mathrm{End}_{\mathbb{R}}(\Sigma)$ is faithful iff $\mathrm{Cl}(V, h)$ is simple as an associative $\mathbb{R}$-algebra, which happens when $p - q \not\equiv_8 1, 5$ (the simple case). The spinor volume form $\omega = \eta(\nu)$ is proportional to id_S iff we are in the non-simple case $p - q \equiv_8 1, 5$. In the simple case, all real irreps of $\mathrm{Cl}(V, h)$ are equivalent. In the non-simple case, $\mathrm{Cl}(V, h)$ admits two inequivalent irreps, which can be realized in the same space Σ. In each of these, the Clifford volume form $\nu \in \mathrm{Cl}(V, h)$ defined by a given orientation of V satisfies:

$$\omega = \eta(\nu) = \epsilon_\eta \mathrm{id}_S \,, \qquad \eta_+ = \eta_- \circ \pi \,,$$

where $\pi : \mathrm{Cl}(V, h) \to \mathrm{Cl}(V, h)$ is the main automorphism, which satisfies $\pi(\nu) = -\nu$. In terms of the Schur algebra $\mathbb{S}(\eta)$ of η, we have the following possibilities:

- If $(p - q) \bmod(8) = 0, 1, 2$, then $\mathbb{S}(\eta) = \mathbb{R}$ (real case).
- If $(p - q) \bmod(8) = 3, 7$, then $\mathbb{S}(\eta) = \mathbb{C}$ (complex case).
- If $(p - q) \bmod(8) = 4, 5, 6$, then $\mathbb{S}(\eta) = \mathbb{H}$ (quaternionic case).

Let η be an irreducible real representation of $\mathrm{Cl}(V, h)$.

Theorem (C. Lazaroiu and C.S.S.)

The following homotopy equivalences of groups hold:

1. *In the normal simple case,* $\mathrm{L}_\eta \simeq \mathrm{Pin}(V, h)$.
2. *In the almost complex case,* $\mathrm{L}_\eta \simeq \mathrm{Spin}^o_{\alpha_{p,q}}(V, h)$, *where* $\alpha_{p,q} = (-1)^{\frac{p-q+1}{4}}$.
3. *In the quaternionic case,* $\mathrm{L}_\eta \simeq \mathrm{Pin}^q(V, h) = \mathrm{Pin}(V, h) \cdot \mathrm{Sp}(1)$
4. *In the normal non-simple case,* $\mathrm{L}_\eta \simeq \mathrm{Spin}(V, h)$.
5. *In the quaternionic non-simple case,* $\mathrm{L}_\eta \simeq \mathrm{Spin}^q(V, h) = \mathrm{Spin}(V, h) \cdot \mathrm{Sp}(1)$.
6. *In the complex even case,* $\mathrm{L}_\eta \simeq \mathrm{Pin}^c(V, h) = \mathrm{Pin}(V, h) \cdot \mathrm{U}(1)$.
7. *In the complex odd case,* $\mathrm{L}_\eta \simeq \mathrm{Spin}^c(V, h) = \mathrm{Spin}(V, h) \cdot \mathrm{U}(1)$.

Definition

$$\mathrm{Spin}^o_\alpha(V, h) \overset{\text{def.}}{=} \mathrm{Spin}(V, h) \cdot \mathrm{Pin}_2(\alpha) \overset{\text{def.}}{=} [\mathrm{Spin}(V, h) \times \mathrm{Pin}_2(\alpha)]/\{-1, 1\}.$$

$p-q$ mod 8	type	$\Lambda(V,h)$
$0,2$	normal simple	$\mathrm{Pin}(V,h)$
$3,7$	complex simple	$\mathrm{Spin}^o_-(V,h), \mathrm{Spin}^o_+(V,h)$
$4,6$	quaternionic simple	$\mathrm{Pin}^q(V,h)$
1	normal non-simple	$\mathrm{Spin}(V,h)$
5	quaternionic non-simple	$\mathrm{Spin}^q(V,h)$

Table: Real canonical spinor groups

Let $\sigma := \sigma_{p,q} \stackrel{\text{def.}}{=} (-1)^{q+\left[\frac{d}{2}\right]}$ and $\alpha := \alpha_{p,q} = (-1)^{\frac{p-q+1}{4}}$ and η an irreducible real Clifford representation.

Theorem (C. Lazaroiu and C.S.S.)

- *In the normal simple case ($p - q \equiv_8 0, 2$), the following statements are equivalent:*
  - a. *(M, g) admits a spinor bundle (S, Γ) of type η.*
  - b. *(M, g) admits an untwisted $\mathrm{Pin}(V, h)$ structure.*
  - c. *We have $\mathrm{w}_2^+(M) + \mathrm{w}_2^-(M) + \mathrm{w}_1^\sigma(M)^2 + \mathrm{w}_1^-(M)\mathrm{w}_1^+(M) = 0$.*
- *In the complex case, the following statements are equivalent:*
  - a. *(M, g) admits a spinor bundle (S, Γ) of type η.*
  - b. *(M, g) admits a Spin^o structure.*
  - c. *We have $\mathrm{w}_1(M) = \mathrm{w}_1(E)$ and $\mathrm{w}_2^+(M) + \mathrm{w}_2^-(M) =$ $\mathrm{w}_2(E) + \mathrm{w}_1(E)(p\mathrm{w}_1^+(M) + q\mathrm{w}_1^-(M)) + \frac{1}{2}\left[p(p+\alpha) + q(q+\alpha)\right]\mathrm{w}_1(E)^2$.*

Theorem (Continuation)

- *In the quaternionic simple case ($p - q \equiv_8 4, 6$), the following statements are equivalent:*
  - (a) *(M, g) admits a spinor bundle (S, Γ) of type η.*
  - (b) *(M, g) admits an untwisted Pin^q structure.*
  - (c) *The exists a principal $\mathrm{SO}(3)$-bundle E over M such that* $\mathrm{w}_2^+(M) + \mathrm{w}_2^-(M) + \mathrm{w}_1^\sigma(M)^2 + \mathrm{w}_1^-(M)\mathrm{w}_1^+(M) = \mathrm{w}_2(E)$.
- *In the normal non-simple case ($p - q \equiv_8 1$), the following statements are equivalent:*
  - (a) *(M, g) admits a spinor bundle (S, Γ) of type η.*
  - (b) *(M, g) admits a Spin structure.*
  - (c) *We have $\mathrm{w}_1(M) = 0$ and $\mathrm{w}_2^+(M) + \mathrm{w}_2^-(M) = 0$.*
- *In the quaternionic non-simple case ($p - q \equiv_8 5$), the following statements are equivalent:*
  - (a) *(M, g) admits a spinor bundle (S, Γ) of type η.*
  - (b) *(M, g) admits a Spin^q structure.*
  - (c) *The exists a principal $\mathrm{SO}(3)$-bundle E over M such that* $w_1(M) = 0 \;\; \text{and} \;\; \mathrm{w}_2^+(M) + \mathrm{w}_2^-(M) = \mathrm{w}_2(E)$.

A closer look at Spin^o_α structures

The twisted adjoint representation λ is the group morphism:

$$\lambda_\eta \colon \mathrm{Spin}^o_\alpha(V,h) \to \mathrm{O}(V,h)\,,$$

defined as follows:

$$\lambda_\eta([a,u]) \stackrel{\mathrm{def.}}{=} \det(\mathrm{Ad}^{(2)}(u))\mathrm{Ad}(a)\,,$$

A Lipschitz structure of this type must be equivariant with respect to this precise morphism. Note its twisted nature. We have short exact sequences:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}^o_\alpha(V,h) \stackrel{\lambda_\eta\times \mathrm{Ad}^{(2)}}{\longrightarrow} \mathrm{S}[\mathrm{O}(V,h)\times \mathrm{O}(2)] \longrightarrow 1\,,$$
$$1 \to \mathrm{Spin}^c(V,h) \stackrel{i}{\to} \mathrm{Spin}^o_\alpha(V,h) \xrightarrow{\tilde{\eta}_\alpha} \mathbb{Z}_2 \to 1\,,$$

whence $\mathrm{Spin}^o_\alpha(V,h)$ is a non-central extension of $\mathbb{Z}_2$ by $\mathrm{Spin}^c(V,h)$.

Examples

Proposition

Let X be a $(2k+1)$-dimensional manifold which is oriented and spin and let Y be an embedded $(2k-1)$-dimensional submanifold of X.

1. *Assume that $2k-1 \equiv_8 7$ and that X is endowed with a Riemannian metric g. Then $(Y, g|_Y)$ admits a Spin^o_+ structure whose characteristic $\mathrm{O}(2)$-bundle E is the orthogonal frame bundle of the normal bundle to Y in X.*
2. *Assume that $2k-1 \equiv_8 3$ and that X is endowed with a negative Riemannian metric g. Then $(Y, g|_Y)$ admits a Spin^o_- structure whose characteristic $\mathrm{O}(2)$-bundle E is the orthogonal frame bundle of the normal bundle to Y in X.*

Also, various examples on Grassmanians, see Arxiv:1809.09084 for more details.

Open problems and further directions

- Classification of all Lipschitz groups associated to weakly faithful real Clifford modules.
- Study of Lipschitz norms and conditions for existence of admissible bilinear forms.
- Classification of spinor bundles (S, Γ) modulo isomorphism.
- Applications to supergravity. Classification of manifolds admitting supersymmetric solutions and study of moduli spaces of supersymmetric solutions.
- Generalized Killing spinors, stabilizers of Lipschitz structures and orbit space: holonomy of spinor connections.
- Dirac operators and index theory on non-spin manifolds.

Thank you for your consideration!