Some results in homogeneous geometry: Invariant Einstein metrics and homogeneous geodesics

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Homogeneous geometry

According to F. Klein's Erlangen program, geometry studies those properties of a space M which remain invariant under the action of a group of transformations G.
A particularly important case is when M is a manifold and G a Lie group acting transitively on M. Then fixing a point $o \in M$, we have that $M \cong G/H$, where $H = \{g \in G : g \cdot o = o\}$ (also a Lie group).
The space G/H is called a *homogeneous space*.
If $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H respectively, then the study of M reduces to the study of the pair $(\mathfrak{g}, \mathfrak{h})$.
Various geometrical properties of M are expressed algebraically using the Lie brackets of $\mathfrak{g}$ $\mathfrak{h}$.

The objects of the talk:

- Invariant Einstein metrics
- Homogeneous geodesics and some generalizations

on certain classes of homogeneous spaces

Based on various joint works with

D. Alekseevksy, I. Chrysikos, Yu. Nikonorov, Y. Sakane, M. Statha, N. Souris

INVARIANT EINSTEIN METRICS

(M, g) Riemannian manifofd is called Einstein $\Leftrightarrow \mathrm{Ric}(g) = \lambda g, \quad \lambda \in \mathbb{R}$

The scalar λ is called the *Einstein constant* or the *cosmological constant*. If the Riemannian manifold (M, g) is compact, then a result of Hilbert states that g is an Einstein metric if and only if g is a critical point of the scalar curvature functional $T : \mathcal{M}_1 \to \mathbb{R}$ given by $T(g) = \int_M S(g) d\mathrm{Vol}_g$, on the set $\mathcal{M}_1$ of Riemannian metrics of unit volume.

As it is well known, for $n = \dim M \leq 3$ an Einstein manifold has constant sectional curvature.
Topological obstructions to the existence of compact Einstein 4-manifolds are known. In dimensions $n \geq 5$, compact simply connected Einstein manifolds with positive Einstein constant are also topologically obstructed, whereas for negative Einstein constant no such obstruction is known.

The problems in this field are generally rather involved. For example, till now necessary and sufficient conditions for a manifold to admit an Einstein metric are still unknown

The Einstein equation is a difficult PDE equation, hence one tries to solve it by making some natural assumptions. In fact, many examples of compact Einstein manifolds have been constructed using bundle, symmetry and holonomy assumptions.

See for example:

A. Besse *Einstein Manifolds*,
M. Wang: *Einstein metrics from symmetry and bundle constructions*, In: Surveys in Differential Geometry: Essays on Einstein Manifolds, 1999,
M. Wang: *Einstein metrics from symmetry and bundle constructions: A sequel*, In: Differential Geometry: Under the Influence of S.-S. Chern, in: Advanced Lectures in Mathematics, 2012, and
D.D. Joyce: *Compact Manifolds with Special Holonomy*, Oxford Mathematical Monographs, 2000 and references therein.

In the homogeneous case we consider G-invariant Einstein metrics on a homogeneous space G $/H$, and the Einstein equation becomes a subtle system of algebraic equations, and the following general questions are still open in both, the compact and non compact cases:

- *Which homogeneous spaces G/H admit a G-invariant Einstein Riemannian metric?*
- *In case of existence, is the set of G-invariant Einstein metrics on G/H finite or infinite?*

Due to the presence of isometric transitive group the G-invariant metric g and Ric are determined by their values at one point.

- $\lambda > 0$ $M = G/H$ is compact with $\pi_1(M) < \infty$
- $\lambda = 0$ M is Ricci flat
- $\lambda < 0$ G/H is non compact.

In the non compact homogeneous case, the only known examples until now are all of a very particular kind, namely, solvable Lie groups endowed with a left-invariant metric (so called *solvmanifolds*). We refer to the works Heber, Lauret, et al.

D. V. Alekseevskii conjecture is still open: A connected homogeneous Einstein manifold of negative scalar curvature is diffeomorphic to $\mathbb{R}^n$

(cf. recent work of R. Arroyo - R. Lafuente: *The Alekseevskii conjecture in low dimensions*, Math. Ann. 2017).

For results < 1987: A. Besse: *Einstein Manifolds*,

For results < 1999: M. Wang: *Einstein metrics from symmetry and bundle constructions*, In: Surveys in Differential Geometry: Essays on Einstein Manifolds. Surv. Differ. Geom. VI, Int. Press, Boston, MA 1999.

For results < 2013: M. Wang: *Einstein metrics from symmetry and bundle constructions: A sequel*, In: Differential Geometry: Under the Influence of S.S. Chern, in: Advanced Lectures in Mathematics, vol. 22, Higher Education Press/International Press, 2012, pp. 253–309.

For results < 2016: A. Arvanitoyeorgos: *Progress on homogeneous Einstein manifolds and some open probrems*, Bull. Greek Math. Soc. 58 (2010-15) 75–97.

Some low dimensional classifications

- $n = 2, 3$ spaces of constant curvature.
- $n = 4$ symmetric spaces (Jensen)
- $n = 5$ Alekseevsky - Dotti - Ferraris
- $n = 6, 7$ Nikonorov - Rodionov (except $\mathrm{SU}(2) \times \mathrm{SU}(2)$)
- $n \leq 11$ Böhm - Kerr (existence)

The problem is even more difficult for the case of a compact Lie group, wherein we need to prove existence of left-invariant Einstein metrics, and then try to find all non isometric left-invariant Einstein metrics. Even for the compact Lie groups $\mathrm{SU}(3)$ and $\mathrm{SU}(2) \times \mathrm{SU}(2)$ the number of left-invariant Einstein metrics is still unknown. (For the last example see some recent result by F. Belgun et al. *Left-invariant Einstein metrics on* $\mathbb{S}^3 \times \mathbb{S}^3$, J. Geom. Phys. 2018).
D'Atri and Ziller in *Naturally Reductive Metrics and Einstein Metrics on Compact Lie Groups*, Memoirs AMS 1979 found a large number of left-invariant Einstein metrics, which are naturally reductive, on the compact simple Lie groups $G = \mathrm{SU}(n), \mathrm{SO}(n)$ and $\mathrm{Sp}(n)$ (all with $\dim G > 3$. In the same article they posed the following question:

Does a compact Lie group admit left-invariant Einstein metrics which are not naturally reductive?

We will not discuss this problem in this talk.

Some examples

- Spheres $S^n \cong \mathrm{SO}(n+1)/\,\mathrm{SO}(n)$
- Projective spaces $\mathbb{R}P^n = \mathrm{O}(n)/(\mathrm{O}(1) \times \mathrm{O}(n-1))$
- Grasmmann manifolds $\mathrm{Gr}_k\mathbb{R}^n = \mathrm{O}(n)/(\mathrm{O}(k) \times \mathrm{O}(n-k))$
- Isotropy irreducible spaces
- Generalized flag manifolds $G/C(T)$, $C(T)$ the cetralizer of a torus T in G
- Stiefel manifolds $V_k\mathbb{R}^n \cong \mathrm{SO}(n)/\,\mathrm{SO}(n-k)$
- Compact simple Lie groups with a bi-invariant metric

Further examples

- 1985 Wang - Ziller Classified normal homogeneous Einstein manifolds
- SU(4)/ SU(2) admits no invariant Einstein metric.

Here SU(2) is a maximal subgroup of SO(4) and SO(4) is a natural subgroup of SU(4).

- Böhm - Wang - Ziller 2004 $G/H \longrightarrow \Gamma_{G/H}$ graph.

  Existence for $c > 0 \longleftrightarrow$ a property of the graph.
- Böhm 2003 $G/H \longrightarrow$ simplicial complex $\Delta_{G/H}$.

  Existence for $c > 0 \longleftrightarrow \Delta_{G/H}$ not contractible.
- Graev 2006, 2007, 2011

$M = G/H \longrightarrow$ Newton polytope P_M.

The number $\mathcal{E}(M)$ of complex solutions of the Einstein equation satisfies $\mathcal{E}(M) \leq \nu(P_M)$, the integer volume of P_M.

We can make a logical separation of compact homogeneous manifolds G/H in <u>two major classes</u>.

Those for which the isotropy representation decomposes into a sum of non equivalent irreducible subrepresentations (these G/H are called *monotypic*) and those for which the isotropy representation contains some equivalent subrepresentations. A well known conjecture of Ziller states that

If the isotropy representation of G/H is monotypic, then the number of Einstein metrics is finite.

Even though the conjecture has been verified for some classes of homogeneous manifolds (e.g. generalized flag manifolds with up to six isotropy summands), it is still open in general.

Some major classes of compact homogeneous spaces that has been progress towards the classification of invariant Einstein metrics:

- Generalized flag manifolds $G/C(T)$ (monotypic)

(Alekseevsky, Kimura, Arvanitoyeorgos, Arv. - Chrysikos, Arv. - Chrysikos - Sakane, Wang - Zhao)

- Stiefel manifolds (not monotypic)

(Jensen, Sagle, Arv., Nikonorov, Sakane, Statha)

- Generalized Wallach spaces G/K (both types)

(Older terminology *three-locally symmetric spaces*
$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3, \ [\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{h}$).
(Lomshakov - Nikonorov - Firsov, Nikonorov (classification for G semisimple), Chen - Kang - Liang (classification for G simple), Chen - Nikonorov)

Preliminaries on homogeneous spaces

$M = G/H$, G a compact semisimple Lie group, H a closed subgroup of G. Let $\mathfrak{g}, \mathfrak{h}$ be corresponding Lie algebras. Assume that M is reductive, i.e. there exists a subspace $\mathfrak{m} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathrm{Ad}(h)\mathfrak{m} \subset \mathfrak{m}$ for all $h \in H$. Here

$$\begin{aligned} \mathrm{Ad} : G &\to \mathrm{Aut}(\mathfrak{g}) \\ \mathrm{Ad}(g) &= d_e(I_g), \quad \text{where} \quad I_g : G \to G, x \mapsto gxg^{-1} \\ \mathrm{ad} : \mathfrak{g} &\to \mathrm{End}(\mathfrak{g}) \end{aligned}$$

Since $\pi : G \to G/H$ is a submersion it is $\mathrm{Ker}(d\pi_e) = \mathfrak{h}$, so

$$\mathfrak{m} \cong T_o(G/H), \quad X \mapsto \frac{d}{dt}(\exp tX \cdot o)\Big|_{t=0},$$

thus study the pair $(\mathfrak{g}, \mathfrak{h})$.

Preliminaries on homogeneous spaces

For $\alpha \in G$, let $\tau_\alpha : G/H \to G/H$, $\tau_\alpha(gH) = \alpha g H$ be the left translation. The homomorphism

$$\begin{array}{rcl} \chi = \mathrm{Ad}^{G/H} : H & \to & \mathrm{Aut}(\mathfrak{m}) \\ h & \mapsto & (d\tau_h)_o \end{array}$$

is called the <u>isotropy representation</u> of G/H.

Proposition

$\mathrm{Ad}^G|_H = \mathrm{Ad}^H \oplus \chi$

Preliminaries on homogeneous spaces

Definition

A Riemannian metric on $M = G/H$ is called G-invariant if for all $\alpha \in G$ $\tau_\alpha : G/H \to G/H$ is an isometry.

- There is 1-1 correspondence between
  - G-invariant metrics on $M = G/H$
  - χ-invariant inner products $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$, i.e.

$$\langle \chi(h)X, \chi(h)Y \rangle = \langle X, Y \rangle, \quad \text{for all} \quad X, Y \in \mathfrak{m}$$

  - Q self-adjoint, positive definite, equivariant operators $A : \mathfrak{m} \to \mathfrak{m}$ (Q a fixed $\mathrm{Ad}(H)$-invariant inner product on $\mathfrak{g}$), s.t.

$$\langle X, Y \rangle = Q(AX, Y).$$

Preliminaries on homogeneous spaces

Let $B = -$ Killing form of $\mathfrak{g}$.
If $\chi = \chi_1 \oplus \cdots \oplus \chi_s$, that is $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$ as $\mathrm{Ad}(H)$-modules, and $\mathfrak{m}_i$ are non equivalent, then any G-invariant Riemannian metric is determined by

$$\langle \cdot, \cdot \rangle = x_1 B|_{\mathfrak{m}_1} + \cdots + x_s B|_{\mathfrak{m}_s}, \quad x_i > 0$$

$$A = \begin{pmatrix} x_1 \mathrm{Id}_{\mathfrak{m}_1} & & \\ & \ddots & \\ & & x_s \mathrm{Id}_{\mathfrak{m}_s} \end{pmatrix}.$$

A formula for the Ricci tensor

Let $d_i = \dim \mathfrak{m}_i$. Let $\{e_i^\alpha\}$ be an B–o.n. basis of $\mathfrak{m}$ adapted to the decomposition above (note that all $\mathfrak{m}_i$ are non-equivalent). This means:
In every $\mathfrak{m}_i$ choose a B–o.n. basis $e_i^1, e_i^2, \ldots, e_i^{d_i}$ $(i \leq i \leq s)$, and consider the numbers

$$\begin{bmatrix} i \\ jk \end{bmatrix} = \sum_{\substack{1\leq\alpha\leq d_i \\ 1\leq\beta\leq d_j \\ 1\leq\gamma\leq d_k}} B([e_i^\alpha, e_j^\beta], e_k^\gamma)^2, \qquad \text{(Wang–Ziller 1986).}$$

It is $\begin{bmatrix} i \\ jk \end{bmatrix} \geq 0$ and symmetric in all three indices. Then the Ricci tensor of such a G-invariant metric is given by

$$\mathrm{Ric} = r_1 x_1 B|_{\mathfrak{m}_1} + \cdots + r_j x_j B|_{\mathfrak{m}_s}.$$

A formula for the Ricci tensor

The following proposition is due to Park–Sakane (1997).

Proposition

The components $r_1, \ldots, r_q$ of the Ricci tensor r of the metric $\langle \cdot, \cdot \rangle = x_1 B|_{\mathfrak{m}_1} + \cdots + x_s B|_{\mathfrak{m}_s}$ on G/H are given by

$$r_k = \frac{1}{2x_k} + \frac{1}{4d_k} \sum_{j,i} \frac{x_k}{x_j x_i} \begin{bmatrix} k \\ ji \end{bmatrix} - \frac{1}{2d_k} \sum_{j,i} \frac{x_j}{x_k x_i} \begin{bmatrix} j \\ ki \end{bmatrix} \quad (k = 1, \ldots, q)$$

where the sum is taken over $i, j = 1, \ldots, q$.

Therefore, the Einstein equation is equivalent to the algebraic system of equations

$$r_1 = r_2 = \cdots = r_j = c$$

for x_i, c.

However, if $\chi = \chi_1 \oplus \cdots \oplus \chi_s$, that is $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$ contains some equivalent $\mathrm{Ad}(H)$-modules $\mathfrak{m}_i$, then G-invariant Riemannian metrics are determined by

$$A = \begin{pmatrix} x_1 \mathrm{Id}_{\mathfrak{m}_1} & a_{12} & \cdots & a_{1s} \\ a_{12} & x_2 \mathrm{Id}_{\mathfrak{m}_2} & \cdots & a_{2s} \\ \vdots & \vdots & & \vdots \\ a_{1s} & \cdots & & x_s \mathrm{Id}_{\mathfrak{m}_s} \end{pmatrix},$$

so Ricci tensor is more complicated to be computed, as we may have $\mathrm{Ric}(\mathfrak{m}_i, \mathfrak{m}_j) \neq 0$.

Isotropy representation is monotypic

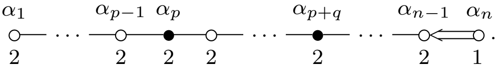
Typical Example: Generalized flag manifolds

Joint works with I. Chrysikos and Y. Sakane
2010, 2011, 2013

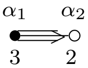
For G compact, semisimple, $M = G/K = G/C(T) \cong \mathrm{Ad}(G)w, w \in \mathfrak{g}$ (They exhaust all compact simply connected homogeneous Kähler manifolds). For G simple they can be classified by the painted Dynkin diagrams.

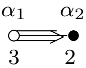
Examples

1) $\mathrm{Sp}(n)/U(p) \times U(q) \times \mathrm{Sp}(n-p-q)$ $(n \geq 3,\ p, q \geq 1)$.

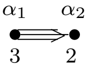


2) $G_2/U(2)$





2) $G_2/T\mathrm{max}$



Generalized flag manifolds

The number of black roots equals to $\dim Z(K) = b_2(M)$.

• It is possible to describe them in Lie terms

• They admit a finite number of Kähler-Einstein metrics

• There exists an 1–1 correspondence between isotropy irreducible submodules $\mathfrak{m}_\xi$ of $\mathfrak{m}^{\mathbb{C}}$ and T-roots $\xi = \alpha|_{Z(\mathfrak{k}^{\mathbb{C}})\cap\mathfrak{h}}$ (α a root of $\mathfrak{g}$ with respect to Cartan subalgebra $\mathfrak{h}$. This is given by

$$\Delta_{\mathfrak{t}} \ni \xi \mapsto \mathfrak{m}_\xi = \sum_{\kappa(\alpha)=\xi} \mathfrak{g}_\alpha^{\mathbb{C}}.$$

Thus we have a decomposition of the $\mathrm{Ad}_G(K)$-module $\mathfrak{m}^{\mathbb{C}}$:

$$\mathfrak{m}^{\mathbb{C}} = \sum_{\xi\in\Delta_{\mathfrak{t}}} \mathfrak{m}_\xi.$$

This induces a decomposition of the $\mathrm{Ad}_G(K)$-module $\mathfrak{m}$ into irreducible submodules:

$$\mathfrak{m} = \sum_{\xi\in\Delta_{\mathfrak{t}}^+} \left(\mathfrak{m}_\xi + \mathfrak{m}_{-\xi}\right)^\tau.$$

Generalized flag manifolds

Let $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q$

$q = 1$ Isotropy irreducible Hermitian symmetric space, Unique Einstein metric (up to scalar)
$q = 2$ A.A.-Chrysikos 2011 ($b_2(M) = 1$.)
$q = 3$ Alekseevski, Kimura, A.A. 1987, 1990, 1993 ($b_2(M) = 1$ or 2).
$q = 4$ A.A.-Chrysikos-Sakane (combined works) 2010, 2011 ($b_2(M) = 1$ or 2).
$q = 5$ A.A.-Chrysikos-Sakane (combined works) 2010, 2011, 2013, 2014 ($b_2(M) = 1$ or 2).
$q = 6$ A.A.-Chrysikos-Sakane (combined works) 2010, 2011, 2013, 2014 ($b_2(M) = 1, 2$ or 3). But not completely classified yet. For example G_2-type $\mathfrak{t}$-roots and BC_2-type $\mathfrak{t}$-roots. This includes $G_2/T_{\max}$
$q = 6$ Y. Wang-G.Zhao 2015 Quotients of F_4, E_6 and E_8.

Other works on $G/T_{\max}$: Wang–Ziller: g_B is Einstein $\Leftrightarrow$ $G \in \{SU(n), \mathrm{SO}(2n), E_6, E_7, E_8\}$
Sakane, Dos Santos, Negreiros.

Open problem: Find all homogeneous Einstein metrics on generalized flag manifolds G/K with $q \geq 6$.

Generalized flag manifolds

When $s \geq 4$ three major difficulties arise:

1. Computation of $\begin{bmatrix} i \\ jk \end{bmatrix}$.
2. Solve polynomial systems of equations, especially when coefficients are parameters (or at least prove existence of positive solutions).
3. Check that the solutions obtained correspond to non isometric Einstein metrics.

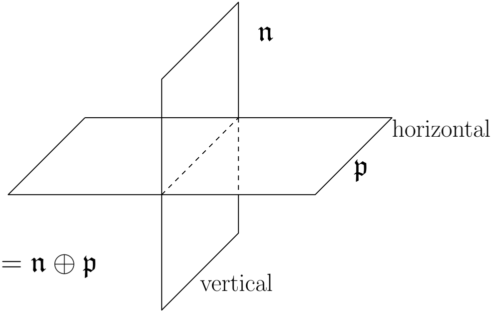
Riemannian submersions

Let $K \subset L \subset G$ and consider the sumbersion

$$L/K \longrightarrow G/K \longrightarrow G/L$$

$$\swarrow \qquad\qquad \downarrow \qquad\qquad \searrow$$

$$\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{n} \qquad \mathfrak{g} = \mathfrak{k} \oplus \underbrace{\mathfrak{n} \oplus \mathfrak{p}}_{\mathfrak{m}} \qquad \mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$$



$$\mathfrak{m} = \mathfrak{n} \oplus \mathfrak{p}$$

Riemannian submersions

Consider the submersion metric g^{sub} on $\mathfrak{m}$ $g^{\text{sub}} = \check{g} + \hat{g}$, where $\check{g} = G$-invariant metric on G/L and $\hat{g} = L$-invariant metric on L/K. Decompose

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_\ell, \quad \mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_s$$

into $\mathrm{Ad}(L)$-submodules and $\mathrm{Ad}(K)$-submodules respectively. Since $K \subset L$, each $\mathfrak{p}_i$ can be further decomposed into $\mathrm{Ad}(K)$-modules.

Thus the submersion metric g^{sub} is given by

$$g^{\text{sub}} = \underbrace{y_1 B|_{\mathfrak{p}_1} + \cdots + y_\ell B|_{\mathfrak{p}_\ell}}_{\check{g}} + \underbrace{z_1 B|_{\mathfrak{n}_1} + \cdots + z_s B|_{\mathfrak{n}_s}}_{\hat{g}} \tag{1}$$

and this is a special case of any G-invariant metric on $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q$

$$\langle \cdot, \cdot \rangle = x_1 B|_{\mathfrak{m}_1} + \cdots + x_q B|_{\mathfrak{m}_q} \tag{$*$}$$

We now decompose each irreducible component $\mathfrak{p}_j$ into irreducible $\mathrm{Ad}(K)$-modules

$$\mathfrak{p}_j = \mathfrak{m}_{j,1} \oplus \cdots \oplus \mathfrak{m}_{j,\,k_j},$$

chosen from the decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q$.
We assume that the $\mathrm{Ad}(K)$-modules $\mathfrak{m}_{j,t}$ $(j = 1, \ldots, \ell,\ t = 1, \ldots, k_j)$ are mutually non equivalent. Then the metric (1) can be written as

$$g^{\mathrm{sub}} = y_1 \sum_{t=1}^{k_1} B|_{\mathfrak{m}_{1,t}} + \cdots + y_\ell \sum_{t=1}^{k_\ell} B|_{\mathfrak{m}_{\ell,t}} + z_1 B|_{\mathfrak{n}_1} + \cdots + z_s B|_{\mathfrak{n}_s} \tag{2}$$

Ricci tensor

It is $r^{g^{\text{sub}}} = \check{r} + \hat{r}$, where $\check{r}$ is the Ricci tensor for the base (this will be known in our case), and $\hat{r}$ is the Ricci tensor for the fiber.

The components of the Ricci tensor $r^{g^{\text{sub}}}$ are given as follows:

Proposition (A.A.–Chrysikos–Sakane)

Let $d_{j,t} = \dim(\mathfrak{m}_{j,t})$. Then the components $r_{(j,\,t)}$ $(j = 1, \ldots, \ell,\ t = 1, \ldots, k_j)$ of the Ricci tensor $r^{g^{\text{sub}}}$ for the metric *(2)* on G/K are given by

$$r_{(j,\,t)} = \check{r}_j - \frac{1}{2d_{j,\,t}} \sum_{i=1}^{s} \sum_{j',\,t'} \frac{z_i}{y_j y_{j'}} \begin{bmatrix} & i & \\ (j,t) & & (j',t') \end{bmatrix}, \tag{3}$$

where $\check{r}_j$ are the components of Ricci tensor $\check{r}$ for the metric $\check{g}$ on G/L.

Riemannian submersions

Since metric (2) is a special case of the metric $(*)$, it is $r^{\langle\cdot,\cdot\rangle} = r^{g^{\mathrm{sub}}}$, hence

$$\check{r} = \text{horizontal part of } r^{\langle\cdot,\cdot\rangle},$$

where $r^{\langle\cdot,\cdot\rangle}$ is given by previous formula of the Ricci tensor by Park–Sakane. Therefore, we can find some of the $\begin{bmatrix} k \\ ij \end{bmatrix}$ for the metric $\langle\cdot,\cdot\rangle$.

By considering appropriate such submersions and by using the known Kähler-Einstein metrics on G/K we can find all $\begin{bmatrix} k \\ ij \end{bmatrix}$ for the metric $\langle\cdot,\cdot\rangle$.

Application: Flag manifolds G/K with G_2-type $\mathfrak{t}$-roots

(6 isotropy summands)
Flag manifolds with G_2-type $\mathfrak{t}$-root system satisfy
$\Pi \setminus \Pi_0 = \{\alpha_i, \alpha_j : \mathrm{Mrk}(\alpha_i) = 2, \mathrm{Mrk}(\alpha_j) = 3\}$ and are the following:

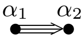
$$F_4/U(3) \times U(1) \quad R_{\mathfrak{t}} = \left\{\xi_1, \xi_2, \xi_1 + \xi_2, \xi_1 + 2\xi_2, \xi_1 + 3\xi_2, 2\xi_1 + 3\xi_2\right\}$$

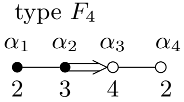
$$E_6/U(3) \times U(3)$$

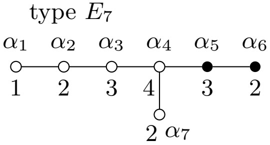
$$E_7/U(6) \times U(1) \quad R_{\mathfrak{t}} = \left\{\xi_5, \xi_6, \xi_5 + \xi_6, 2\xi_5 + \xi_6, 3\xi_5 + \xi_6, 3\xi_5 + 2\xi_6\right\}$$

$$E_8/E_6 \times U(1) \times U(1)$$

$$G_2/T \qquad R_{\mathfrak{t}} = R$$







Riemannian submersions

Isotropy representation: $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \cdots \oplus \mathfrak{m}_6$.
The only structure constants which are non zero are $\begin{bmatrix} 3 \\ 12 \end{bmatrix}$, $\begin{bmatrix} 6 \\ 15 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 23 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 24 \end{bmatrix}$, $\begin{bmatrix} 6 \\ 34 \end{bmatrix}$ (and their symmetries).

- G-invariant metrics:

$$\langle \cdot, \cdot \rangle = x_1(-B)|_{\mathfrak{m}_1} + \cdots + x_6(-B)|_{\mathfrak{m}_6}. \tag{4}$$

- Corresponding Ricci tensor is given by formula of Park-Sakane.

Consider $K \subset L_1 \subset G$ such that $\mathfrak{l}_1 = \mathfrak{k} \oplus \mathfrak{m}_2$ and the submersion

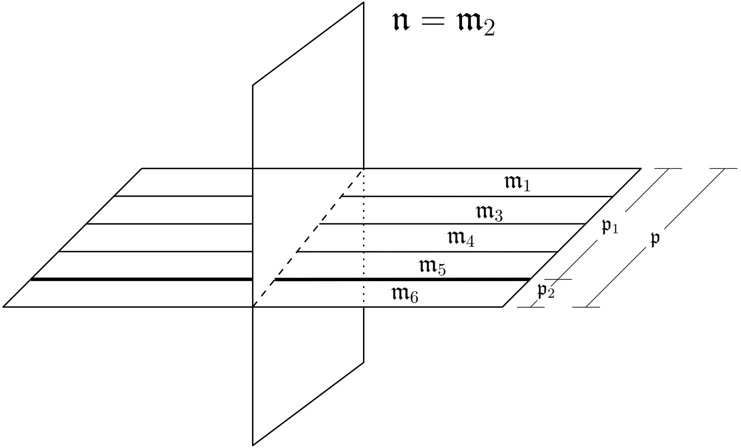
$$L_1/K \longrightarrow G/K \longrightarrow G/L_1.$$

Note that G/L_1 is a flag manifold with 2 isotropy summands. For example, for

$G/K = F_4/U(3) \times U(1)$, it is

$$Sp(3) \times U(1)/U(3) \times U(1) \to F_4/U(3) \times U(1) \to F_4/Sp(3) \times U(1).$$

Decompose $\mathfrak{m} = \mathfrak{n} \oplus \mathfrak{p}$ as follows:



Riemannian submersions

Thus

$$g_1 = \underbrace{y_1\big(B|_{\mathfrak{m}_1} + B|_{\mathfrak{m}_3} + B|_{\mathfrak{m}_4} + B|_{\mathfrak{m}_5}\big) + y_2 B|_{\mathfrak{m}_6}}_{\check{g_1}} + z_1 B|_{\mathfrak{m}_2}. \qquad (5)$$

Since metric (5) is a special case of metric (4) we obtain that (by Park-Sakane)

$$\begin{aligned}
r_1 &= \frac{1}{2y_1} - \frac{1}{2d_1}\begin{bmatrix}3\\12\end{bmatrix}\frac{z_1}{{y_1}^2} - \frac{1}{2d_1}\begin{bmatrix}6\\15\end{bmatrix}\frac{y_2}{{y_1}^2} && (6)\\
r_3 &= \frac{1}{2y_1} - \frac{1}{2d_3}\begin{bmatrix}3\\12\end{bmatrix}\frac{z_1}{{y_1}^2} - \frac{1}{2d_3}\begin{bmatrix}4\\23\end{bmatrix}\frac{z_1}{{y_1}^2} - \frac{1}{2d_3}\begin{bmatrix}6\\34\end{bmatrix}\frac{y_2}{{y_1}^2} && (7)\\
r_4 &= \frac{1}{2y_1} - \frac{1}{2d_4}\begin{bmatrix}4\\23\end{bmatrix}\frac{z_1}{{y_1}^2} - \frac{1}{2d_4}\begin{bmatrix}5\\24\end{bmatrix}\frac{z_1}{{y_1}^2} - \frac{1}{2d_4}\begin{bmatrix}6\\34\end{bmatrix}\frac{y_2}{{y_1}^2} && (8)\\
r_5 &= \frac{1}{2y_1} - \frac{1}{2d_5}\begin{bmatrix}5\\24\end{bmatrix}\frac{z_1}{{y_1}^2} - \frac{1}{2d_5}\begin{bmatrix}6\\15\end{bmatrix}\frac{y_2}{{y_1}^2}. && (9)
\end{aligned}$$

Riemannian submersions

The base G/L_1 is a flag manifold with 2 isotropy summands, so we know the Ricci tensor for $\check{g}_1$, i.e.

$$\check{r_1} = \frac{1}{2y_1} - \frac{y_2}{2y_1^2}\frac{\tilde{d_2}}{\tilde{d_1}+4\tilde{d_2}}, \ \tilde{d_i} = \dim(\mathfrak{p}_i).$$

For the above example it is $\frac{\tilde{d_2}}{\tilde{d_1}+4\tilde{d_2}} = 1/18$. So we have that

$$\begin{aligned} r_1 &= \frac{1}{2y_1} - \frac{1}{2d_1}\begin{bmatrix}3\\12\end{bmatrix}\frac{z_1}{y_1^2} - \frac{1}{2d_1}\begin{bmatrix}6\\15\end{bmatrix}\frac{y_2}{y_1^2} \\ r_{(1,1)} &= \frac{1}{2y_1} - \frac{y_2}{2y_1^2}\frac{1}{18} + \text{ fiber part.} \end{aligned}$$

Since $r_1 = r_{(1,1)}$, we equate red parts and obtain that $\begin{bmatrix}6\\15\end{bmatrix} = \frac{1}{9}$.

Question: Does this method work for any generalized flag manifold?

That is, can we compute Ricci tensor of generalized flag manifolds inductively?

Isotropy representation is monotypic

Typical examples: Real, Complex and Quaternionic Stiefel manifolds $V_k\mathbb{F}^n = G(n)/G(n-k)$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $G = \mathrm{SO}, \mathrm{Sp}, \mathrm{SU}$ respectively.

For example, for $G/H = \mathrm{SO}(n)/\mathrm{SO}(n-k)$ it is given by

$$\chi = \underbrace{1 \oplus \cdots \oplus 1}_{\binom{k}{2}} \oplus \underbrace{\lambda_{n-k} \oplus \cdots \oplus \lambda_{n-k}}_{k},$$

where $\lambda_{n-k} : \mathrm{SO}(n-k) \to \mathrm{Aut}(\mathbb{R}^{n-k})$ is the standart representation of $\mathrm{SO}(n-k)$.

Full system of equations expressing Einstein condition is extremely complicated.

Real Stiefel manifolds $V_k\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-k)$

- S. Kobayashi 1963 existence for $T_1S^n = \mathrm{SO}(n)/\mathrm{SO}(n-2) = V_2\mathbb{R}^n$ (as $\mathbb{S}^1$-bundle over Kähler manifold $\mathrm{SO}(n)/(\mathrm{SO}(n-2)\times\mathrm{SO}(2))$).
- A. Sagle 1970 existence for $V_k\mathbb{R}^n, k \geq 3$
- G. Jensen 1973 $V_k\mathbb{R}^n, k \geq 3$ admits at least two invariant Einstein metrics.
- A.A., Y. Sakane, M. Statha 2015 Left-invariant metrics Einstein on $V_n\mathbb{R}^n = \mathrm{SO}(n)$, which are not naturally reductive.
- A. Back and W.Y. Hsiang 1987 proved that for $n \geq 5$, $V_2\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-2)$ admits exactly one homogeneous Einstein metric. The same result was obtained in 1998 by M. Kerr (the diagonal metrics are the only homogeneous Einstein metrics).
- D. V. Alekseevsky, I. Dotti, C. Ferraris 1996 $\mathrm{SO}(4)/\mathrm{SO}(2)$ admits exactly two invariant Einstein metrics. One is a diagonal metric and the other is non diagonal metric (product metric on $\mathbb{S}^3\times\mathbb{S}^2$, as $\mathrm{SO}(4)/\mathrm{SO}(2)$ is diffeomorphic to $\mathbb{S}^3\times\mathbb{S}^2$).

Real Stiefel manifolds

- A.A., V.V. Dzhepko and Yu. G. Nikonorov 2009 proved that for $s > 1$ and $\ell > k \geq 3$, the Stiefel manifolds $\mathrm{SO}(sk+\ell)/\mathrm{SO}(\ell)$ admit at least four $\mathrm{SO}(sk+\ell)$-invariant Einstein metrics that are also $\mathrm{Ad}(\mathrm{SO}(k)^s \times \mathrm{SO}(sk+\ell))$-invariant (ADN metrics), and two of which are Jensen's metrics.
- A.A., Y. Sakane and M. Statha 2014, 2015 proved that $V_4\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-4)$ for $n \geq 6$ and $V_5\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-5)$ for $n \geq 7$ admit at least four invariant Einstein metrics. Two are Jensen's metrics and the other two are $\mathrm{Ad}(\mathrm{SO}(3) \times \mathrm{SO}(n-4))$-invariant and $\mathrm{Ad}(\mathrm{SO}(4) \times \mathrm{SO}(n-5))$-invariant respectively.
- A.A., Y. Sakane and M. Statha 2019 proved that for $2 \leq p \leq \dfrac{2}{5}n - 1$ the Stiefel manifolds $V_{2p}\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-2p)$ admit at least four $\mathrm{Ad}(\mathrm{U}(p) \times \mathrm{SO}(n-2p))$-invariant Einstein. Two of the metrics are Jensen's metrics and the other two are different from Jensen's metrics.

The above metrics are different from ADN metrics.

Quaternionic Stiefel manifolds $V_k\mathbb{H}^n = \mathrm{Sp}(n)/\mathrm{Sp}(n-k)$

- G. Jensen 1973 The first invariant Einstein metrics on $V_k\mathbb{H}^n$ using Riemannian submersions.
- W. Ziller 1982 $V_1\mathbb{H}^n = \mathbb{S}^{4n-1}$
- A.A., Y. Sakane and M. Statha 2015 Left-invariant metrics Einstein on $V_n\mathbb{H}^n = \mathrm{Sp}(n)$, which are not naturally reductive.
- A.A., V.V. Dzhepko and Yu. G. Nikonorov 2007 proved that for $s > 1$ and $\ell > k \geq 1$, the Stiefel manifolds $\mathrm{Sp}(sk+\ell)/\mathrm{SO}(\ell)$ admit at least four $\mathrm{Sp}(sk+\ell)$-invariant Einstein metrics that are also $\mathrm{Ad}(\mathrm{Sp}(k)^s \times \mathrm{Sp}(sk+\ell))$-invariant (ADN metrics), two of which are Jensen's metrics.
- A.A., Y. Sakane, M. Statha 2018 obtained new invariant Einstein metrics on $V_p\mathbb{H}^n$ different from Jensen's and ADN metrics. We viewed $V_p\mathbb{H}^n$ as a total space over generalized Wallach space and generalized flag manifold as follows: Let $n = k_1 + k_2 + k_3, p = k_1 + k_2$. Then write

$$\begin{aligned}
\mathrm{Sp}(k_1)\times\mathrm{Sp}(k_2) \to \quad & \mathrm{Sp}(n)/\mathrm{Sp}(n-p) \quad \to \mathrm{Sp}(n)/(\mathrm{Sp}(k_1)\times\mathrm{Sp}(k_2)\times\mathrm{Sp}(k_3))\\
\mathrm{U}(k) \to \quad & \mathrm{Sp}(p)/\mathrm{Sp}(n-p) \quad \to \mathrm{Sp}(n)/(\mathrm{U}(p)\times\mathrm{Sp}(n-p)).
\end{aligned}$$

Quaternionic Stiefel manifolds

Theorem (A.A. - Sakane - Statha (2018))

For $n = 3, 4$ the Stiefel manifold $V_2\mathbb{H}^n$ admits
(1) eight $\mathrm{Ad}(\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1))$-invariant Einstein metrics. Four of them are new, two are Jensen metrics and the other two are ADN metrics.
(2) eight $\mathrm{Ad}(\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(2))$-invariant Einstein metrics. Four of them are new, two are Jensen metrics and the other two are ADN metrics.

Theorem (A.A. - Sak. - St. (2018))

3) For $2 \leq p \leq \frac{3n}{4}$ there exist two $\mathrm{Ad}(\mathrm{U}(p) \times \mathrm{Sp}(n-p))$-invariant Einstein metrics on $\mathrm{Sp}(n)/\mathrm{Sp}(n-p)$ different from Jensen's metrics.

Equivalent isotropy summands

How do we approach the problem when the isotropy representation χ has some equivalent summands?

The basic approach is to use an appropriate subgroup K of G so that
(i) the set of $\mathrm{Ad}(K)$-invariant inner products is a subset of $\mathrm{Ad}(H)$-invariant inner products, and
(ii) the set of $\mathrm{Ad}(K)$-invariant inner products consists of diagonal such products.
(Such technique was originally used by A.A. – Dzhepko – Nikonorov in *Invariant Einstein metrics on some homogeneous spaces of classical Lie groups*, Canad. J. Math. 2009).

Now, a choice of K so that (i) holds is the following:

$$H \subset K \subset N_G(H) \subset G$$

A way to achieve this, is by using

Proposition

Let K be a subgroup of G with $H \subset K \subset G$ and such that $K = L \times H$, for some subgroup L of G. Then K is contained in $N_G(H)$.

Equivalent isotropy summands

Also from the inclusion $H \subset K \subset G$ we have the fibration

$$L \to G/H \to G/(L \times H)$$

so the tangent space of G/H can be written as a direct sum of two subspaces: the vertical, that is $T_o(L)$ and the horizontal, that is $T_o(G/K)$, which in our cases are both of them $\mathrm{Ad}(K)$-invariant.
Further, a choice of K so that (ii) holds, is that G/K is monotypic.

Example 1.
$G/H = \mathrm{SO}(k_1 + k_2 + k_3)/\mathrm{SO}(k_3)$, $L = \mathrm{SO}(k_1) \times \mathrm{SO}(k_2)$,
$\tilde{H} = \mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \times \mathrm{SO}(k_3)$. Here we have

$$\mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \to G/H \to G/\tilde{H} \quad \text{(base is a generalized Wallach space)}$$

Then $\mathfrak{m} = \mathfrak{so}(k_1) \oplus \mathfrak{so}(k_2) \oplus \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{23}$.
Example 2.

$$\mathrm{U}(k_1 + k_2) \to \mathrm{Sp}(k_1 + k_2 + k_3)/\mathrm{Sp}(k_3) \to \mathrm{Sp}(k_1 + k_2 + k_3)/(\mathrm{U}(k_1 + k_2) \times \mathrm{Sp}(k_3)).$$

Base is a generalized flag manifold
Then $\mathfrak{m} = \mathfrak{u}(1) \oplus \mathfrak{su}(k_1 + k_2) \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2$.

The Stiefel manifolds $V_{2p}\mathbb{R}^n = \mathrm{SO}(n)/\,\mathrm{SO}(n-2p)$

We consider the generalized flag manifold $G/K = \mathrm{SO}(n)/\,\mathrm{U}(p)\times\mathrm{SO}(n-2p)$, where the embedding of K in G is diagonal. The tangent space $\mathfrak{p}$ of G/K decomposes into two $\mathrm{Ad}(K)$-submodules

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2,$$

with

$$\mathfrak{p}_1 = \left\{ \begin{pmatrix} 0 & A_{12} \\ -{}^tA_{12} & 0 \end{pmatrix} \right\} \text{ and } \mathfrak{p}_2 \subset \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix},$$

where $A_{12} \in M(n-2p, 2p)$ ($M(p,q)$ the set of all $p \times q$ matrices) and $A_{11} \in \mathfrak{so}(2p)$.

Note that the irreducible $\mathrm{Ad}(H)$-submodules $\mathfrak{p}_1$, $\mathfrak{p}_2$ are non equivalent.

The Stiefel manifolds $V_{2p}\mathbb{R}^n = \mathrm{SO}(n)/\,\mathrm{SO}(n-2p)$

The tangent space $\mathfrak{m}$ of the Stiefel manifold can be written as follows

$$\mathfrak{m} = \mathfrak{h}_0 \oplus \mathfrak{su}(p) \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \tag{10}$$

where $\mathfrak{h}_0$ is 1 dimensional center of $\mathfrak{su}(p)$ and $\mathfrak{p}_1 \oplus \mathfrak{p}_2$ is the tangent space of generalized flag manifolds $B(\ell, p)$ or $D(\ell, p)$.

Note that $\mathrm{Ad}(\mathrm{U}(p) \times \mathrm{SO}(n-2p))$-modules in the decomposition (10) are mutually non equivalent.

For simplicity, we rewrite the decomposition (10) as

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3. \tag{11}$$

We write the invariant metrics on Stiefel manifold $\mathrm{SO}(n)/\,\mathrm{SO}(n-2p)$ defined by the $\mathrm{Ad}(\mathrm{U}(p) \times \mathrm{SO}(n-2p))$-invariant inner products on $\mathfrak{m}$ as

$$\langle \cdot, \cdot \rangle = u_0(-B)|_{\mathfrak{m}_0} + u_1(-B)|_{\mathfrak{m}_1} + u_2(-B)|_{\mathfrak{m}_2} + u_3(-B)|_{\mathfrak{m}_3}, \tag{12}$$

where $u_i \in \mathbb{R}_+$ $(i = 0, 1, 2, 3)$.

We set $d_1 = \dim(\mathfrak{m}_1)$, $d_2 = \dim(\mathfrak{m}_2)$ and $d_3 = \dim(\mathfrak{m}_3)$ (and $d_0 = \dim(\mathfrak{m}_0) = 1$.) It easy to see that the following relations hold:

$$[\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_3, \quad [\mathfrak{m}_3, \mathfrak{m}_3] \subset \mathfrak{m}_0 \oplus \mathfrak{m}_1, \quad [\mathfrak{m}_2, \mathfrak{m}_3] \subset \mathfrak{m}_2.$$

Hence, we see that the only non zero triples (up to permutation of indices) for the metric corresponding to (12) are

$$A_{220}, \quad A_{330}, \quad A_{111}, \quad A_{122}, \quad A_{133}, \quad A_{322}.$$

From A. A., K. Mori and Y. Sakane, we have the following:

Lemma

The triples A_{ijk} are given as follows:

$$A_{220} = \frac{d_2}{(d_2 + 4d_3)}, \quad A_{330} = \frac{4d_3}{(d_2 + 4d_3)} \quad A_{111} = \frac{2d_3(2d_1 + 2 - d_3)}{(d_2 + 4d_3)}$$
$$A_{122} = \frac{d_1 d_2}{(d_2 + 4d_3)} \quad A_{133} = \frac{2d_3(d_3 - 2)}{(d_2 + 4d_3)} \quad A_{322} = \frac{d_2 d_3}{(d_2 + 4d_3)}.$$

Proposition

The components of the Ricci tensor r for the invariant metric $\langle\cdot,\cdot\rangle$ on Stiefel manifold $G/H = \mathrm{SO}(n)/\mathrm{SO}(n-2p)$ defined by (12) are given as follows:

$$\begin{aligned}
r_0 &= \frac{u_0}{4u_2^2}\,\frac{d_2}{(d_2+4d_3)} + \frac{u_0}{4u_3^2}\,\frac{4d_3}{(d_2+4d_3)} \\
r_1 &= \frac{1}{4d_1u_1}\,\frac{2d_3(2d_1+2-d_3)}{(d_2+4d_3)} + \frac{u_1}{4u_2^2}\,\frac{d_2}{(d_2+4d_3)} + \frac{u_1}{2d_1u_3^2}\,\frac{d_3(d_3-2)}{(d_2+4d_3)} \\
r_2 &= \frac{1}{2u_2} - \frac{u_3}{2u_2^2}\,\frac{d_3}{(d_2+4d_3)} - \frac{1}{2u_2^2}\left(u_0\,\frac{1}{(d_2+4d_3)} + u_1\,\frac{d_1}{(d_2+4d_3)}\right) \\
r_3 &= \frac{1}{u_3}\left(\frac{1}{2} - \frac{1}{2}\,\frac{d_2}{(d_2+4d_3)}\right) + \frac{u_3}{4u_2^2}\,\frac{d_2}{(d_2+4d_3)} \\
&\quad -\frac{1}{u_3^2}\left(u_0\,\frac{2}{(d_2+4d_3)} + u_1\,\frac{d_3-2}{(d_2+4d_3)}\right)
\end{aligned}$$

The metric of the form (12) is Einstein if and only if the system of equations

$$r_0 = r_1, \quad r_1 = r_2, \quad r_2 = r_3, \tag{13}$$

has positive solutions.

After normalizing with $u_3 = 1$ the system of equations (13) becomes

$$\begin{aligned} f_1 &= u_0 u_1 (n-2p) - {u_1}^2 (n-2p) + 2(p-1) u_0 u_1 {u_2}^2 \\ &\quad -(p-2){u_1}^2{u_2}^2 - p{u_2}^2 = 0, \\ f_2 &= {u_1}^2\left(np-p^2-1\right) - 2(n-2)pu_1u_2 + p^2{u_2}^2 \\ &\quad +(p-2)p{u_1}^2{u_2}^2 + (p-1)pu_1 + u_0u_1 = 0, \\ f_3 &= 2(n-2)pu_2 + p(-n+p+1) + 2(p-2)(p+1)u_1{u_2}^2 \\ &\quad -(p-1)(p+1)u_1 - 4(p-1)p{u_2}^2 + 4u_0{u_2}^2 - u_0 = 0. \end{aligned} \tag{14}$$

We consider a polynomial ring $R = \mathbb{Q}[n,p][z, u_0, u_1, u_2]$ and an ideal J generated by $\{f_1, f_2, f_3, z\, u_0\, u_1\, u_2 - 1\}$ to find positive solutions of equation (14).

We take the lexicographic order $>$ with $z > u_0 > u_1 > u_2$ for a monomial ordering on R.

Then, by an aid of computer, we see that a Gröbner basis for the ideal J contains the polynomial $\left(2(p-1){u_2}^2 - 2(n-2)u_2 + n - 1\right) \times G_{n,p}(u_2)$,
where the polynomial $G_{n,p}(u_2)$ of degree 8 is given by

$$
\begin{aligned}
G_{n,p}(u_2) &= 8\left(5(p-2)^3+22(p-2)^2+29(p-2)+8\right)(p-2)(p-1){u_2}^8 \\
&-8(n-2)(p-2)(3p-1)\left(p^2-p+2\right){u_2}^7+\Big(\big(68(p-2)^4+312(p-2)^3 \\
&+484(p-2)^2+288(p-2)+64\big)(n-2p)+16(p-2)^5+100(p-2)^4 \\
&+296(p-2)^3+452(p-2)^2+320(p-2)+96\Big){u_2}^6 \\
&-8(n-2)\big(4(p-2)^3+15(p-2)^2+21(p-2)+8\big)(n-2p){u_2}^5 \\
&+2(n-2p)(p-1)\big(p(21p-26)(n-2p)+10(p-2)^3+42(p-2)^2 \\
&+80(p-2)+48\big){u_2}^4-2(n-2)\left(7p^2-8p+4\right)(n-2p)^2{u_2}^3 \\
&+\Big(p(11p-12)(n-2p)+8(p-2)^3+33(p-2)^2+56(p-2) \\
&+30\Big)(n-2p)^2{u_2}^2-\left(2p(n-2p)+4(p-1)p\right)(n-2p)^3u_2 \\
&+p(n-2p)^4+\left(p^2-p+1\right)(n-2p)^3
\end{aligned}
$$

(coefficients of even degree are positive and coefficients of odd degree are negative).

If $\big(2(p-1){u_2}^2 - 2(n-2)u_2 + n - 1\big) = 0$ we obtain Jensen's Einstein metrics.
From the above expression we see that if the equation $G_{n,p}(u_2) = 0$ has real solutions, then these are positive for $2 \leq p \leq n/2$.

Now we take the lexicographic order $>$ with $z > u_1 > u_2 > u_0$ for a monomial ordering on R. Then, by the aid of computer, we see that a Gröbner basis for the ideal J contains the polynomial
$(u_0 - 1)H_{n,p}(u_0)$ where

$$H_{n,p}(u_0) = \sum_{k=0}^{8} a_k(n,p){u_0}^k.$$

For example,

$$
\begin{aligned}
a_0(n,p) &= 64(p-2)p^3\Big(n-\frac{5(p+1)}{2}\Big)^9 + 96(p-2)p^3(11p+7)\Big(n-\frac{5(p+1)}{2}\Big)^8 \\
&+576(p-2)p^3(p+1)(13p+5)\Big(n-\frac{5(p+1)}{2}\Big)^7 + \big(30004(p-2)^7 + 427488(p-2)^6 \\
&+2515972(p-2)^5 + 7835288(p-2)^4 + 13626592(p-2)^3 + 12555872(p-2)^2 + 4791744(p \\
&+512\big)\Big(n-\frac{5(p+1)}{2}\Big)^6 + 4\big(18789(p-2)^6 + 248313(p-2)^5 + 1295975(p-2)^4 \\
&+3340687(p-2)^3 + 4256148(p-2)^2 + 2146136(p-2) + 928\big)p^2\Big(n-\frac{5(p+1)}{2}\Big)^5 \\
&+\big(122679(p-2)^7 + 1995834(p-2)^6 + 13365614(p-2)^5 + 47170188(p-2)^4 \\
&+92568607(p-2)^3 + 95837594(p-2)^2 + 40950780(p-2) + 44512\big)p^2\Big(n-\frac{5(p+1)}{2}\Big)^4 \\
&+2\big(65563(p-2)^9 + 1402351(p-2)^8 + 12986098(p-2)^7 + 68013362(p-2)^6 \\
&+220420041(p-2)^5 + 452819519(p-2)^4 + 576171516(p-2)^3 + 415509440(p-2)^2 \\
&+130343854(p-2) + 282848\big)p\Big(n-\frac{5(p+1)}{2}\Big)^3
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}\big(354867(p-2)^{10}+8723046(p-2)^{9}\\
&+94326525(p-2)^{8}+589028208(p-2)^{7}+2341360757(p-2)^{6}+6145435310(p-2)^{5}\\
&+10655244691(p-2)^{4}+11775095484(p-2)^{3}+7534446728(p-2)^{2}+2135654080(p-2)\\
&+8047168\big)p\Big(n-\frac{5(p+1)}{2}\Big)^{2}+\frac{1}{4}\big(3936(p-2)^{8}+75201(p-2)^{7}+603308(p-2)^{6}\\
&+2638758(p-2)^{5}+6806032(p-2)^{4}+10370489(p-2)^{3}+8665044(p-2)^{2}+3082288(p-\\
&+18432\big)p(p+1)(5p+1)(7p+11)\Big(n-\frac{5(p+1)}{2}\Big)+\frac{1}{16}\big(92455(p-2)^{11}+2535522(p-2)^{10}\\
&+30740389(p-2)^{9}+217376892(p-2)^{8}+994341005(p-2)^{7}+3078217258(p-2)^{6}\\
&+6539013051(p-2)^{5}+9423500944(p-2)^{4}+8830656620(p-2)^{3}+4872616168(p-2)^{2}\\
&+1215054848(p-2)+10612736\big)(p+1)^{2}.
\end{aligned}
$$

Thus we see that, if the equation $H_{n,p}(u_0) = 0$ has real solutions u_0, then these are positive for $n - 5(p+1)/2 \geq 0$, that is, $p \leq 2n/5 - 1$.

Now we take a third lexicographic order $>$ with $z > u_0 > u_2 > u_1$ for a monomial ordering on R. Then, by the aid of computer, we see that a Gröbner basis for the ideal J contains the polynomial $(u_1 - 1)F_{n,p}(u_1)$. We will write the polynomial $F_{n,p}(u_1)$ of degree 8 explicitly next.

Moreover, the Gröbner basis contains polynomials of the form such that $b(n,p)u_2 - X(u_1)$ and $c(n,p)u_0 - Y(u_1)$, where $X(u_1)$ and $Y(u_1)$ are polynomials of degree 7 with coefficients in $\mathbb{Q}[n,p]$, and $b(n,p)$ and $c(n,p)$ are in $\mathbb{Q}[n,p]$.

We also can show that $b(n,p)$ and $c(n,p)$ are non zero for $n - 2p \geq 0$ and $p \geq 2$. In particular, if u_1 is real, then the solutions u_2 and u_0 are real for $b(n,p)u_2 = X(u_1)$ and $c(n,p)u_0 = Y(u_1)$.

In the above we have seen that, if u_2 and u_0 are real solutions, these are positive.

$$
\begin{aligned}
&F_{n,p}(u_1) = (2n-p-1)^4(p-2)^2(p+1)^2{u_1}^8 \\
&-2(2n-p-1)^3(p-2)(p+1)\left(-10p^3+np^2+25p^2-20p+4n\right){u_1}^7 \\
&-2(2n-p-1)\big(-182p^7-83np^6+1407p^6+505n^2p^5-1587np^5-2570p^5-284n^3p^4 \\
&-603n^2p^4+5381np^4+504p^4+36n^4p^3+710n^3p^3-1440n^2p^3-4595np^3+1377p^3-64n \\
&-616n^3p^2+2768n^2p^2-250np^2+202p^2+32n^4p+48n^3p-688n^2p+344np-480p+32n \\
&-112n^3+112n^2+64n+32\big){u_1}^6+(2n-p-1)^2\big(140p^6-10np^5-726p^5-99n^2p^4+414n \\
&+1057p^4+44n^3p^3+68n^2p^3-930np^3-178p^3-112n^3p^2+312n^2p^2+420np^2-596p^2 \\
&+80n^3p-208n^2p-280np+816p-64n^3+272n^2-304n-64\big){u_1}^5+\big(46p^8+1262np^7 \\
&-2706p^7-2685n^2p^6+2570np^6+7173p^6+828n^3p^5+10434n^2p^5-31616np^5+5706p^5 \\
&+1996n^4p^4-20068n^3p^4+34511n^2p^4+15458np^4-15743p^4-1728n^5p^3+9248n^4p^3 \\
&+7624n^3p^3-75052n^2p^3+63050np^3-14418p^3+384n^6p^2+864n^5p^2-22832n^4p^2 \\
&+65888n^3p^2-54384n^2p^2+16444np^2-1358p^2-896n^6p+6336n^5p-12288n^4p+2384n^3 \\
&+7616n^2p-6256np+1784p+256n^6-2048n^5+5632n^4-6624n^3+4336n^2-1632n+24
\end{aligned}
$$

$$-2\big(-330p^8+137np^7+2129p^7+852n^2p^6-4328np^6-2144p^6-1120n^3p^5+2522n^2p^5$$
$$+6676np^5-2146p^5+704n^4p^4-1288n^3p^4-2508n^2p^4-4002np^4+5788p^4-304n^5p^3$$
$$+1096n^4p^3-2600n^3p^3+10706n^2p^3-14817np^3+3701p^3+64n^6p^2-224n^5p^2+1104n^4p$$
$$-6616n^3p^2+13188n^2p^2-6654np^2+1114p^2-64n^6p-128n^5p+3008n^4p-8600n^3p$$
$$+8552n^2p-4428np+1000p+128n^6-1024n^5+2944n^4-3952n^3+3024n^2-1288n+25$$
$$+\big(444p^8-782np^7-1226p^7+363n^2p^6+2566np^6-121p^6+232n^3p^5-2758n^2p^5+708np$$
$$+1854p^5-292n^4p^4+1524n^3p^4+447n^2p^4-5066np^4+439p^4+80n^5p^3-160n^4p^3-194$$
$$+5916n^2p^3-1638np^3-646p^3-96n^5p^2+896n^4p^2-1760n^3p^2-1512n^2p^2+4028np^2$$
$$-1020p^2-320n^4p+1792n^3p-2768n^2p+600np+80p-224n^3+896n^2-848n+320\big)u_1$$
$$+2\big(3p^2-2np+p-2\big)\big(18p^6-35np^5-5p^5+31n^2p^4-17np^4-10p^4-16n^3p^3+37n^2p^3$$
$$-25np^3+62p^3+4n^4p^2-18n^3p^2+26n^2p^2-53np^2+9p^2+4n^3p-16n^2p+44np-56p$$
$$+\big(p^2-np+p-1\big)^2\left(3p^2-2np+p-2\right)^2.$$

Now we see that,

$$F_{n,p}(1) > 0 \ \text{ for } \ 3 \leq p \leq \frac{2}{5}n - 1 \text{ and } F_{n,2}(2n) > 0 \ \text{ for } \ p = 2,$$

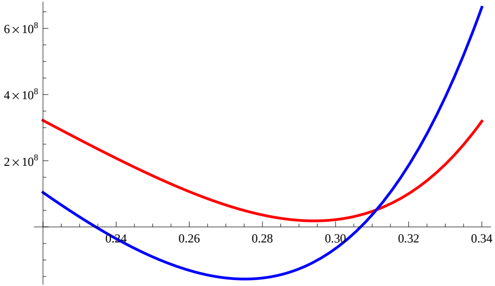
and $F_{n,p}(0) = \left(p^2 - np + p - 1\right)^2 \left(3p^2 - 2np + p - 2\right)^2 > 0.$

Also, we see that

$$F_{n,p}\left(\frac{1}{4}\right) < 0 \ \text{ for } \ 2 \leq p \leq \frac{2}{5}n - 1.$$

Thus we obtain at least two positive solutions for the equation $F_{n,p}(u_1) = 0$ for $2 \leq p \leq \dfrac{2}{5}n - 1$. Hence, we obtain two $\mathrm{Ad}(\mathrm{U}(p) \times \mathrm{SO}(n-2p))$-invariant Einstein metrics on the Stiefel manifolds $V_{2p}\mathbb{R}^n = \mathrm{SO}(n)/\,\mathrm{SO}(n-2p)$ which are not Jensen's Einstein metrics.

For $n = 31$, we have $2 \le p \le 2n/5 - 1 = 62/5 - 1 = 11.4$, but for $p = 12, 13$, we see the following:
Except Jensen's Einstein metrics,
$V_{26}\mathbb{R}^{31} = \mathrm{SO}(31)/\mathrm{SO}(5)$, has no $\mathrm{Ad}(\mathrm{U}(13) \times \mathrm{SO}(5)$-invariant Einstein metrics,
$V_{24}\mathbb{R}^{31} = \mathrm{SO}(31)/\mathrm{SO}(7)$, has two more $\mathrm{Ad}(\mathrm{U}(12) \times \mathrm{SO}(7)$-invariant Einstein metrics.



$FU1(u1, 31, 13) = 491774976{u_1}^8 + 1682093952{u_1}^7 + 4011833808{u_1}^6 + 2082493764{u_1}^5 + 1342556360{u_1}^4 - 2795832361{u_1}^3 + 1093464243{u_1}^2 - 193555008u_1 + 15968016$,
$FU1(u1, 31, 12) =$
$24356284225{u_1}^8 + 71363530420{u_1}^7 + 235478881736{u_1}^6 + 125628595904{u_1}^5 + 221500487082{u_1}^4 - 235075487612{u_1}^3 + 75786327156{u_1}^2 - 12840182320u_1 + 1073676289.$

Complex Stiefel manifolds $V_k\mathbb{C}^n = \mathrm{SU}(n)/\,\mathrm{SU}(n-k)$

Einstein metrics have not been studied before. We just know the following:

- Irreducible symmetric space $V_1\mathbb{C}^n = \mathrm{SU}(n)/\,\mathrm{SU}(n-1) = \mathbb{S}^{2n-1}$
- Mori 1994 ($n \geq 6$) and A.A., Y. Sakane, M. Statha 2017 ($n \geq 5$)
  Left-invariant Einstein metrics on $V_n\mathbb{C}^n = \mathrm{SU}(n)$, which are not naturally reductive.

Theorem (A.A.–Sakane–Statha 2017)

For $m \geq 6$ and $n \geq m/2$, or $m = 3, 4, 5$ and $n \geq 3$, the complex Stiefel manifold $V_m\mathbb{C}^{m+n} = \mathrm{SU}(m+n)/\,\mathrm{SU}(n)$ admits four invariant Einstein metrics which are $\mathrm{Ad}(\mathrm{S}(\mathrm{SO}(m) \times \mathrm{U}(1) \times \mathrm{U}(n)))$-invariant. Two are of Jensen's type and the other two metrics are new. The Stiefel manifolds $V_2\mathbb{C}^{n+2} = \mathrm{SU}(n+2)/\,\mathrm{SU}(n)$ admit only two invariant Einstein merics, which are of Jensen's type.

Here we used the fibration

$$\mathrm{SO}(m) \times \mathrm{U}(1) \to \mathrm{SU}(m+n)/\,\mathrm{SU}(n) \to \mathrm{SU}(m+n)/(\mathrm{S}(\mathrm{SO}(m) \times \mathrm{U}(1)) \times \mathrm{U}(n)),$$

and the decomposition of the tangent space of $\mathrm{SU}(m+n)/\,\mathrm{SU}(n)$ as

$$\mathfrak{m} = \mathfrak{h}_0 \oplus \mathfrak{so}(m) \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2.$$

Note that $\dim \mathfrak{h}_0 = 1$.

Remark. For the case of the complex Stiefel manifold $\mathrm{SU}(p+n)/\,\mathrm{SU}(n)$ some of the $\mathrm{SU}(p+n)$-invariant Einstein metrics are obtained from solutions of quadratic equations. We call these Einstein metrics of *Jensen's type*, because they are of the form $g = B|_{\mathfrak{m}} + s^2 B|_{\mathfrak{h}_0} + t^2 B|_{\mathfrak{su}(p)}$, on the total space of fibrations of the form $\mathrm{SU}(p+n)/\,\mathrm{SU}(n) \to \mathrm{SU}(p+n)/\,\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(n))$, where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{s}(\mathfrak{u}(p) + \mathfrak{u}(n))$ in $\mathfrak{su}(p+n)$, $\mathfrak{h}_0$ is the center of the Lie algebra of $\mathfrak{s}(\mathfrak{u}(p) + \mathfrak{u}(n))$ and B is the negative of the Killing form of $\mathfrak{g}$.

Recently we proved the following:

Theorem (A.A.–Sakane–Statha 2018)

1) The complex Stiefel manifold $V_2\mathbb{C}^4 = \mathrm{SU}(4)/\mathrm{SU}(2)$ admits two $\mathrm{Ad}(\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(2))$-invariant Einstein metrics which are of Jensen's type.
2) The complex Stiefel manifold $V_3\mathbb{C}^5 = \mathrm{SU}(5)/\mathrm{SU}(2)$ admits four $\mathrm{Ad}(\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(2)\times\mathrm{U}(2))$-invariant Einstein metrics, two of these are of Jensen's type.
3) The complex Stiefel manifold $V_4\mathbb{C}^6 = \mathrm{SU}(6)/\mathrm{SU}(2)$ admits eight $\mathrm{Ad}(\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(2)\times\mathrm{U}(2))$-invariant Einstein metrics, two of these are of Jensen's type.
4) The complex Stiefel manifolds $V_{2m}\mathbb{C}^{2m+n}$ $(m\geq 2)$ admit at least two $\mathrm{Ad}(\mathrm{S}(\mathrm{U}(m)\times\mathrm{U}(m)\times\mathrm{U}(n)))$-invariant Einstein metrics which are not of Jensen's type, for certain infinite values of m and n.

$V_{\ell+m}\mathbb{C}^{\ell+m+n} = \mathrm{SU}(\ell+m+n)/\,\mathrm{SU}(n)$

Our approach is the following:
We consider the generalized flag manifold
$G/H = \mathrm{SU}(\ell+m+n)/\,\mathrm{S}(\mathrm{U}(\ell)\times\mathrm{U}(m)\times\mathrm{U}(n))$ whose tangent space decomposes into a direct sum of irreducicle and inequivalent submodules $\mathfrak{m} = \mathfrak{m}_{12}\oplus\mathfrak{m}_{13}\oplus\mathfrak{m}_{23}$.
We decompose the Lie algebra of H into its center $\mathfrak{h}_0$ (**2-dimensional**) and simple ideals $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3$.
Then the tangent space of the Stiefel manifold $G/K = \mathrm{SU}(\ell+m+n)/\,\mathrm{SU}(n)$ decomposes as $\mathfrak{p} = \mathfrak{h}_0\oplus\mathfrak{h}_1\oplus\mathfrak{h}_2\oplus\mathfrak{m}_{12}\oplus\mathfrak{m}_{13}\oplus\mathfrak{m}_{23}$. Then we parametrize all scalar products in the center $\mathfrak{h}_0$ by further decomposing $\mathfrak{h}_0 = \mathfrak{h}_4\oplus\mathfrak{h}_5$ into one-dimensional ideals, and we then consider appropriate $\mathrm{Ad}(\mathrm{S}(\mathrm{U}(\ell)\times\mathrm{U}(m)\times\mathrm{U}(n))$-invariant scalar products on $\mathfrak{p}$ that depend on positive parameters a, b, c, d, u_1, u_2, v_4, v_5 and $x_{(6)}, x_{(7)}, x_{(8)}$. These scalar products determine G-invariant metrics on G/H.
The Ricci tensor r of such metrics has components r_0, r_4, r_5 for the center $\mathfrak{h}_0$ (non diagonal part) and r_1, r_2, r_6, r_7, r_8 diagonal part).

$V_{2m}\mathbb{C}^{2m+n} = \mathrm{SU}(2m+n)/\,\mathrm{SU}(n)$

Theorem (A.A. - Y. Sakane - M. Statha)

The complex Stiefel manifolds $V_{2m}\mathbb{C}^{2m+n}$ admit at least two $\mathrm{Ad}(\mathrm{S}(\mathrm{U}(m) \times \mathrm{U}(m) \times \mathrm{U}(n)))$-invariant Einstein metrics, which are not of Jensen's type, for the following values of m and n:

$m \geq 8$	$n \geq m/2$
$m = 6,\ 7$	$n \geq 4$
$m = 4,\ 5$	$n \geq 3$
$m = 2,\ 3$	$n \geq 2$

.

HOMOGENEOUS GEODESICS

Let $(M = G/H, g)$ be a homogeneous Riemannian manifold. A geodesic $\gamma(t)$ through $o = eK$ is called <u>homogeneous</u> if it is an orbit of a 1-parameter subgroup G, i.e. $\gamma(t) = \exp tX \cdot o$, $0 \neq X \in \mathfrak{g}$ (X = geodesic vector).

• $M = G/K$ is called <u>g.o. space</u> (or space with homogeneous geodesics) if any geodesic γ of M is homogeneous.
Notice: Property depends on the representation of $M = G/K$.

• A Riemannian manifold (M, g) is called <u>g.o. manifold</u> (or a manifold with homogeneous geodesics) if any geodesic γ of M is an orbit of a 1-parameter subgroup of the full isometry group of (M, g).
Terminology was indroduced by O. Kowalski – L. Vanhecke (1991).

Homogeneous geodesics appear in physics as well:
- The equation of montion of many systems of classical mechanics reduces to the geodesic equation in an appropriate Riemannian manifold M. Homogeneous geodesics in M correspond to relative equilibriums of the corresponding system
- In Lorentzian geometry, homogeneous spaces such that all their *null* geodesics are homogeneous, are candidates for constructing solutions to the 11-dimensional supergravity, which preserve more that 24 of the available 32 supersymmetries.

Examples - Some history

• Riemannian symmetric spaces

• Normal homogeneous spaces $(M = G/K, g)$ (Riemannian metric g is induced by a bi-invariant metric on G.

• Naturally reductive Riemannian manifolds (include previous examples) (Note: Their classification was given for $\dim \leq 5$ by Kowalski – Vanhecke (1985), for $\dim \leq 6$ by I. Agricola – A. Ferreira (2016) and for $\dim \leq 8$ by R. Storm (2017))

• There are g.o. spaces which are in no way naturally reductive (A. Kaplan 1983).

• Weakly symmetric spaces (Selberg 1956).

• Classification of g.o. spaces in $\dim \leq 6$ was given by Kowalski – Vanhecke (1991).

• Genelarized normal homogeneous Riemannian manifolds (δ-homogeneous manifolds) (Yu. Nikonorov – V. Berestovskii).

• g.o. Riemannian nilmanifolds (C. Gordon 1996).

- Classification of g.o. spaces fibered over irreducible symmetric spaces (H. Tamaru 1998).
- Examples of g.o. manifolds in dimension 7 (Z. Dušek – O. Kowalski – S. Nikčević 2007).
- Classification of simply connected generalized flag manifolds admiting non normal g.o. metrics (D. Alekseevsky – A.A. 2007).
- Classification of compact homogeneous g.o. manifolds with positive Euler characteristic (D. Alekseevsky – Yu. Nikonorov 2007).
- Homogeneous geodesics in Heisenberg groups and other pseudo-Riemannian manifolds (Z. Dušek – O. Kowalski 2002-08).
- Classification of compact, simply connected g.o. spaces with two isotropy summands (Z. Chen – Yu. Nikonorov 2019).
- For a given g.o. space some totally geodesic submanifolds were described as well as the nilradical and the radical of the isometry group (Yu. Nikonorov 2017).

• Geometric and algebraic characterization of g.o. manifolds that are diffeomorphic to $\mathbb{R}^n$ (Gordon – Nikonorov 2018).

• Examples of left-invariant Einstein metrics on compact simple Lie groups which are not g.o. (Chen-Chen-Deng 2018, Nikonorov 2018).
Note that there are examples of homogeneous Einstein metrics that are neither naturally reductive, nor g.o. (e.g. $\mathrm{SU}(3)/T_{\max}$, or Aloff-Wallach spaces $\mathrm{SU}(3)/\mathbb{S}^2_{k,l}$).

• A Ledger-Obata space $(F \times F \times \cdots \times F)/\mathrm{diag}(F)$ (F a connected, compact, simple Lie group) is a g.o. space if and only if it is naturally reductive (Nikolayevsky – Nikonorov 2019)

• The notion of homogeneous geodesics has been extended to geodesics which are orbits of a product of two or more exponential factors (A. Arvanitoyeorgos, N. Souris, G. Calvaruso 2015, 2016, 2018)).

For G be a compact, connected and semisimple Lie group. Let $(M = G/K, g)$ be a homogeneous Riemannian manifold, where g is a G-invariant metric. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be a reductive decomposition of the Lie algebra of G with respect to B, the Killing form of $\mathfrak{g}$.

Recall the 1-1 correspondence between:

$\diamond$ G-invariant metrics g on M, $\diamond$ $\mathrm{Ad}(K)$-invariant scalar products $\langle\ ,\ \rangle$ on $\mathfrak{m}$,

$\diamond$ $\mathrm{Ad}(K)$-equivariant positive definite symmetric operators $\Lambda : \mathfrak{m} \to \mathfrak{m}$.

Lemma (Geodesic Lemma, Kowalski–Vanhecke)

A non zero vector $X \in \mathfrak{g}$ is a geodesic vector if and only if $\langle [X, Y]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = 0$, for all $Y \in \mathfrak{m}$.

Proposition (Alekseevsky – A.A)

Let $(M = G/K, g)$ be a compact homogeneous Riemannian manifold, Λ associated operator. Let $a \in \mathfrak{k}, x \in \mathfrak{m}$. Then the following are equivalent:

1) The orbit $\gamma(t) = \exp t(a + x) \cdot o$ is a geodesic.

2) $[a + x, \Lambda(x)] \in \mathfrak{k}$.

3) $\langle [a + x, y], x \rangle = 0$ for all $y \in \mathfrak{m}$.

Corollary (Alekseevsky – A.A, Souris)

$(M = G/K, g)$ is a g.o. space if and only if for all $x \in \mathfrak{m}$ there exists $a(x) \in \mathfrak{k}$ such that $[a(x) + x, \Lambda(x)] = 0$.

The general problem is the following:

Problem Let G/K be a compact homogeneous space. Find all G-invariant Riemannian metrics g so that $(G/K, g)$ is a g.o. space.

By using the correspondence of G-invariant metrics with endomorphisms $\Lambda : \mathfrak{m} \to \mathfrak{m}$ the problem can be restated as follows:

Problem Let G/K be a compact homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ with respect to some Ad-invariant scalar product B on $\mathfrak{g}$. Find all endomorphisms $\Lambda : \mathfrak{m} \to \mathfrak{m}$ such that for all $x \in \mathfrak{m}$ there exists $a(x) \in \mathfrak{k}$ such that

$$[a(x) + x, \Lambda(x)] = 0.$$

A) Generalized Wallach spaces

A genelarized Wallach space is a compact homogeneous space G/K with G compact and connected, such that there exists an $\mathrm{Ad}(K)$-invariant decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ with $[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{k}$, $\mathfrak{m}_i$ irreducible.
(Older terminology: 3-locally symmetric space).
Their classification was achieved independently by Yu. Nikonorov (2016) and Z. Chen – Y. Kang – K. Wang (2016) as follows:

Theorem (N, CKW)

Let G/K be a connected and simply connected compact homogeneous space. Then G/K is a generalized Wallach space if and only if it is one of the following types:
1) G/K is a direct product of three irreducible symmetric spaces of compact type.
2) The group is simple and the pair $(\mathfrak{g}, \mathfrak{k})$ is one of the pairs in the next table.
3) $G = F \times F \times F \times F$ and $K = \mathrm{diag}(F) \subset G$ for some connected, compact, simple Lie group F, with the following description on the Lie algebra level:

$$(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{f} \oplus \mathfrak{f} \oplus \mathfrak{f} \oplus \mathfrak{f}, \mathrm{diag}(\mathfrak{f})) = \{(X, X, X, X) \mid X \in f\},$$

where $\mathfrak{f}$ is the Lie algebra of F, and (up to permutation)
$\mathfrak{m}_1 = \{(X, X, -X, -X) \mid X \in f\}$, $\mathfrak{m}_2 = \{(X, -X, X, -X) \mid X \in f\}$,
$\mathfrak{m}_3 = \{(X, -X, -X, X) \mid X \in f\}$.

$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{g}$	$\mathfrak{h}$
$\mathfrak{so}(\ell+m+n)$	$\mathfrak{so}(\ell)\oplus\mathfrak{so}(m)\oplus\mathfrak{so}(n)$	$\mathfrak{e}_7$	$\mathfrak{so}(8)\oplus 3\mathfrak{sp}(1)$
$\mathfrak{su}(\ell+m+n)$	$\mathfrak{su}(\ell)\oplus su(m)\oplus\mathfrak{su}(n)$	$\mathfrak{e}_7$	$\mathfrak{su}(6)\oplus\mathfrak{sp}(1)\oplus\mathbb{R}$
$\mathfrak{sp}(\ell+m+n)$	$\mathfrak{sp}(\ell)\oplus\mathfrak{sp}(m)\oplus\mathfrak{sp}(n)$	$\mathfrak{e}_7$	$\mathfrak{so}(8)$
$\mathfrak{su}(2\ell), \ell\geq 2$	$\mathfrak{u}(\ell)$	$\mathfrak{e}_8$	$\mathfrak{so}(12)\oplus 2\mathfrak{sp}(1)$
$\mathfrak{so}(2\ell), \ell\geq 4$	$\mathfrak{u}(\ell)\oplus\mathfrak{u}(\ell-1)$	$\mathfrak{e}_8$	$\mathfrak{so}(8)\oplus\mathfrak{so}(8)$
$\mathfrak{e}_6$	$\mathfrak{su}(4)\oplus 2\mathfrak{sp}(1)\oplus\mathbb{R}$	$\mathfrak{f}_4$	$\mathfrak{so}(5)\oplus 2\mathfrak{sp}(1)$
$\mathfrak{e}_6$	$\mathfrak{so}(8)\oplus\mathbb{R}^2$	$\mathfrak{f}_4$	$\mathfrak{so}(8)$
$\mathfrak{e}_6$	$\mathfrak{sp}(3)\oplus\mathfrak{sp}(1)$		

Every generalized Wallach space admits a 3-parameter familly of invariant Riemannian metrics determined by the following $\mathrm{Ad}(K)$-invariant scalar products

$$\langle\ ,\ \rangle = \lambda_1 B(\ ,\ )|_{\mathfrak{m}_1} + \lambda_2 B(\ ,\ )|_{\mathfrak{m}_2} + \lambda_3 B(\ ,\ )|_{\mathfrak{m}_3}, \lambda_i > 0.$$

Which of GWS are g.o. spaces?

Theorem (A.A. – Y. Wang)

Let $(G/K, g)$ be a generalized Wallach space. Then
1) If $(G/K, g)$ is a space of type 1) then this is a g.o. space for any $\mathrm{Ad}(K)$-invariant Riemannian metric.
2) If $(G/K, g)$ is a space of type 2) or 3) then this is a g.o. space if and only if g is the standard metric.

However, to find all homogeneous geodesics in G/K is difficult. It suffices to find all real solutions of a system of $d_1 + d_2 + d_3$ ($d_i = \dim \mathfrak{m}_i$) quadratic equations for the variables x_i, a_i, b_i, c_i, where

$$X = \sum_1^{\dim \mathfrak{k}} x_i e_i^0 + \sum_1^{d_1} a_j e_j^1 + \sum_1^{d_2} b_k e_k^2 + \sum_1^{d_3} c_s e_s^3 \in \mathfrak{g} \setminus \{0\}$$

is a geodesic vector.

Some simple cases: $\mathrm{SU}(2)$, $\mathrm{SO}(4)/\mathrm{SO}(2)$.

Theorem (A.A. – Y. Wang, R.A. Marinosci 2002)

For the generalized Wallach space $\mathrm{SU}(2)/\{e\}$ the only geodesic vectors for a given metric $(\lambda_1, \lambda_2, \lambda_3)$ are the following:
1) If $\lambda_i = \lambda_j \neq \lambda_k$ $(i, j, k \in \{1, 2, 3\})$, then any vector $X \in \mathfrak{m}_k \backslash \{0\}$ or $X \in (\mathfrak{m}_i \oplus \mathfrak{m}_j) \backslash \{0\}$.
2) If $\lambda_1, \lambda_2, \lambda_3$ are distinct, then any vector $X \in \mathfrak{m}_1 \cup \mathfrak{m}_2 \cup \mathfrak{m}_3$.

For the real Stiefel manifold $\mathrm{SO}(n)/\mathrm{SO}(n-2)$ $n \geq 4$ the system is not easy to solve.
We found explicitely all homogeneous geodesics in $\mathrm{SO}(4)/\mathrm{SO}(2)$ for all values of $\lambda_1, \lambda_2, \lambda_3$.
The result was generalized by N. Souris (Ph.D. Thesis 2017):

Theorem (N.Souris, in Ph.D. Thesis 2018)

The real Stiefel manifold $(\mathrm{SO}(n)/\mathrm{SO}(n-k), g)$ is a g.o. space if and only if g is the standard metric.

B) M-spaces

Let G/K be a generalized flag manifold with $K = C(S) = S \times K_1$, where S is a torus and K_1 is the semisimple part of K.

The associated M-space is G/K_1 (H.C. Wang 1954).

Examples:

$G/K = \mathrm{SO}(2n+1)/\,\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SO}(2n-3), \quad G_2/\,\mathrm{U}(2)$

$G/K_1 = \mathrm{SO}(2n+1)/\,\mathrm{SU}(2) \times \mathrm{SO}(2n-3), \quad G_2/\,\mathrm{SU}(2).$

Let G/K be a generalized flag manifold with tangent space decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$, into $\mathrm{Ad}(K)$-invariant and irreducible K-modules.

Then the tangent space of the corresponding M-space G/K_1 is $\mathfrak{n} = \mathfrak{s} \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$, as $\mathrm{Ad}(K_1)$-modules.

Lemma

Assume that an $\mathrm{Ad}(K)$-irreducible submodule $\mathfrak{m}_i$ $(i \in \{1, \ldots, s\})$ is $\mathrm{Ad}(K_1)$-reducible. Then we have a decomposition $\mathfrak{m}_i = \mathfrak{n}_1^i \oplus \mathfrak{n}_2^i$, where $\mathfrak{n}_1^i$ and $\mathfrak{n}_2^i$ are equivalent irreducible $\mathrm{Ad}(K_1)$-invariant submodules.

Theorem A (A.A. – Y. Wang – G. Zhao)

Let G/K be a generalized flag manifold with $s \geq 3$ in the decomposition. Let G/K_1 be the corresponding M-space. If $(G/K_1, g)$ is a g.o. space, then

$$g = \langle \cdot, \cdot \rangle = \Lambda \mid_{\mathfrak{s}} + \lambda B(\cdot, \cdot) \mid_{\mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \cdots \oplus \mathfrak{m}_s}, \ (\lambda > 0),$$

where Λ is the operator associated to the metric g.

For the proof we used previous work of A.A. – D. Alekseevsky on the classification of non standard g.o. metrics on generalized flag manifolds, the above lemma, and the following fact:

Proposition (Yu. Nikonorov 2017)

The scalar product $\langle \cdot, \cdot \rangle$ generating a g.o. metric g on a Riemannian space $(G/H, g)$, is not only $\mathrm{Ad}(H)$-invariant but also $\mathrm{Ad}(N_G(H_0))$-invariant, where $N_G(H_0))$ is the normalizer of the unit component H_0 of H in G.

In our case, K_1 is connected and we can see that $K \subset N_G(K_1)$. As a consequence, a g.o. G-invariant metric g on the M-space G/K_1 (in general block diagonal) is not only $\mathrm{Ad}(K_1)$-invariant but also $\mathrm{Ad}(K)$-invariant (hence diagonal).

Corollary (A.A. – Y. W. – G. Z.)

Let G/K be a generalized flag manifold with $s \geq 3$ in the decomposition. Let G/K_1 be the corresponding M-space.
If $\dim \mathfrak{s} = 1$ and there exists some $j \in \{1, \ldots, s\}$ such that $\mathfrak{m}_j$ is reducible as an $\mathrm{Ad}(K_1)$-module, then $(G/K_1, g)$ is a g.o. space if and only if g is the standard metric.

If $s = 1$, that is if $\mathfrak{n} = \mathfrak{s} \oplus \mathfrak{m}$, we have the following:

Theorem C (A.A. – Y. Wang – G. Zhao)

Let G/K be a generalized flag manifold with $s = 1$ and $(G/K_1, g)$ be the corresponding M-space. Then
1) If $\mathfrak{m}$ is irreducible as $\mathrm{Ad}(K_1)$ module, then g.o. metrics on G/K_1 have been classified by Z. Chen and Yu. Nikonorov in *Geodesic orbit Riemannian spaces with two isotropy summands. I*, Geom. Dedicata (2019).
2) If $\mathfrak{m}$ is reducible as $\mathrm{Ad}(K_1)$ module, then $(G/K_1, g)$ is a g.o. space if and only if g is the standard metric.

If $s = 2$, that is if $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$, we have the following:

Theorem B (A.A. – Y. Wang – G. Zhao)

Let G/K be a generalized flag manifold with two isotropy summands and $(G/K_1, g)$ be the corresponding M-space. Then
1) If $\dim \mathfrak{m}_2 = 2$, then the standard metric is the only g.o. metric on M-space $(G/K_1, g)$, unless $G/K_1 = \mathrm{SO}(5)/\mathrm{SU}(2)$, or $\mathrm{Sp}(n)/\mathrm{Sp}(n-1)$, $(n \geq 2)$.
2) If $\dim \mathfrak{m}_2 \neq 2$ and the M-space $(G/K_1, g)$ is a g.o. space, then $g = \langle \cdot, \cdot \rangle = \mu B(\cdot, \cdot) \mid_{\mathfrak{s}} + \lambda B(\cdot, \cdot) \mid_{\mathfrak{m}_1 \oplus \mathfrak{m}_2}$, $(\mu, \lambda > 0)$, unless $G/K_1 = \mathrm{SO}(2n+1)/\mathrm{SU}(n), (n > 2)$.

C) G.O. metrics in homogeneous spaces with equivalent isotropy summands

For a homogeneous space G/K it is often that the isotropy representation contains equivalent subrepresentations (e.g. Stiefel manifolds).

For the corresponding decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$ of the tangent space of G/K N. P. Souris studied the effect of the g.o. condition on the simplification of the endomorphism $\Lambda : \mathfrak{m} \to \mathfrak{m}$, associated to a G-invariant metric g.

This is achived by considering the alternative decompostion $\mathfrak{m} = S_0 \oplus \cdots \oplus S_N$, $N < s$ into classes of equivalent submodules (isotypical summands).

Under certain assumtions on S_i a g.o. metric on G/K is scalar on each S_k.
As a consequence:

Theorem (N.P. Souris 2018)

The complex Stiefel manifolds $\mathrm{U}(n)/\,\mathrm{U}(n-k)$ admit exactly one family of $\mathrm{U}(n)$-invariant g.o. metrics.

Here it is $\mathfrak{m} = S_0 \oplus S_1$.

D) TWO-STEP HOMOGENEOUS GEODESICS

Recently, we initiated the study of geodesics of the form

$$\gamma(t) = \exp tX \exp tY \cdot o, \quad X, Y \in \mathfrak{g}$$

in a homogeneous space G/K, which we called two-step homogeneous geodesics.

Such geodesics were first considered by H.C. Wang in *Discrete nilpotent subgroups of Lie groups*, J. Differential Geometry 3 (1969) 481–492, as geodesics in a semisimple Lie group G equipped with a metric induced by a Cartan involution of $\mathfrak{g}$.

Also, geodesics in a simple Lie group G equipped with a left-invariant metric, which is G_1-naturally reductive with respect to some $G_1 \subset G$ (J.E. D'Atri - W. Ziller: *Naturally reductive metrics and Einstein metrics on compact Lie groups*, Memoirs Amer. Math. Soc. 19 (215) (1979)).

Finally, R. Dohira in *Geodesics in reductive homogeneous spaces*, Tsukuba J. Math. 19(1) (1995) 233–243 proved that

if the tangent space $\mathfrak{m} = T_o(G/K)$ of G/K splits as $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$, such that $\mathfrak{m}_1, \mathfrak{m}_2$ satisfy the algebraic conditions $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{k} \oplus \mathfrak{m}_2$, $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1$, $[\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{k}$ and G/K is equipped with G-invariant metrics of the form $g = B|_{\mathfrak{m}_1} + cB|_{\mathfrak{m}_2}$, then all geodesics are of the form

$$\gamma(t) = \exp t(X_1 + cX_2) \exp(1 - c)tX_2 \cdot o,$$

with $\dot{\gamma}(0) = X_1 + X_2$, $\gamma(0) = o$.

If every geodesic of G/K passing though $o = eK$ is two step-homogeneous, then G/K is called a two-step homogeneous space.

We can also define n-step homogeneous geodesic
$\gamma(t) = (\exp tX_1 \exp tX_2 \cdots \exp tX_n) \cdot o$, $X_i \in \mathfrak{g}$.

We considered generalized Wallach spaces G/K equipped with G-invariant metrics $\langle\ ,\ \rangle = \lambda_1 B|_{\mathfrak{m}_1} + \lambda_2 B|_{\mathfrak{m}_2} + \lambda_3 B|_{\mathfrak{m}_3}$ and searched for geodesics of the form

$$\gamma(t) = \exp tX \exp tY \exp tZ \cdot o, \quad X \in \mathfrak{m}_1, Z \in \mathfrak{m}_2, Z \in \mathfrak{m}_3,$$

satisfying $\gamma(0) = o$, $\dot\gamma(0) = X_1 + X_2 + X_3 \in \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3$ and $X = a_1X_1 + a_2X_2 + a_3X_3$, $Y = b_1X_1 + b_2X_2 + b_3X_3$, $Z = \cdots$.
We proved that if $[X_i, X_j] \neq 0$ for $i \neq j$ then one of the following holds:

- $\langle\ ,\ \rangle = (1, 1, c)$ and $\gamma(t) = \exp t(X_1 + X_2 + cX_3) \exp t(1 - c)X_3 \cdot o$,
- $\langle\ ,\ \rangle = (1, c, 1)$ and $\gamma(t) = \exp t(X_1 + cX_2 + X_3) \exp t(1 - c)X_2 \cdot o$, or
- $\langle\ ,\ \rangle = (c, 1, 1)$ and $\gamma(t) = \exp t(cX_1 + X_2 + X_3) \exp t(1 - c)X_1 \cdot o$.

Next, we prove that the above curves are indeed geodesics:

Theorem (A.A. – N.P. Souris)

Let $M = G/K$ be a generalized Wallach space with a G invariant metric of the form $(1, 1, c)$, $(1, c, 1)$, or $(c, 1, 1)$ $(c > 0)$. Then the unique geodesic $\gamma(t)$ through $o = eK$ with $\dot\gamma(0) = X_1 + X_2 + X_3$ $(X_i \in \mathfrak{m}_i)$ is given by

- $\gamma(t) = \exp t(X_1 + X_2 + cX_3) \exp t(1 - c)X_3 \cdot o$,
- $\gamma(t) = \exp t(X_1 + cX_2 + X_3) \exp t(1 - c)X_2 \cdot o$ or
- $\gamma(t) = \exp t(cX_1 + X_2 + X_3) \exp t(1 - c)X_1 \cdot o$ respectively.

Proposition (A.A. – N.P. Souris)

Let G/K be a generalized Wallach space. If any of the relations $[\mathfrak{m}_i, \mathfrak{m}_j] = 0$ $(i \neq j)$ holds, then G/K is a g.o. space with respect to any G-invariant metric $(\lambda_1, \lambda_2, \lambda_3)$. That is, every geodesic is homogeneous.

Why do the three exponetial factors reduce to two?

For every generalized Wallach space there exit fibrations

$$G_i/K \to G/K \to G/G_i,$$

where G_i/K and G/G_i are locally symmetric.

The key ingredient is an alternatice characterization of geodesics in a Riemannian homogeneous space $(G/K, g)$

Let $\pi : G \to G/K$ be the natural projection, and let $g \in G, X \in \mathfrak{g}$, $X^L_g = (dL_g)_e X$ left-invariant v.f.. For $\alpha : I \subset \mathbb{R} \to G$ smooth curve, $\gamma = \pi \circ \alpha : I \to G/K$ is a smooth curve in G/K.
We can extend $\dot{\gamma}$ to a vector field locally in G/K.

Definition

1) For $W \in \mathfrak{m}$ we define vector field $\hat{W}$ in a neighborhood of G/K by

$$\hat{W}_{\pi(\alpha(t)g)} d\pi_{\alpha(t)g}(W^L_{\alpha(t)g}).$$

2) Define the function $G_W : \mathbb{R} \to \mathbb{R}$ by $G_W(t) = g(\hat{W}_{\gamma(t)}, \nabla_{\dot{\gamma}(t)}\dot{\gamma}(t))_{\gamma(t)}$.

By Koszul's formula we have the following:

Proposition

The curve $\gamma = \pi \circ \alpha : I \to G/K$ is a geodesic in $(G/K, g)$ if and only if $G_W(t) = 0$ for all $t \in \mathbb{R}$ and for all $W \in \mathfrak{m}$.

Let $Z, Y \in \mathfrak{m}$. We define the function $T : \mathbb{R} \to \mathrm{Aut}(\mathfrak{g})$ by

$$T \equiv T(t) = \mathrm{Ad}(\exp(-tZ)\exp(-tY)).$$

Then we have the following:

Proposition (A.A. – N.P. Souris)

Let $\gamma(t) = \exp tX \exp tY \exp tZ \cdot o$ in G/K, where $X, Y, Z \in \mathfrak{g}$. Then $\gamma(t)$ is a geodesic in G/K through o if and only if

$$G_W(t) = g((TX)_{\mathfrak{m}} + (TY)_{\mathfrak{m}} + Z_{\mathfrak{m}}, [W, TX + TY + Z]_{\mathfrak{m}})_o$$
$$+ g(W, [TX, TY + Z]_{\mathfrak{m}} + [TY, Z]_{\mathfrak{m}})_o,$$

is identically zero for all $t \in \mathbb{R}$ and for all $W \in \mathfrak{m}$.

Note that for $X = Y = 0$ we obtain the geodesic lemma (of Kowalski – Vanhecke).

The following theorem provides a general method to construct two-step homogeneous geodesics:

Theorem (A.A. – N.P. Souris)

Let $M = G/K$ be a homogeneous space admitting a naturally reductive Riemannian metric. Let B be the corresponding inner product on $\mathfrak{m} = T_o(G/K)$. We assume that $\mathfrak{m}$ admits an $\mathrm{Ad}(K)$-invariant orthogonal decomposition

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \cdots \oplus \mathfrak{m}_s, \tag{15}$$

with respect to B. We equip G/K with a G-invariant Riemannian metric g corresponding to the $\mathrm{Ad}(K)$-invariant positive definite inner product $\langle \cdot, \cdot \rangle = \lambda_1 \, B|_{\mathfrak{m}_1} + \cdots + \lambda_s \, B|_{\mathfrak{m}_s}$, $\lambda_1, \ldots, \lambda_s > 0$. If $(\mathfrak{m}_a, \mathfrak{m}_b)$ is a pair of submodules in the decomposition (15) such that

$$[\mathfrak{m}_a, \mathfrak{m}_b] \subset \mathfrak{m}_a, \tag{16}$$

then any geodesic γ of $(G/K, g)$ with $\gamma(0) = o$ and $\dot{\gamma}(0) \in \mathfrak{m}_a \oplus \mathfrak{m}_b$, is a two-step homogeneous geodesic.

Theorem (cont.)

In particular, if $\dot{\gamma}(0) = X_a + X_b \in \mathfrak{m}_a \oplus \mathfrak{m}_b$, then for every $t \in \mathbb{R}$ this geodesic is given by $\gamma(t) = \exp t(X_a + \lambda X_b) \exp t(1-\lambda)X_b \cdot o$, where $\lambda = \lambda_b/\lambda_a$. Moreover, if either $\lambda_a = \lambda_b$ or $[\mathfrak{m}_a, \mathfrak{m}_b] = \{0\}$ holds, then γ is a homogeneous geodesic, that is $\gamma(t) = \exp t(X_a + X_b) \cdot o$, for any $t \in \mathbb{R}$.

The case $s = 2$ of the previous theorem can be used to construct various g.o. metrics on

- Lie groups equipped with an one-parameter family of left-invariant metrics.
- Generalized flag manifolds equipped with certain one-parameter families of diagonal metrics.
- Generalized Wallach spaces equipped with three different types of diagonal metrics.
- k-symmetric spaces where k is even, endowed with an one parameter family of diagonal metrics.

III) HOMOGENEOUS GEODESICS IN PSEUDO-RIEMANNIAN MANIFOLDS

Joint work with G. Calvaruso and N.P Souris

Any homogeneous Riemannian manifold is reductive, but this is not the case for pseudo-Riemannian manifolds in general. There exist pseudo-Riemannian

manifolds not admitting reductive decomposition, so one has to consider these two cases separately.

Due to the existence of null vectors in pseudo-Riemannian manifolds, the definition of homogeneous geodesic has to be modified, by requiring that $\nabla_{\dot\gamma}\dot\gamma = k(\gamma)\dot\gamma$.

(cf. also interesting historical comments in J. Mikeš, E. Stepanova and A. Vanžurová: *Differential Geometry of Special Mappings*, Olomouc (2015)).

It turns out that $k(\gamma)$ is a constant function (cf. Z. Dušek – O. Kowalski 2007).

The analogue of the geodesic lemma was known to physicists (J. Figueroa et al 2005, S. Philip 2006), but a formal proof was given in 2007 by Z. Dušek and O. Kowalski.

Lemma (Z. Dušek and O. Kowalski)

Let $M = G/H$ be a reductive homogeneous pseudo-Riemannian space with reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, and $X \in \mathfrak{g}$. Then the curve $\gamma(t) = \exp(tX) \cdot o$ is a geodesic curve with respect to some parameter s if and only if

$$\langle [X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = k \langle X_{\mathfrak{m}}, Z_{\mathfrak{m}} \rangle, \quad \text{for all} \quad Z \in \mathfrak{m},$$

where k is some real constant. Moreover, if $k = 0$, then t is an affine parameter for this geodesic. If $k \neq 0$, then $s = e^{kt}$ is an affine parameter for the geodesic. This occurs only if the curve $\gamma(t)$ is a null curve in a (properly) pseudo-Riemannian space.

Recently we initiated a systematic study of two-step homogeneous geodesics and two-step g.o. pseudo-Riemannian spaces.

Definition (A.A. – G. Calvaruso – N.P. Souris)

1) Let $(G/H, \langle\ ,\rangle)$ be a homogeneous pseudo-Riemannian space and consider a point $o \in G/H$. A geodesic $\gamma : I \to G/H$ through o, with affine parameter s, is called *two-step homogeneous* if there exists a parametrization $t = \phi(s)$ of γ and vectors X, Y in the Lie algebra $\mathfrak{g}$ of G, such that

$$\gamma(t) = \exp(tX)\exp(tY)\cdot o \quad \text{for all } t \in \phi(I).$$

2) A *two-step geodesic orbit space* (or two-step go space) is a pseudo-Riemannian homogeneous space $(G/H, \langle\ ,\rangle)$ such that every geodesic through a point $o \in G/H$ is two-step homogeneous.

For $W \in \mathfrak{g}$, we introduce the function $\mathcal{G}^W : J \to \mathbb{R}$ defined by

$$\mathcal{G}^W(t) = \langle \nabla_{\dot\gamma}\dot\gamma - k\dot\gamma, d\pi_* W^L\rangle_{\gamma(t)},$$

Definition (A.A. – G. Calvaruso – N.P. Souris)

Let $(G/H, \langle\ ,\rangle)$ be a homogeneous pseudo-Riemannian space and let $\gamma : J \to \mathbb{R}$ be a curve in G/H. Then γ is a geodesic up to reparametrization if and only if $\mathcal{G}^W(t) = 0$ for any $W \in \mathfrak{g}$ and for any $t \in J$.

The following theorem provides a general characterization of two-step homogeneous geodesics in pseudo-Riemannian spaces, possibly non reductive.

Theorem (A.A. – G. Calvaruso – N.P. Souris)

Let $(G/H, \langle\ ,\rangle)$ be a homogeneous pseudo-Riemannian space, $o = eK$. Let $\gamma : J \to G/H$ be the curve $\gamma(t) = \exp(tX)\exp(tY) \cdot o, \quad X, Y \in \mathfrak{g}$.
Let $T : J \to \mathrm{Aut}(\mathfrak{g})$ be the map

$$T(t) = \mathrm{Ad}(\exp(-tY)) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,\mathrm{ad}^n(-Y).$$

Then γ is a geodesic up to reparametrization (i.e. a two-step homogeneous geodesic) if and only if there exists a function $k : J \to \mathbb{R}$ such that

$$\begin{aligned}\mathcal{G}^W(t) \quad &= \quad \langle \pi_*(T(t)X + Y), \pi_*([W, T(t)X + Y])\rangle_o + \langle \pi_*(W), \pi_*([T(t)X, Y])\rangle_o \\ &\qquad -k(t)\langle \pi_*(W), \pi_*(T(t)X + Y)\rangle_o = 0,\end{aligned}$$

for any $W \in \mathfrak{g}$ and for any $t \in J$.

For the <u>reductive case</u> the previous theorem simplifies in the following way.

Generalized Geodesic Lemma (A.A. – G. Calvaruso – N.P. Souris)

Let $(G/H, \langle\ ,\rangle)$ be a pseudo-Riemannian reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ of the Lie algebra of G.
Then the curve $\gamma(t) = \exp(tX)\exp(tY) \cdot o$ is a geodesic up to reparametrization if and only if there exists a function $k : J \to \mathbb{R}$, such that

$$\mathcal{G}^W(t) = \langle (T(t)X+Y)_{\mathfrak{m}}, [W, T(t)X+Y]_{\mathfrak{m}}\rangle + \langle W, [T(t)X, Y]_{\mathfrak{m}}\rangle - k(t)\langle W, (T(t)X+Y)_{\mathfrak{m}}\rangle$$
$$= 0,$$

for any $W \in \mathfrak{m}$ and for any $t \in J$.

By setting $X = 0$ in the above equation this reduces to

$$\langle Y_{\mathfrak{m}}, [W, Y]_{\mathfrak{m}}\rangle = k(t)\langle W, Y_{\mathfrak{m}}\rangle, \quad \text{for all } W \in \mathfrak{m}, t \in J.$$

This implies that $k(t)$ is independent of t and so, $k(t) = k$ is a constant. Hence, for $X = 0$ the above lemma implies that the curve γ with $\gamma(t) = \exp(tY) \cdot o$ is a geodesic up to some parameter if and only if there exists a constant k such that

$$\langle Y_{\mathfrak{m}}, [W, Y]_{\mathfrak{m}}\rangle = k\langle W, Y_{\mathfrak{m}}\rangle \quad \text{for all } W \in \mathfrak{m}.$$

This is exactly the geodesic lemma of Dušek – Kowalski (pseudo-Rieman. case).