An overview on G_2-structures and special metrics

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1. G_2-structures
2. Known examples of compact manifolds with closed G_2-structures
3. Laplacian flow and ERP condition

G_2-structures

Definition

A G_2-structure on M^7 is given by a 3-form φ with pointwise stabilizer isomorphic to G_2.

Proposition (Fernández, Gray)

The following are equivalent:

(a) $\nabla^{LC}\varphi = 0$;

(b) $d\varphi = 0$ and $d(*\varphi) = 0$;

(c) $Hol(g_\varphi)$ is isomorphic to a subgroup of G_2.

Metrics induced by parallel G_2-structures are Ricci-flat [Bonan].

Intrinsic torsion forms

The intrinsic torsion of a G_2-structure can be identified with $\nabla^{LC}\varphi$ and it is encoded in the exterior derivatives $d\varphi$, $d*\varphi$ as

$$\begin{aligned} d\varphi &= \tau_0 * \varphi + 3\tau_1 \wedge \varphi + *\tau_3, \\ d*\varphi &= 4\tau_1 \wedge *\varphi + \tau_2 \wedge \varphi \end{aligned}$$

where τ_i is an intrinsic torsion i-form.

Example

- nearly parallel G_2-structure: $d\varphi = \tau_0 * \varphi \Leftrightarrow \tau_1 = \tau_2 = \tau_3 = 0$
- closed G_2-structure: $d\varphi = 0 \Leftrightarrow \tau_0 = 0,\ \tau_1 = 0,\ \tau_3 = 0$

Scalar curvature

The scalar curvature $Scal(g_\varphi)$ may be expressed in terms of τ_i.

Example

- For a nearly-parallel G_2-structure ($d\varphi = \tau_0 * \varphi$), the associated metric g_φ is Einstein with $Scal(g_\varphi) = \frac{21}{8}\tau_0^2 > 0$.
- For a closed G_2-structure ($d\varphi = 0$), $Scal(g_\varphi) = -\frac{1}{2}|\tau_2|^2 \leq 0$

Theorem (Bryant; Cleyton, Ivanov)

No compact 7-manifolds can admit a closed (non-parallel) G_2-structure φ such that g_φ is Einstein.

A spinorial description

G_2-structures can be defined using globally defined unit spinors σ:

σ induces the G_2-form

$$\varphi(X, Y, Z) := \langle X \cdot Y \cdot Z \cdot \sigma, \sigma \rangle,$$

where dots denote Clifford multiplication and $\langle \cdot, \cdot \rangle$ is the scalar product in the spinor bundle.

Definition

There exists an endomorphism S of TM satisfying

$$\nabla_X \sigma = S(X) \cdot \sigma,$$

for every tangent vector X on M, called the intrinsic endomorphism of (M, g, σ).

Fernández-Gray classes can be described in terms of the intrinsic endomorphism S [Agricola, Chiossi, Friedrich, Höll].

In particular:

• φ is nearly parallel if and only if $S = \lambda Id$
(i.e. σ is a Killing spinor)
[Friedrich, Kath, Moroianu, Semmelmann]

• φ is closed if and only if $S \in \mathfrak{g}_2$.

Remark

For a closed G_2-structure φ defined by σ we have
• S is skew-symmetric;
• σ is harmonic, i.e. $D\sigma = 0$, where D is the Riemannian Dirac operator.

Nearly parallel G_2-structures

Theorem (Friedrich, Kath, Moroianu, Semmelmann)

A 7-dim simply-connected spin manifold (M, g) admits a Killing spinor σ if and only if $\exists$ a (nearly)-parallel G_2-structure φ (i.e. $d\varphi = -8a * \varphi$). Moreover, φ is defined by σ in a unique way.

If $a \neq 0$ and $M \neq S^7$ then there are three types, distinguished by the dimension m_a of the space of all Killing spinors.

In particular, $1 \leq m_a \leq 3$, $m_{-a} = 0$ and M is either 3-Sasakian ($m_a = 3$), a Sasaki-Einstein ($m_a = 2$) or a proper nearly parallel ($m_a = 1$) [Friedrich, Kath].

Remark

Compact examples for any type are known [Boyer, Galicki; Friedrich, Kath, Moroianu, Semmelmann].

The 7-sphere

The first example of proper nearly parallel G_2-structure is the standard one on (S^7, g_{can}).

Theorem (Friedrich)

Let (S^7, φ, g_{can}) be a nearly parallel G_2-structure on the standard 7-sphere. Then φ is conjugated, under the action of the isometry group $SO(8)$, to the standard nearly parallel G_2-structure of S^7.

Problem

Does S^7 admits closed G_2-structures?

Closed G_2-structures

A G_2-structure φ is closed (or calibrated) if $d\varphi = 0$. Then

$$d * \varphi = \tau \wedge \varphi,$$

where $\tau \in \Lambda^2_{14} \cong \mathfrak{g}_2$, i.e. $\tau \wedge \varphi = - * \tau$ and $\tau \wedge *\varphi = 0$.

Remark

- $\tau = d^*\varphi \Rightarrow d^*\tau = 0 \Rightarrow d\tau = \Delta_\varphi \varphi$, where $\Delta_\varphi = dd^* + d^*d$ is the Hodge Laplacian.
- φ defines a calibration on M (i.e. $\varphi|_\xi \leq vol_\xi$, $\forall$ tg oriented 3-plane ξ) [Harvey-Lawson].

Associative 3-fold: $N^3 \subset M$ calibrated by φ, i.e. $\varphi|_{N^3} = dV_{N^3}$.
Coassociative 4-fold: $N^4 \subset M$ is coassociative if $\varphi|_{N^4} = 0$.

Ricci tensor

The Ricci tensor and the scalar curvature of g_φ can be expressed in terms of τ:

$$Ric(g_\varphi) = \frac{1}{4}|\tau|^2 g_\varphi - \frac{1}{4} j(d\tau - \frac{1}{2} *(\tau \wedge \tau)),$$

where $j : \Lambda^3 \to S^2$ is defined by

$$j(\beta)(X, Y) = *(\iota_X\varphi \wedge \iota_Y\varphi \wedge \beta)$$

The scalar curvature is given by

$$Scal(g_\varphi) = -\frac{1}{2}|\tau|^2 \leq 0.$$

ERP condition

Theorem (Cleyton-Ivanov; Bryant)

If M is compact with a closed G_2-structure φ, then
$\int_M [Scal(g_\varphi)]^2 dV_\varphi \leq 3 \int_M |Ric(g_\varphi)|^2 dV_\varphi$

[Bryant]: equality holds if and only if

$$d\tau = \frac{|\tau|^2}{6}\varphi + \frac{1}{6} *_\varphi (\tau \wedge \tau),$$

in such a case, φ is called extremally Ricci pinched (ERP).

Automorphism group

Remark

General results on the existence of closed G_2-structures on (compact) 7-manifolds are still not known.

$Aut(M, \varphi) := \{f \in Diff(M) | f^*\varphi = \varphi\} \Rightarrow$
its Lie algebra is $\mathfrak{aut}(M, \varphi) = \{X \in \chi(M) | L_X\varphi = 0\}$.

Theorem (Podestá, Raffero)

M compact with φ closed non-parallel. If $X \in \mathfrak{aut}(M, \varphi)$, then the 2-form $i_X\varphi$ is harmonic. Consequently:

- $\dim \mathfrak{aut}(M, \varphi)) \leq b_2(M)$;
- $\mathfrak{aut}(M, \varphi)$ *is abelian with dim* ≤ 6.

Known examples of manifolds with closed G_2-structures

$\Rightarrow$ There are no compact homogeneous examples and no compact examples of any cohomogeneity in semisimple case.

Examples

- Solvable (in particular nilpotent) Lie groups G with left-invariant closed G_2-structures $\hookrightarrow$ compact locally homogeneous $\Gamma\backslash G$ [Fernández, Conti-Fernández, Freibert, Lauret, ...]
- Complete closed G_2-structures which are invariant under the cohomogeneity one action of a compact simple Lie group [Cleyton, Swann].
- Non-solvable Lie groups with left-invariant closed G_2-structures [F, Raffero].

A compact example using orbifold resolution

Motivation: Compact Joyce examples with $Hol(g_\varphi) = G_2$ obtained as orbifold resolutions of $\mathbb{T}^7/F$ with $F \subset G_2$ finite subgroup and by a perturbation argument.

Idea: instead of $\mathbb{T}^7$ start with a nilmanifold $M = \Gamma\backslash N$ with an invariant closed closed G_2-structure and $N \cong \mathbb{R}^7$ 3-step nilpotent with structure equations

$$[e_1, e_2] = -e_4,\ [e_1, e_3] = -e_5,\ [e_1, e_4] = -e_6,\ [e_1, e_5] = -e_7$$

and $\Gamma \cong 2\mathbb{Z} \times \mathbb{Z}^6$.

Remark

$M = \Gamma\backslash N$ is diffeomorphic to a mapping torus M_ν of $\mathbb{T}^6$ by a diffeomorphism ν of $\mathbb{T}^6$ induced by a linear automorphism of $\mathbb{R}^6$ with projection

$$[(x_1, \ldots, x_7)] \in M \mapsto x_1 + 2\mathbb{Z} \in S^1 = \mathbb{R}/2\mathbb{Z}.$$

Consider the action of $F = \mathbb{Z}_2$ generated by

$$\rho : (x_1, \ldots, x_7) \in N \mapsto (-x_1, -x_2, x_3, x_4, -x_5, -x_6, x_7) \in N$$

Then

$$\rho(ab) = \rho(a)\rho(b), \ \forall a, b \in N,$$

$\hookrightarrow \rho$ induces an action of $\mathbb{Z}_2$ on $M = \Gamma\backslash N$.

Theorem (Fernández, F, Kovalev, Munoz)

- *$\hat{M} = M/\mathbb{Z}_2$ is a compact 7-orbifold with $b_1(\hat{M}) = 1$ and an orbifold closed G_2-form $\hat{\varphi}$. The singular locus S of $\hat{M}$ is the disjoint union of 16 copies of T^3.*
- *$\exists$ a resolution $\pi : (\tilde{M}, \tilde{\varphi}) \to (\hat{M}, \hat{\varphi})$ with $\tilde{M}$ compact smooth manifold, $b_1(\tilde{M}) = 1$ and $\tilde{\varphi} = \pi^*\hat{\varphi}$ in the complement of a small neighborhood of the exceptional locus $E = \pi^{-1}(S)$.*
- *Moreover, $\pi_1(\tilde{M}) = \mathbb{Z}$ and $\tilde{M}$ is formal.*

Associative 3-folds of $(\tilde{M}, \tilde{\varphi})$

One can construct examples of associative 3-folds of $\tilde{M}$ applying

Proposition (Joyce)

Let (Y, φ) with a closed G_2 form φ and $\sigma \neq id_Y$ be an involution of Y such that $\sigma^* \varphi = \varphi$. Then the fixed point set P is an embedded associative 3-fold. Furthermore, if Y is cpt then so is P.

Using the involution on N:

$$\sigma : (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \to (-x_1, -x_2, x_3, x_4, -x_5, \tfrac{1}{2} - x_6, x_7) \Rightarrow$$

Theorem (Fernández, F, Kovalev, Munoz)

- *$(\tilde{M}, \tilde{\varphi})$ has associative calibrated 3-tori.*
- *For each of those 3-tori, $\exists$ a 3-dimensional family of non-trivial associative deformations.*

Laplacian flow

Idea: use a geometric flow to deform closed G_2-structures and eventually obtain a parallel one

Definition (Bryant)

Let φ_0 be a a closed G_2-structure on M^7. The Laplacian flow (LF) is

$$\begin{cases} \partial_t \varphi(t) = \Delta_{\varphi(t)} \varphi(t), \\ d\varphi(t) = 0, \\ \varphi(0) = \varphi_0. \end{cases}$$

where $\Delta_{\varphi(t)}$ is the Hodge Laplacian of $g_{\varphi(t)}$.

if $\varphi(t)$ solves the LF, then $\varphi(t) \in [\varphi_0]$ and

$$\partial_t g_{\varphi(t)} = -2Ric(g_{\varphi(t)}) + l.o.t.$$

Remark

If M is compact, then

• stationary points are parallel G_2-structures, i.e. $d\varphi = 0$, $d * \varphi = 0$.

• the LF is the gradient flow of Hitchin's volume functional $\mathcal{V} : \varphi \in [\varphi_0] \mapsto \int_M \varphi \wedge *\varphi$.

$\mathcal{V}$ is monotonically increasing along the LP, its critical points are parallel G_2-structures and they are strict maxima.

Theorem (Bryant, Xu)

Assume that (M, φ_0) is compact. Then the LF has a unique solution for short time $t \in [0, \epsilon)$, with ϵ depending on $\varphi_0 = \varphi(0)$.

Recent developments

- If φ_0 is near a torsion-free G_2-structure $\tilde{\varphi}$, then the LF converges to a torsion-free G_2-structure which is related to $\tilde{\varphi}$ via a diffeomorphism [Xu, Ye; Lotay, Wei].
- long-time existence result, uniqueness and compactness theory, stability of critical points, real analicity [Lotay-Wei].
- non-collapsing under the assumption of bounded Scal [G. Chen].
- lower dimensional reduction of LF [Fine-Yao; F-Raffero].

Solutions to the LF

Study of explicit solutions on

- simply connected solvable Lie groups with left-invariant closed G_2-structure [Fernández-F-Manero, Lauret, F-Raffero].
- $\mathbb{T}^7$ with cohomogeneity one closed G_2-structure [Huang-Wang-Yao].

Remark

Self-similar solutions $\varphi(t) = \rho(t) f_t^* \varphi$ of the LF $\Longleftrightarrow$ closed G_2-structures φ satisfying

$$\Delta_\varphi \varphi = \lambda \varphi + L_X \varphi$$

for some $\lambda \in \mathbb{R}$ and vector field X.

According to the sign of λ, a Laplacian soliton is called shrinking ($\lambda < 0$), steady ($\lambda = 0$), or expanding ($\lambda > 0$).

Theorem (Lin; Lotay-Wei)

On a compact manifold any Laplacian soliton φ (which is not torsion-free) must have $\lambda > 0$ and $X \neq 0$.

In particular, on a compact 7-manifold the only steady Laplacian solitons are given by parallel G_2-structures.

Remark

The existence of non-trivial expanding Laplacian solitons on compact manifolds is still an open problem.

$\exists$ steady, shrinking and expanding (homogeneous) solitons on non-compact manifolds [Lauret, Nicolini; F-Raffero].

ERP condition

Problem

Study the behaviour of the LF starting from an ERP φ_0.

Recall: a closed G_2-structure φ is ERP if

$$d\tau = \frac{|\tau|^2}{6}\varphi + \frac{1}{6} *_\varphi (\tau \wedge \tau) \Longleftrightarrow$$

$$\int_M [Scal(g_\varphi)]^2 dV_\varphi = 3 \int_M |Ric(g_\varphi)|^2 dV_\varphi$$

Proposition (Bryant)

M compact with φ ERP, then

- the norm $|\tau|$ is constant and $\tau^3 = 0$;
- τ^2 (resp. $*_\varphi(\tau^2)$) is a non-zero closed simple 4-form (resp. 3-form) of constant norm;
- $TM = P \oplus Q$ where

$$P := \{X \in TM \,|\, \iota_X(\tau^2) = 0\}, \quad Q := \{X \in TM \,|\, \iota_X *_\varphi (\tau^2) = 0\}.$$

Moreover, the P-leaves are associative submanifolds, while the Q-leaves are coassociative submanifolds.

- $\mathrm{Ric}(g_\varphi) = -\frac{1}{6}|\tau|^2 g_\varphi|_P$ non-positive with eigenvalues $-\frac{1}{6}\,|\tau|^2_\varphi$ of multiplicity three and 0 of multiplicity four.

Theorem (F, Raffero)

M compact with an ERP closed G_2-structure φ. Then the solution of LP with $\varphi(0) = \varphi$ is

$$\varphi(t) = \varphi + f(t)\, d\tau,$$

with $f(t) = \frac{6}{|\tau|^2_\varphi} \left(\exp\left(\frac{|\tau|^2_\varphi}{6}\, t \right) - 1 \right)$.

In particular:

- $\varphi(t)$ *is ERP for all* $t \in \mathbb{R}$ *with* $\tau(t) = \exp\left(\frac{|\tau|^2_\varphi}{6}\, t \right) \tau$.
- *LF has constant velocity* $|\Delta_{\varphi(t)}|_{\varphi(t)} = \frac{1}{\sqrt{6}} |\tau|^2$.
- $\mathrm{Ric}(g_{\varphi(t)}) = \mathrm{Ric}(g_\varphi)$.

Asymptotic behaviour

Remark

- If $\varphi(0)$ is not ERP, then $\varphi(t)$ cannot become ERP in finite time.
- The total volume $Vol_{g_{\varphi(t)}}(M) = \int_M dV_{\varphi(t)} = \exp\left(\frac{|\tau|^2}{3} t\right) Vol_{g_\varphi}(M)$.

Proposition (F, Raffero)

- when $t \to +\infty$, the the volume of the P-leaves goes to zero relative to the volume of the manifold, while the volume of the Q-leaves and the volume of the manifold grow at the same rate.
- when $t \to -\infty$, the volume of the Q-leaves and the volume of the manifold tend to zero at the same rate.

Examples

• A compact quotient of the non-compact homogeneous $M = SL(2,\mathbb{C}) \ltimes \mathbb{C}^2/SU(2)$ by $\Gamma \subset Aut(M, \varphi)$ [Bryant].

• Left-invariant ERP on solvable Lie groups which are necessarily steady Laplacian soliton [Lauret, Nicolini].

• non-locally homogeneous examples [Ball].

Remark

$\exists$ a steady Laplacian soliton on a solvable Lie group which is not ERP [F, Raffero].

THANK YOU VERY MUCH FOR THE ATTENTION!!