# Differential operators in different geometries 

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Dirac operators in differential geometry and global analysis In memory of Thomas Friedrich (1949-2018)

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## $\mathbb{G}$-gradients

Assume $E$ is a vector bundle over a manifold $M$. Assume that $\nabla$ is a covariant derivative in $E$, i.e.,

$$
C^{\infty}(E) \quad \xrightarrow{\nabla} \quad C^{\infty}\left(T^{*} M \otimes E\right)
$$

Assume $\mathbb{G}$ is a Lie group acting both on $T^{*} M$ and $E$ (such a group is always strictly associated to the geometric structure considered on $M$ ). Split both the origin bundle $E$ and the target bundle $F=T^{*} M \otimes E$ onto direct sums of $\mathbb{G}$-irreducible invariant subbundles.
The restriction of $\nabla$ to any one of such subbundles of $E$ composed with the projection onto any one of $F$ is just a $\mathbb{G}$-gradient.

| $V_{1}$ |  |  |  |  | $W_{1}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\oplus$ |  |  |  |  | $\oplus$ |
| $\vdots$ |  |  |  |  |  |
| $\oplus$ | $\searrow$ |  |  |  | $\nearrow$ |
| $V_{\mu}$ | $\rightarrow$ | $E$ | $\square$ | $F$ | $\rightarrow$ |
| $\oplus$ | $\nearrow$ |  |  |  | $W_{\nu}$ |
| $\vdots$ |  |  |  |  | $\searrow$ |
| $\oplus$ |  |  |  |  |  |
|  |  |  | $\vdots$ |  |  |
| $V_{r}$ |  |  |  |  |  |

So, for any $\mu, \nu$ the first order differential operator

$$
\nabla^{\mu \nu}=P_{\mu \nu}=\pi_{\nu} \circ \nabla \circ j_{\mu}: C^{\infty}\left(V_{\mu}\right) \longrightarrow C^{\infty}\left(W_{\nu}\right)
$$

is a $\mathbb{G}$-gradient.

## The Stein-Weiss gradients

Here, we will be mainly interested in $S O(n)$-gradients, i.e., in the case $\mathbb{G}=S O(n)$.
$S O(n)$-gradients were introduced first in
$\square$ E. Stein, G. Weiss, Generalization of the Cauchy-Riemann equations and representations of the rotation group, Amer. J. Math. 90 (1968) 163-196.

Many natural first order linear differential operators in geometry are either gradients or their compositions. For example, the exterior and interior derivatives $d$ and $\delta$, respectively, the Cauchy-Riemann operator $\bar{\partial}$. The classical Dirac operator on exterior forms $d+\delta$ is their sum.
Gradients depend on the geometry of $M$ (the group $\mathbb{G}$ ) and this is obvious, but, on the other hand, they can themselves, e.g., by their spectral properties, determine, to some extent the geometry (dimension, volume, area of the boundary, scalar curvature,...).
$S O(n)$-gradients

$$
\nabla^{\mu \nu}=P_{\mu \nu}=\pi_{\nu} \circ \nabla \circ j_{\mu}: C^{\infty}\left(V_{\mu}\right) \longrightarrow C^{\infty}\left(W_{\nu}\right)
$$

are often called Stein-Weiss gradients.
Without loss of generality we confine considerations to the case when the origin bundle is irreducible:
The splitting receives then a simpler form, namely:

$$
\nabla=G_{1}+\cdots+G_{\nu}+\cdots+G_{r}
$$

$S O(n)$-gradients are characterized by their conformal covariance. Fegan proved namely

囯 H. D. Fegan, Conformally invariant first order differential operators, Quart. J. Math. Oxford (2), 27 (1976), 371-378.

Theorem. Each $\mathrm{SO}(\mathrm{n})$-gradient $G$ is conformally covariant, in the sense that there are constants $c$ and $c^{*}$ with

$$
G=\Omega^{-(c+1)} \underline{G} \Omega^{c}, \quad G^{*}=\Omega^{-\left(c^{*}+1\right)} \underline{G}^{*} \Omega^{c^{*}}
$$

whenever we have two conformally equivalent metrics, i.e. metrics $g$ and $\underline{g}$ related by $g=\Omega^{2} g$ for some positive smooth function $\Omega$ on $M$.
Conversely, any conformally covariant first order linear differential operator from an irreducible bundle is a composition of a gradient and a bundle map.

## Examples

Example: the bundle of skew-symmetric forms
For $k=1, \ldots, n-1$ consider the bundle $\bigwedge^{k}$. It is $\mathbb{G}$-irreducible (when some exceptional cases are excluded). The target bundle $T^{*} \otimes \bigwedge^{k}$ splits into three r $S O(n)$-irreducible invariant subbundles and the resulting splitting of the covariant derivative is:

$$
\nabla=G_{1}^{a}+G_{2}^{a}+G_{3}^{a} .
$$

where

$$
G_{1}^{a}=\frac{1}{k+1} d^{a}, \quad G_{3}^{a}=\frac{-1}{n-k+1} \operatorname{cotr}^{a} \circ d^{a *} .
$$

and

$$
G_{2}^{a}=\nabla-\frac{1}{k+1}^{a}+\frac{1}{n-k+1}^{a} \circ d^{a *}
$$

where $d^{a}$ and $d^{a *}$ are the usual operators of exterior derivative and coderivative, respectively. Here $\operatorname{cotr}^{a}$ is the operator adjoint to the trace operator defined by the metric.

In the particular case $k=1$ we have

$$
\nabla=\frac{1}{2} d^{a}+S-\frac{1}{n} g d^{a *}
$$

The operator $S$ is is given by

$$
S \alpha=\nabla^{s} \alpha+\frac{1}{n} \delta \alpha \cdot g, \quad \alpha \in C^{\infty}\left(\bigwedge^{1}\right)
$$

where $\nabla^{s}$ is the symmetrized $\nabla$,
$S$ is known as the Cauchy-Ahlfors operator.
國 A. Pierzchalski, Ricci curvature and quasiconformal deformations of a Riemanian manifold, Manuscripta Math. 66 (1989) 113-127.
Composition with the adjoint leads to the scond order strongly elliptic operator called the Ahlfors Laplacian:

$$
S^{*} S=\frac{1}{2} \delta d+\frac{1}{n} d \delta-\text { ric }
$$

Here (ric is the Ricci action on forms. Notice that its leading part is a special case of the so called weighted Laplacian

$$
\Delta_{a b}=a \delta d+b d \delta \quad a, b>0
$$

## Example: the bundle of symmetric forms

Let $S^{k}$ be the bundle of all symmetric k-tensors (forms) $S=\bigoplus_{k \geqslant 0} S^{k}$. Let $S_{0}^{k}$ be the subbundle of trace free symmetric k-tensors and let $\mathrm{S}_{0}=\bigoplus_{k \geqslant 0} \mathrm{~S}_{0}^{k}$.
Excluding some exceptional dimensions there are three irreducible subbundles of $T^{*} \otimes \mathrm{~S}_{0}^{k}$.
Define the Stein-Weiss gradients

$$
G_{j}^{s}=\pi_{j}^{s} \circ \nabla: C^{\infty}(\mathrm{S})_{0}^{k} \longrightarrow C^{\infty}\left(T^{*} \otimes \mathrm{~S}_{0}^{k}\right), \quad j=1,2,3
$$

where $\pi_{j}^{s}, j=1,2,3$ are the projections onto the irreducible subbundles.

One can check that then

$$
\begin{gathered}
\nabla=G_{1}^{s}+G_{2}^{s}+G_{3}^{s} \\
G_{1}^{s} \zeta=\frac{1}{k+1}\left(d^{s} \zeta+\frac{2}{n+k-1} g \odot d^{s *} \zeta\right) \\
G_{2}^{s} \zeta=\nabla \zeta-\frac{1}{k+1} d^{s} \zeta-\frac{2}{(n+k-1)(k+1)} g \odot d^{s *} \zeta+\frac{1}{n+k-1} \operatorname{cotr}^{s}\left(d^{s *} \zeta\right) \\
G_{3}^{s} \zeta=\frac{-1}{n+k-1} \operatorname{cotr}^{s}\left(d^{s *} \zeta\right)
\end{gathered}
$$

for $\zeta \in \mathrm{S}_{0}^{k}$.
Here $d^{s}$ is the symmetrized covariant derivative, $d^{s *}$ the adjoint to $d^{s}$ and $\operatorname{cotr}^{s}$ the adjoint to tr .

Notice that in the particular case $k=1$ the splitting of $\nabla$ reduces to that given previously for the skew-symmetric bundle and that

$$
G_{1}^{s}=G_{2}^{a}, \quad G_{2}^{s}=G_{1}^{a}, \quad G_{3}^{s}=G_{3}^{a}
$$

Here $G_{1}^{s}$ is a symmetric counterpart of the Cauchy-Ahlfors operator.

Finally notice that $P_{1}^{s}$ is an elliptic operator. More exactly, the only elliptic one of the three considered gradients.

## The problem of ellipticity

$\nabla$ is an elliptic operator in the sense of injectivity of its symbol. The question arises:
which gradients in the splitting:

$$
\nabla^{\mu \nu}=\pi_{\nu} \circ \nabla \circ j_{\mu}: C^{\infty}\left(V_{\mu}\right) \longrightarrow C^{\infty}\left(W_{\nu}\right)
$$

or, in particular - when the origin bundle is irreducible - in the splitting:

$$
\nabla=G_{1}+\cdots+G_{\nu}+\cdots+G_{r}
$$

are elliptic.
Notice that if $G_{\nu}$ is elliptic then the second order differential operator

$$
G_{\nu}^{*} G_{\nu}
$$

where $G_{\nu}^{*}$ denotes the operator formally adjoint to $G_{\nu}$ is strongly elliptic.

The problem was completely solved within the three years 1995-1997: in the three following papers:
$\square$ J. Kalina, A. Pierzchalski, P. Walczak, Only one of the generaized gradients can be elliptic, Ann. Polon. Math. 67 (1997), 111-120.

J. Kalina, B. Ørested, A. Pierzchalski, P. Walczak, G. Zang, Elliptic gradients and highest weights, Bull. Polon. Acad. Sci. Ser. Math. 44 (1996), 511-519.
T. Branson, Stein-Weiss operators and ellipticity, J. Funct. Anal. 151 (1997), 334-383.

## The operators of Laplace type

The Laplace operator

$$
\Delta^{\mathrm{s}}=d^{s^{*}} d^{s}-d^{s} d^{s^{*}}
$$

on symmetric tensors was defined by Sampson in 1971. Similarly to its counterpart: the Hodge-de Rham Laplacian

$$
\Delta^{a}=d^{a^{*}} d^{a}+d^{a} d^{a^{*}}
$$

(here $d^{a}$ is the exterior derivative) that acts on skew-symmetric forms, it is strongly elliptic.
The Lichnerowicz Laplacian - an operator acting on arbitrary tensors, not necessarily symmetric or skew symmetric - was introduced in 1961:

$$
\begin{equation*}
\left(\Delta_{L} t\right)_{\mu_{1} \cdots \mu_{k}}=\nabla^{*} \nabla t_{\mu_{1} \cdots \mu_{k}}+q(t) \tag{1}
\end{equation*}
$$

where

$$
q(t)=\sum_{i=1}^{k} \mathrm{R}_{\mu_{i}}{ }^{\rho} t_{\mu_{1} \cdots \rho \cdots \mu_{k}}-\sum_{i \neq j} \mathrm{R}_{\mu_{i}}{ }^{\rho} \mu_{j}{ }^{\sigma} t_{\mu_{1} \cdots \rho \cdots \sigma \cdots \mu_{k}}
$$

and where $q$ is a zero order operator depending on the curvature. Here $\mathrm{R}_{\mu \nu \rho \sigma}$ and $\mathrm{R}_{\mu \rho}$ and the curvature and the Ricci tensors, respectively. Notice also that since $\nabla_{\rho} g_{\mu \nu}=0$, the indices in the above formulas (1) and (14) can be freely risen and lowered.

The Lichnerowicz Laplacian preserves the type of symmetry of tensors. Moreover, when restricted to the subbundle of skew-symmetric tensors, it coincides with the Hodge-de Rham Laplacian $\Delta^{a}$.
A. Lichnerowicz, Propagateurs et commutateurs en relativité généralé, Inst. Hautes Études Sci. Publ. Math. 10 (1961), 1-56.

When restricted to the subbundle of symmetric tensors $\Delta_{L}$ relates to the Sampson Laplacian $\Delta^{\mathrm{s}}$ by a curvature term (zero order operator). Aside from is nice properties and interesting geometry the Lichnerowicz Laplacian is an elliptic operator.

Some important papers on operators in the bundle of symmetric tensors are:

J.H. Sampson, On a theorem of Chern, Transactions of the American Matheematical Society, 177 (1973) 141-153.
R M. Boucetta, Spectra and symmetric eigentensors of the Lichnerowicz Laplacian on $S^{n}$, Osaka Journal of Mathematics, 146, 235-254.
國
N.S. Dairbekov, V.A. Sharafutdinov, On Conformal Killing Symmetric Tensor Fields, Siberian Advances in Mathematics, 21 2011, 1-41.K. Heil, A. Moroianu, U. Semmelmann, Killing and conformal Killing tensors Journal of Geometry and Physics 106 (2016) 1-6.

## Another attempt

In analogy to the classical operators of the gradient and the divergence on functions and vector fields, respectively one can define the gradient and the divergence in the bundle of symmetric forms. The definition is analogous to that for skew-symmetric forms given by Rummler in 1988.
The operator has recently been investigated in detail by A. Kimaczyńska:A. Kimaczyńska, The differential operators in the bundle of symmetric tensors on a Riemannian manifold, PhD thesis, Lodz University (2016)

The gradient operator grad : $C^{\infty}\left(S^{k}\right) \rightarrow C^{\infty}\left(S^{k} \otimes T\right)$ is defined by

$$
\operatorname{grad}=\mathfrak{a} d^{s}-d^{s} \mathfrak{a}
$$

where $d^{5}$ is the symmetrized covariant derivative and the operator $\mathfrak{a}: S^{k} \rightarrow S^{k-1} \otimes T$ is given locally by

$$
\mathfrak{a} \varphi=\sum_{i=1}^{n} \iota_{e_{i}} \varphi \otimes e_{i}
$$

where $e_{1}, \ldots, e_{n}$ is an orthonormal basis in $T$ and $\varphi \in S^{k}$
The gradient is a differentiation of the symmetric algebra in the following sense:
Proposition. For $\varphi \in C^{\infty}\left(S^{k}\right)$ and $\psi \in C^{\infty}\left(S^{\prime}\right)$ we have

$$
\operatorname{grad}(\varphi \odot \psi)=\operatorname{grad} \varphi \odot \psi+\varphi \odot \operatorname{grad} \psi
$$

The divergence operator div: $C^{\infty}\left(S^{k} \otimes T\right) \rightarrow C^{\infty}\left(S^{k}\right)$ defined by

$$
\begin{equation*}
\operatorname{div}=\operatorname{tr} \mathrm{d}^{\mathrm{s}}-\mathrm{d}^{\mathrm{s}} \operatorname{tr} . \tag{2}
\end{equation*}
$$

acts on the symmetric product as follows:

Proposition. For $\varphi \in C^{\infty}\left(S^{k}\right) \mathbf{i} \psi \in C^{\infty}\left(S^{\prime} \otimes T\right)$ we have

$$
\operatorname{div}(\varphi \odot \psi)=\varphi \odot \operatorname{div} \psi+\operatorname{grad} \varphi \odot \psi .
$$

Moreover:
Theorem. The differential operators - grad : $C^{\infty}\left(S^{k}\right) \rightarrow C^{\infty}\left(S^{k} \otimes T\right)$ and div : $C^{\infty}\left(S^{k} \otimes T\right) \rightarrow C^{\infty}\left(S^{k}\right)$ are formally adjoint with respect to the global (integral) scalar product.

Theorem (Weitzenböck Formula) The following formula holds

$$
\Delta^{\mathrm{s}}=-\operatorname{div} \operatorname{grad}-\mathfrak{R}
$$

where the Ricci type tensor $\mathfrak{R}$ is locally defined by

$$
\Re=\sum_{i, j=1}^{n} e_{j}^{*} \odot \iota_{e_{i}} \mathrm{R}_{e_{i}, e_{j}}
$$

where $e_{1}, \ldots, e_{n}$ is a local orthonormal frame on $M$ and $e_{1}^{*}, \ldots, e_{n}^{*}$ is the dual frame.

The Weitzenböck formulas for generalized gradients
The pioneer work on the Weitzenböck formulas for the Stein-Weiss gradients was done by Branson in:
T. Branson, Stein-Weiss operators and ellipticity, J. Funct. Anal. 151 (1997), 334-383.

He proved that there are constants $b_{i}$ that certain linear combinations of $G_{i}^{*} G_{i}$ are not second order operators but sections of the endomorphism bundle $\operatorname{End}\left(E_{\rho}\right)$ depending on the curvature, namely

$$
\Sigma_{i} b_{i} G_{i}^{*} G_{i}=\{\text { curvature endomorphism }\}
$$

Each formula of this form is called then a Weitzenböck formula. He also showed that if the number of irreducible bundles for $T^{*} \otimes_{\mathbb{C}} E \rho$ is $N$ then the number of independent Weitzenböck formulas is [ $N / 2$ ].

The methods for a concrete construction of Weitzenböck formulas were given next in
Y. Homma, Bochner-Weitzenböck formula and curvature action on Riemannian manifold, Trans. Am. Math. Soc 358, (2006) 87-114.
U. Semmelmann, G. Weinart, The Weitzenböck machine, Compos. Math. 146, (2010) 507-540.

Let us recall here that the Weitzenböck formulas are a power tool for differential geometry and global analysis. A majority of important vanishing theorems is just their direct consequence.

## Elliptic boundary conditions

Let us introduce now the notion of ellipticity at the boundary First of all introduce the so called half-geodesic coordinate system:
Let $M$ be an oriented compact Riemannian manifold of dimension $n$ with a nonempty boundary $\partial M$. Near $\partial M$ we let $x=(y, r)$ where $y=\left(y_{1}, \ldots, y_{n-1}\right)$ is a system of local coordinates on $\partial M$ and where $r$ is the normal distance to the boundary. We assume $\partial M=\{x: r(x)=0\}$ and that $\frac{\partial}{\partial_{r}}$ is the inward unit normal. We further normalize the choice of coordinate by requiring the curves $x(r)=\left(y_{0}, r\right)$ for $r \in[0, \delta)$ are unit speed geodesics for any $y_{o} \in \partial M$. The inward geodesic flow identifies a neighborhood of $\partial M$ in $M$ with the collar $\partial M \times[0, \delta)$ for some $\delta>0$. The collaring gives a splitting of $T M=T \partial M \oplus T \mathbb{R}$ and a dual splitting $T^{*} M=T^{*} \partial M \oplus T^{*} \mathbb{R}$. To reflect this splitting we let for $\xi \in T^{*} M$ that $\xi=(\zeta, z)$ where $\zeta \in T^{*} \partial M, z \in T^{*} \mathbb{R}$.

To define the notion of ellipticity of a "selfadjoint" boundary condition for a linear differential operator $L$, we consider the ordinary differential equation:

$$
\sigma_{L}\left(y, 0, \zeta, D_{r}\right) f(r)=\lambda f(r) \quad \text { with } \quad \lim _{r \rightarrow \infty} f(r)=0
$$

where

$$
(\zeta, \lambda) \neq(0,0) \in T^{*} \partial M \times \mathbb{C} \backslash \mathbb{R}_{+}
$$

We say that boundary condition is elliptic with respect to $\mathbb{C} \backslash \mathbb{R}_{+}$if:

- $L$ is elliptic in the interior of $M$,
- on the boundary there always exists a unique solution to this ordinary differential equation satisfying the boundary condition.

A consequence of such the ellipticity is the following

Theorem. Let $V$ be a vector bundle on a compact Riemannian manifold with nonempty boundary Let $P: C^{\infty}(V) \rightarrow C^{\infty}(V)$ be an elliptic differential operator and $B$ a self-adjoint boundary condition.
(a) We can find a complete orthonormal system $\left\{\phi_{n}\right\}, n=1,2, \ldots$ for $L^{2} V$ of eigenvectors of $P: P \phi_{n}=\lambda_{n} \phi$.
(b) The eigenvectors $\phi_{n}$ are smooth and satisfy the boundary condition. Moreover $\lim _{n \rightarrow \infty}\left|\lambda_{n}\right| \rightarrow \infty$.
(c) If we order the eigenvalues $\left|\lambda_{1}\right| \leqslant\left|\lambda_{2}\right| \leqslant \ldots$ then there exists a constant $C$ and an exponent $\delta$ such that $\left|\lambda_{n}\right| \geqslant C n^{\delta}$ if $n$ is sufficiently large. cf.
$\square$ P. B. Gilkey, Invariance theory, the heat equation and the Atiyah Singer index theorem, Publish or Perish, Wilmington, Delaware, 1984

Notice that the theorem enables searching for the solutions by standard methods of harmonic analysis.

The system of natural boundary conditions
Recall that $\nabla$ in an irreducible bundle have the orthogonal splitting:

$$
\nabla \alpha=G_{1} \alpha+\cdots+G_{s} \alpha
$$

Notice the followig universal formula:
$\square$ B. Ørsted, A. Pierzchalski, The Ahlfors Laplacian on a Riemannian manifold with boundary, Mchigan Math. J. 43 (1) (1996), 99-122.
Theorem. For each differential operator $G_{i}, i=1, \ldots, s$ (in fact for each Stein - Weiss gradient) and for any sections $\alpha$ and $\beta$

$$
\left(G_{i} \alpha, \beta\right)-\left(\alpha, G_{i}^{*} \beta\right)=-\int_{\partial M}\langle\alpha, \beta(\nu)\rangle \Omega_{\partial M}
$$

Applying that formula twice we get that:

$$
\begin{aligned}
\left(G^{*} G \alpha_{1}, \alpha_{2}\right)- & \left(\alpha_{1}, G^{*} G \alpha_{2}\right)= \\
& -\int_{\partial M}\left(\left\langle\alpha_{1}, i_{\nu} G \alpha_{2}\right\rangle-\left\langle i_{\nu} G \alpha_{1}, \alpha_{2}\right\rangle\right) \Omega_{\partial M}
\end{aligned}
$$

This is the point for a formulation universal boundary conditions for $\nabla$ and possibly for other important differential operators, in particular for gradients.

To be elliptic at the boundary the conditions must be chosen in a subltle way. They must be namely:

- strong enough to force the self-adjointness of the operator $L=G^{*} G$ i.e, to force the vanishing of the integral over the boundary.
- weak enough to assure the existence of solutions to the considered ordinary differential equation.
A natural system of boundary conditions was suggested in
國 T. P. Branson, A. Pierzchalski Natural boundary conditions for gradients, manuscript, 2004
Let us describe the idea:
At the boundary, the action of the special orthogonal group $S O(n)$ is replaced by the action of of its subgroup that keep the normal vector invariant. So, our irreducible bundle splits now onto, say $s$, orthogonal subbundles. Denote by $p_{1}, \ldots, p_{s}$ the suitable projections on these subbundles.
Now, split both $\alpha$ and $i_{\nu} G \alpha$ by taking the compositions with the projections onto the just obtained (orthogonal and irreducible) subbundles.
We get then that at the boundary:

$$
\alpha=p_{1} \alpha+\cdots+p_{s} \alpha
$$

and, similarly

$$
i_{\nu} G \alpha=p_{1} i_{\nu} G \alpha+\cdots+p_{s} i_{\nu} G \alpha
$$

As a result we get the following decomposition for the scalar products that appear in the integrand:

$$
\begin{aligned}
\left\langle\alpha_{1}, i_{\nu} G \alpha_{2}\right\rangle & = \\
& +\left\langle p_{1} \alpha_{1}, p_{1} i_{\nu} G \alpha_{2}\right\rangle \\
& +\left\langle p_{2} \alpha_{1}, p_{2} i_{\nu} G \alpha_{2}\right\rangle \\
& \cdots \\
& +\left\langle p_{s} \alpha_{1}, p_{s} i_{\nu} G \alpha_{2}\right\rangle .
\end{aligned}
$$

Now, the right hand side of the last equality can be written (symbolically) in a form of a two column matrix

$$
\left[\begin{array}{cc}
p_{1} \alpha_{1} & p_{1} i_{\nu} G \alpha_{2} \\
p_{2} \alpha_{1} & p_{2} i_{\nu} G \alpha_{2} \\
\vdots & \vdots \\
p_{s} \alpha_{1} & p_{s} i_{\nu} G \alpha_{2}
\end{array}\right]
$$

A natural boundary condition will be obtained by the demand that exactly one term of each row of the matrix is equal to zero.
We get that way $2^{s}$ natural boundary conditions.

## Few examples

- the case of operators acting on functions, say

$$
\operatorname{grad}: C^{\infty}(M \otimes R) \rightarrow C^{\infty}(T)
$$

where $T$ is the tangent bundle.
The Stokes formula for the elliptic operator dyw grad says in this case

$$
\int_{M} \text { divgrad } f g \Omega_{M}-\int_{M} f \text { divgrad } g=-\int_{\partial M}\left(f \nabla_{\nu} g-\nabla_{\nu} f g\right) \Omega_{\partial M}
$$

Since the tangent bundle is irreducible, we have a one row matrix

$$
\left[\begin{array}{ll}
* & *
\end{array}\right]
$$

So the natural boundary conditions take the form:

$$
\left[\begin{array}{ll}
0 & *
\end{array}\right] \text { or }\left[\begin{array}{ll}
* & 0
\end{array}\right]
$$

or, explicitly,

$$
f=0 \text { on } \partial M \text { or } \nabla_{\nu} f=0 \text { on } \partial M
$$

what means the Dirichlet condition or the Neumann condition, respectively. They both are elliptic!

- the case of operators acting on exterior forms of arbitrary degree. Also in this case the bundle splits onto two summands at the boundary (so $s=2$ ) and the four ( $=2^{2}$ ) boundary conditions appear naturally. They have an interesting symmetry with respect to the Hodge star operator.
The obtained that way boundary conditions have been successfully tested in the case of a class of second order operators acting on differential skew-symmetric forms of an arbitrary degree in the Euclidean ball in $\mathbb{R}^{n}$ :

T
W. Kozłowski, A. Pierzchalski, Natural boundary value problems for weighted form Laplacians, Ann. Sc. Norm. Sup. Pisa, VII, (2008), 343-367.

The conditions are:
Dirichlet boundary condition ( $\mathcal{D}$ ):

$$
\omega^{T}=0 \quad \text { and } \quad \omega^{N}=0 \quad \text { on } \quad \partial M
$$

Absolute boundary condition $(\mathcal{A})$ :

$$
\omega^{N}=0 \quad \text { and } \quad(d \omega)^{N}=0 \quad \text { on } \quad \partial M
$$

Relative boundary condition $(\mathcal{R})$ :

$$
(\delta \omega)^{T}=0 \quad \text { and } \quad \omega^{T}=0 \quad \text { on } \quad \partial M
$$

Neumann boundary condition $(\mathcal{B})$ :

$$
(\delta \omega)^{T}=0 \quad \text { and } \quad(d \omega)^{N}=0 \quad \text { on } \quad \partial M
$$

Here $\omega^{T}$ and $\omega^{N}$ denote the tangent and the normal parts of $\omega$ at the boundary, respectively.
The first three conditions are known to geometers. In particular they appear in the Weyl's paper mentioned above. The fourth one seems to be unknown. But, being natural, it should have a geometric or physical meaning.

Observe also a surprising symmetry with respect to the Hodge star operator $*$. Namely, by the following known relations:

$$
* *= \pm 1, \quad(* \omega)^{T}= \pm *\left(\omega^{N}\right), \quad(* \omega)^{N}= \pm *\left(\omega^{T}\right)
$$

and

$$
\delta \omega= \pm \star d \star \omega, \quad d \omega= \pm \star \delta \star \omega
$$

It follows easily that the set of all the four boundary conditions $\{\mathcal{D}, \mathcal{A}, \mathcal{R}, \mathcal{B}\}$ is star - invariant.
More precisely, each of the conditions $\mathcal{D}, \mathcal{B}$ is star-invariant, while the conditions $\mathcal{A}$ and $\mathcal{R}$ are star-symmetric: each to the other.
Also in this case
All the four conditions have been proved to be elliptic.

- the case of operators acting on the bundle of symmetric forms of arbitrary degree $k$.

The bundle of symmetric forms splits onto $k+1$ summands at the boundary, so $s=k+1$, and -in contrast to the skew-symmetric case - the number of summands in the splitting depends on the degree of forms.

As a result there are $2^{k+1}$ boundary conditions for the bundle of symmetric tensors of order $k$.

All of them were investigated inA. Kimaczyńska, The differential operators in the bundle of symmetric tensors on a Riemannian manifold, PhD thesis, Lodz University (2016)
andA. Kimaczyńska, A. Pierzchalski, Elliptic operators in the bundle of symmetric tensors, Banach Center Publications, 113 (2017), 193-218

The explicit shape of each of the possible $2^{k+1}$ boundary conditions was described.
They have been investigated in detail. It was proved by an original construction of the so called auxiliary bundle that

Theorem. All the natural $2^{k+1}$ boundary conditions are elliptic.

## The symplectic case

In the case of a symplectic manifold elliptic operators were constructed in

R- L-S. Tseng, S-T. Yau Cohomology and Hodge theory on symplectic manifolds I, J. Differential Geom. 91 (2012) 383-416.
L-S. Tseng, S-T. Yau Cohomology and Hodge theory on symplectic manifolds II, J. Differential Geom. 91 (2012) 417-443.

L-S. Tseng, L. Wang Hodge theory and symplectic boundary conditions, arXiv: 1409.8250 v 1 , pp 35

Let $\left(M^{2 n}, \omega\right)$ be a symplectic manifold, ie. a $2 n$-dimensionl manifold ( $M^{2 n}$ with a skew-symmetric two-form $\omega$ that is assumed to be closed $(d \omega=0)$ and nondegenerte $\left(\omega^{n} \neq 0\right)$. In local coordinates we can write:

$$
\omega=\frac{1}{2} \sum \omega_{i, j} d x^{i} \wedge d x^{j}
$$

Let $\Omega^{k}$ denote the space of smooth $k$ - skew - symmetric forms on $M$. The Lefschetz operator $L: \Omega^{k} \rightarrow \Omega^{k+2}$ and its dual operator $\Lambda: \Omega^{k} \rightarrow \Omega^{k-2}$ are defined by

$$
\begin{gathered}
L(\eta)=\omega \wedge \eta \\
\Lambda(\eta)=\frac{1}{2}\left(\omega^{-1}\right)^{i, j} \iota \partial_{x^{i}} \iota \partial_{x^{j}} \eta
\end{gathered}
$$

where $\omega^{-1}$ denotes the inverse matrix of $\omega$.
The degree counting operator $H: \Omega^{*} \rightarrow \Omega^{k}$ is defined by

$$
H=\sum_{k}(n-k) \prod^{k}
$$

where $\Omega^{*}$ denotes the direct sum of $\Omega^{k}, k=0, \ldots, 2 n$ and $\prod^{k}$ is the projection operator onto forms of degree $k . L$ and $\Lambda$ together with $H$ give a representation of $s /(2)$ algebra acting on $\Omega^{*}$ :

$$
[\Lambda, L]=H, \quad[H, \Lambda]=2 \Lambda, \quad[H, L]=-2 L
$$

This $s l(2)$ representation allows a Lefschetz decomposition of forms in terms of irreducible finite - dimensional $s /(2)$ modules.

Let us recall the action (representation) of the Lie algebra $s /(2)$ in a vector space. Take the standard basis of the algebra:

$$
x=\left[\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right], \quad y=\left[\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right], \quad h=\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right] .
$$

We have then

$$
[h, x]=2 x,[h, y]=-2 x,[x, y]=h
$$

Let V be a vector space. We say that V is an $s /(2)$-module if a homeomorphism of $s l(2)$ into the algebra of endomorphisms of V is given. It can be proved that $h$ acts diagonally on $V . V$ can be then decomposed as the direct sum of eigenspaces

$$
V_{\lambda}=\{v \mid h . v=\lambda v\}
$$

. We accept the convention that $V_{\lambda}$ still make sense (and is 0 ) when $\lambda$ is not an eigenvalue for the endomorphism of $V$ which represents $h$.
We call $\lambda$ a weight of $h$ in $V$ and $V_{\lambda}$ a weight space, if only $V \lambda \neq 0$.

Lemma. If $v \in V_{\lambda}$, then $x . v \in V_{\lambda+2}$ and $y \cdot v \in V_{\{\lambda-2\} .}$.

In the case $\operatorname{dim} V$ is finite, the Lemma implies that $x$ and $y$ are represented by nilpotent endomorphisms of $V$. Moreover, since the sum $V=V_{\lambda}$ is direct, there must exist $V_{\lambda} \neq 0$ such that $V_{\lambda+2}=0$. For such $\lambda$ any nonzero vector in $V_{\lambda}$ is called a maximal vector of weight $\lambda$.
$s /(2)$ irreducile modules are completely characterized in the following

Theorem. Assume that $V$ is an irreducible module for $s /(2)$.
(a) Relative to the action of $h, V$ is the direct sum of weight spaces $V_{\mu}$, $\mu=m, m-2, \ldots,-(m-2),-m$, where $m=\operatorname{dim} V-1$ and $\operatorname{dim} V_{\mu}=1$ for ewch $\mu$.
(b) $V$ has (up to nonzero scalar multiples) a unique maximal vector, whose weight (called the highest weigth of $V$ ) is $m$.
(c) The action of $s /(2)$ on $V$ is given explicitely by the following formulas:

$$
\begin{gathered}
h \cdot v_{i}=(\lambda-2 i) v_{i}, \\
y \cdot v_{i}=(i+1) v_{i+1}, \\
x \cdot v_{i}=(\lambda-i+1) v_{i-1}
\end{gathered}
$$

where $v_{0}$ is a maximal vector in $v_{\lambda}, v_{-1}=0$ and $v_{i}=(1 / i!) y^{i} \cdot v_{0}$.

Comming back to the action of $\mathrm{sl}(2)$ on the space of $k$-forms on $M$ one can prove that
The highest weight states of these irreducible $s /(2)$ modules are the spaces of primitive forms, denoted by $P^{*}$.
Recall that a form $\eta \in \Omega^{k}$ is called primitive if

$$
\Lambda \eta=0
$$

This is equivalent to the condition

$$
L^{n-k+1} \eta=0
$$

By the definition, the degree of primitive the form is constrained to be $k \leqslant n$.

Moreover, for a given $\eta \in \Omega^{k}$, there is a unique Lefschetz decomposition into primitive forms as

$$
\eta=\sum_{r \geqslant \max (k-n, 0)} \frac{1}{r!} L^{r} B_{k-2 r},
$$

where $B_{k-2 r} \in P^{k-2 r}$ can be expressed in terms of $\eta$ :

$$
B_{k-2 r}=\left(\sum_{s=0} a_{r, s} \frac{1}{s!} L^{s} \Lambda^{r+s}\right) \eta
$$

Each term of this decomposition can be labeled by a pair $(r, s)$ corresponding to the space

$$
\mathfrak{L}^{r, s}=\left\{A \in \Omega^{2 r+s}: A=L^{r} B_{s} \text { with } B_{s} \in P^{s}\right\}
$$

Let $d$ be the exterior derivative on $\mathfrak{L}^{r, s}$.

Proposition. The operator $d$ acting on $\mathfrak{L}^{r, s}$ leads at most two terms:

$$
d: \mathfrak{L}^{r, s} \rightarrow \mathfrak{L}^{r, s+1} \oplus \mathfrak{L}^{r+1, s-1}
$$

with

$$
\begin{aligned}
& d L^{r} B_{s}=L^{r}\left(d B_{s}\right)=L^{r} B_{s+1}+L^{r+1} B_{s-1} \text { when } s<n, \\
& d L^{r} B_{n}=L^{r}\left(d B_{n}\right)=L^{r+1} B_{n-1} .
\end{aligned}
$$

Define the first order differential operators

$$
\partial_{+}: \mathfrak{L}^{r, s} \rightarrow \mathfrak{L}^{r, s+1}
$$

and

$$
\partial_{-}: \mathfrak{L}^{r, s} \rightarrow \mathfrak{L}^{r, s-1}
$$

by:

$$
\begin{aligned}
& \partial_{+}\left(L^{r} B_{s}\right)=L^{r} B_{s+1}, \\
& \partial_{-}\left(L^{r} B_{s}\right)=L^{r} B_{s-1}
\end{aligned}
$$

Then $d=\partial_{+}+L \partial_{-}$. Here $B_{s}, B_{s+1}, B_{s-1} \in P^{*}$ and $d B_{s}=B_{s+1}+L B_{s-1}$. Define also another first order differential operator $d^{\wedge}: \Omega^{k} \rightarrow \Omega^{k-1}$ given by

$$
d^{\wedge}=d \Lambda-\wedge d
$$

Let $(M, \omega)$ be a symplectic manifold and $(\omega, J, g)$ a compatible triple. Recal that then $J$ is an almost complex structure on $M$ such that

$$
\omega(J v, J w)=\omega(v, w)
$$

and

$$
\omega(v, J v)>0 \text { for } v \neq 0
$$

Tseng and Yau defined the following Laplacians on primitive forms:

$$
\begin{aligned}
& \Delta_{+}=\partial_{+} \partial_{+}^{*}+\partial_{+}^{*} \partial_{+} \text {on } P^{k}, \text { for } k<n ; \\
& \Delta_{-}=\partial_{-} \partial_{-}^{*}+\partial_{-}^{*} \partial_{-} \text {on } P^{k}, \text { for } k<n ; \\
& \Delta_{++}=\left(\partial_{+} \partial_{-}\right)^{*}\left(\partial_{+} \partial_{-}\right)+\left(\partial_{+} \partial_{+}^{*}\right)^{2}, \text { on } P^{n} ; \\
& \Delta_{--}=\left(\partial_{+} \partial_{-}\right)\left(\partial_{+} \partial_{-}\right)^{*}+\left(\partial_{-}^{*} \partial_{-}\right)^{2}, \text { on } P^{n} .
\end{aligned}
$$

Theorem. Each od the four Laplacians is an elliptic operator.

For any $\eta \in \Omega^{k}$, they also defined defined the following operators:

$$
\begin{aligned}
& \Delta_{d d^{\wedge}}(\eta)=d^{\wedge *} d^{*} d d^{\wedge} \eta+\frac{1}{4}\left(d d^{*}+d^{\wedge} d^{\wedge *}\right)^{2} \eta \\
& \Delta_{d+d^{\wedge}}(\eta)=d d^{\wedge} d^{\wedge *} d^{*} \eta+\frac{1}{4}\left(d^{*} d+d^{\wedge *} d^{\wedge}\right)^{2}
\end{aligned}
$$

Theorem. The operators $\Delta_{d d^{\wedge}}$ and $\Delta_{d+d^{\wedge}}$ are elliptic.

## Symplectic boundary conditions

Let $\left(M^{2 n}, \omega\right)$ be a compact symplectic manifold with smooth boundary $\partial M$.
Let $(\omega, J, g)$ be a compatible triple on it.
Let $\rho$ be a boundary defining function, i. e. such function that

$$
\begin{aligned}
& \rho<0 \text { on } M \text { and } \rho(x)=0 \text { if and only if } x \in \partial M \text {, } \\
& \text { the norm of gradient }|\nabla \rho|=1 \text { on } \partial M .
\end{aligned}
$$

The following boundary conditions have been suggested by L-S. Tseng and L. Wang. A form $\eta$, satisfies:

Dirichlet boundary condition, denoted by $\eta \in D$, if $d(\rho \eta)_{\mid \partial M}=0$; Neumann boundary condition, denoted by $\eta \in N$, if $d^{*}(\rho \eta)_{\mid \partial M}=0$; $J$-Dirichlet boundary condition, denoted by $\eta \in J D$, if $d^{\wedge *}(\rho \eta)_{\mid \partial M}=0$; $J$-Neumann boundary condition, denoted by $\eta \in J N$, if $d^{\wedge}(\rho \eta)_{\mid \partial M}=0$; $\partial_{+}$-Dirichlet boundary condition, denoted by $\eta \in D_{+}$, if $\partial_{+}\left(\rho \eta_{\mid \partial M}\right)=0$; $\partial_{+}$-Neumann boundary condition, denoted by $\eta \in N_{+}$, if $\partial_{+}^{*}\left(\rho \eta_{\mid \partial M}\right)=0$; $\partial_{-}$Dirichlet boundary condition, denoted by $\eta \in D_{-}$, if $\partial_{-}\left(\rho \eta_{\mid \partial M}\right)=0$;
$\partial_{-}$-Neumann boundary condition, denoted by $\eta \in N_{-}$, if $\partial_{-}^{*}\left(\rho \eta_{\mid \partial M}\right)=0$.

Many self-adjoint and elliptic boundary problems can now be formulated. Let us quote the following example:

$$
\begin{aligned}
& \Delta_{+} \Phi=\Psi, \text { on } M \\
& \partial_{+}(\rho \Phi)=0, \text { on } \partial M, \text { and } \partial_{+}\left(\rho \partial_{+}^{*} \Phi\right)=0 \text { on } \partial M
\end{aligned}
$$

Notice that on a symplectic manifold the de Rham-Hodge type Laplcian

$$
\Delta=d^{*} d+d d^{*}
$$

completely degenerates. Here the $d^{*}$ is the operator adjoint to $d$ with respect to the symplectic volume form
The remedy is making use of a compatible triple ( $\omega, J, g$ ) composed of the symplectic form, an almost complex structure and a Riemannian metric. The Riemannian metric gives us then the standard inner product on differential forms. With that product we can define the adjoint operators. They relate to $d^{*}$ by formulas involving the almost complex structure $J$.
Finally second order elliptic operators can be constructed in a usual way. Some operators of this form were constructed and investigated in detail by A. Najberg in her recent PhD dissertation:
䍰 A. Najberg, The gradient and the divergence on symplectic manifolds, PhD dissertation, Lodz University (2019), to appear.

THANK YOU FOR YOUR ATTENTION

