C-projective symmetries of submanifolds in quaternionic geometry

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Dirac operators in differential geometry and global analysis – in memory of Thomas Friedrich (1949-2018)

Joint work with Henrik Winther

Gap phenomenon for parabolic geometries

The algebra of infinitesimal symmetries for parabolic Cartan geometries:

- finite dimensional,
- the maximal dimension achieved for flat structures,
- Gap phenomenon: maximal symmetry dimension for non-flat structures drops significantly.

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Submaximally symmetric structure – structure achieving maximal symmetry dimension in the non-flat case. General theory studied by B. Kruglikov and D. The in *The gap phenomenon in parabolic geometries* , J. reine angew. math. **723**, 153-216 (2017).

Projective structure:

- a class of torsion-free connections which have the same geodesics,
- is parametrized by 1-forms and the equivalence relation is explicitly given by

$$D \sim_p D' \Leftrightarrow \exists_\gamma :\ \forall Y, Z \in \Gamma TM \quad D'_Y Z = D_Y Z + [\![Y, \gamma]\!]_p Z,$$

  where γ is a 1-form and $[\![Y, \gamma]\!]_p Z = \gamma(Y)Z + \gamma(Z)Y$.
- is a parabolic Cartan geometry with $G = PGL(n+1, \mathbb{R})$ and $P_0 = GL_n\mathbb{R}$.

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c-projective and quaternionic structures: complex and quaternionic analogues of projective structure.

(S, J) – a complex manifold equipped with a complex connection ∇.
A curve γ is called *J–planar* if it satisfies

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. C–projective structure:

- the class of complex connections which share the same J–planar curves,
- is parametrized by 1-forms and the equivalence relation is explicitly given by

$$D \sim_c D' \Leftrightarrow \exists_\gamma : \ \forall Y, Z \in \Gamma TM \quad D'_Y Z = D_Y Z + [\![Y, \gamma]\!]_c Z,$$

  where $\gamma \in T^*M$ and
  $[\![Y, \gamma]\!]_c(Z) = \frac{1}{2}(\gamma(Y)Z + \gamma(Z)Y - \gamma(JY)JZ - \gamma(JZ)JY)$.
- is a parabolic Cartan geometry with $G = PGL(n+1, \mathbb{C})$ with reduction of the structure group $GL(m, \mathbb{C}) \subset GL(2m, \mathbb{R})$.

Definition

Let $n \geq 2$. A $4n$-dimensional smooth manifold M is called quaternionic if it is equipped with a rank 3 subbundle $Q \subset \mathrm{End}(TM)$ such that Q is fibrewise generated by three anti-commuting almost complex structures I, J, K satisfying

$$I^2 = J^2 = K^2 = IJK = -1,$$

and such that there exists a torsion-free connection D, called a quaternionic connection, preserving Q (i.e $D_X Q \subset Q$).

There is a notion of q-planar curves and one can show that all quaternionic connections have the same q-planar curves.

For quaternionic manifold (M, Q, D):

- the class of quaternionic connections satisfy the following equivalence:

$$D \sim_q D' \Leftrightarrow \exists_\gamma :\ \forall Y, Z \in \Gamma TM \quad D'_Y Z = D_Y Z + [\![Y, \gamma]\!]_q Z, \qquad (1)$$

  where $\gamma \in T^*M$ and

$$[\![Y, \gamma]\!]_q(Z) = \frac{1}{2}(\gamma(Y)Z + \gamma(Z)Y - \Sigma_{i=1}^{3}(\gamma(I_i Y)I_i Z + \gamma(I_i Z)I_i Y))$$

  where I_1, I_2, I_3 is a pointwise frame of Q.
- it is a parabolic Cartan geometry with $G = PGL(n+1, \mathbb{H})$ with reduction of the structure group $GL(n, \mathbb{H}) \times_{\mathbb{Z}/2} Sp(1) \subseteq GL(4n, \mathbb{R})$.

Definition

Let $(S, J, [D]_c)$ be a c-projective $2n$-manifold. A vector field V is an infinitesimal c-projective symmetry if the Lie derivatives along V of J and of the class $[D]_c$ vanish.

The submaximal c-projective structures have been studied in
B. Kruglikov, V. Matveev, D. The, *Submaximally symmetric c-projective structures*, Int. J. Mat. **27**, No. 3, 1650022 - 34 pp, (2016).

The Weyl curvature of torsion-free c-projective structure decomposes into two pieces: type I and type II.

- type I is the $(2,0)$ part of the tensor,
- type II is the $(1,1)$ part.

In general there is also type III which corresponds to the intrinsic torsion. For the minimal submaximal c-projective structure the Weyl curvature is supported purely in one of these types.

- The generalized Feix–Kaledin construction deals only with manifolds with Weyl curvature of type **II**.

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$$2n^2 - 2n + 4,$$

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  – except for the case $2n = 4$ it is equal to the submaximal dimension for general c-projective structures.
- The submaximal type II structure is **unique**.

Definition

Let $(M, Q, [D]_q)$ be a quaternionic manifold. A vector field V is an infinitesimal quaternionic symmetry if the Lie derivative preserves the space Q. In that case, the flow of V also preserves the class of quaternionic connections.

The submaximal quaternionic structures have been studied in B. Kruglikov, H. Winther, L. Zalabova, *Submaximally Symmetric Almost Quaternionic Structures*, Transformation Groups (2017).

- The Weyl curvature decomposes into two pieces, one of which corresponds to the torsion part.
- The submaximal dimension for quaternionic $4n$ manifolds ($n > 1$) is equal to

$$4n^2 - 4n + 9,$$

- There is an explicit locally hypercomplex submaximal model. However, it is not known if the submaximal quaternionic model is unique.

Generalized Feix–Kaledin construction

A. Borówka, D. Calderbank, *Projective geometry and the quaternionic Feix-Kaledin construction*, Trans. AMS vol. 372 no.7 (2019), 4729-4760:

- Relationship between c-projective geometries with Weyl curvature of type $(1,1)$ and quaternionic geometries.
- Real analytic c-projective manifolds with Weyl curvature of type $(1,1)$ are precisely the maximal totally complex submanifolds of quaternionic manifolds with a local circle action of a special kind.
- The family of quaternionic manifolds containing a fixed c-projective submanifold S with type $(1,1)$ c-projective curvature is parametrized by holomorphic line bundles equipped with compatible real-analytic complex connections on S with type $(1,1)$ curvature.
- An explicit construction of the twistor spaces of such quaternionic manifolds from the c-projective structure and the line bundle
- Any quaternionic manifold with this properties can be obtained in this way.

From this point of view it is natural to ask what the consequences are for the algebras of c-projective and quaternionic symmetries, and in particular for the dimension of the algebra of submaximal symmetries.

Q1: Flat quaternionic and c-projective structures are related by gFK. Is the same true for (some) submaximal models?

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Q3: Do c-projective symmetries on submanifold extend as quaternionic symmetries to the manifold obtained by gFK?

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Q3: Do c-projective symmetries on submanifold extend as quaternionic symmetries to the manifold obtained by gFK?

Q4: What about other quaternionic symmetries? Is there an explanation for the formula $4n^2 - 4n + 9 = 2(2n^2 - 2n + 4) + 1$?

A. Borowka, H. Winther, *C-projective symmetries of submanifolds in quaternionic geometry*, AGAG vol. vol 55 iss 3 (2019), 395–416:

Q1: One can explicitly show (using Maple DG package) that the submaximal quaternionic model given by [KWZ] admits an S^1 action of the type required by gFK and that the induced c-projective structure is submaximally symmetric (which is unique).
Moreover we prove that any submaximally symmetric quaternionic model admits such an action.

Q2: As the quaternionic model is locally hypercomplex we know that the line bundle is $\mathcal{O}(-1)$ (some root of the canonical bundle) with the connection induced by one of the connections in the c-projective class. We show that the submaximally symmetric c-projective class admits a unique invariant (with respect to c-projective symmetries) connection and that this is the connection used for the construction.

Q3: We prove the following theorem:

Theorem

Let V be a symmetry of a real-analytic c-projective manifold S with c-projective curvature of type $(1,1)$, and suppose that V preserves a connection on a holomorphic line bundle on S (associated with the holomorphic tangent bundle) with type $(1,1)$ curvature. Then V extends from the submanifold S to a quaternionic symmetry on the quaternionic manifold obtained by a generalized Feix–Kaledin construction from these data.

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$\Rightarrow$ All c-projective symmetries of submmaximal model extend to the hypercomplex model considered earlier.
$\Rightarrow$ For trvial line bundle all c-projective symmetries extend – note that this is not the Feix case!

In general we are unable to extend all c-projective symmetries (i.e., the ones that do not preserve ∇) when ∇ is not trivial. For example in the case of Grassmannian $Gr_2(\mathbb{C}^4)$, the dimension of all quaternionic symmetries is 15 (and one of them is the S^1 action), whereas it is constructed from S equipped with the flat c-projective structure (and $(\mathcal{L}^{-\frac{1}{2}}, \nabla)$ induced by the Fubini-Study metric), which has dimension of c-projective symmetries equal to 16.

Q4: We may not get orthogonal symmetries even for hypercomplex manifolds:

Example

Consider the Calabi metric on the cotangent bundle $M = T^*\mathbb{CP}^n$. This structure is hyperKähler, and so defines a quaternionic structure on M. If this structure admitted the full amount of possible 'orthogonal' symmetries, then for $n = 2$ would have quaternionic symmetry dimension equal at least to the submaximal quaternionic dimension. This is not possible, as we prove that the Calabi metric on the cotangent bundle of $\mathbb{CP}^n$ has quaternionic symmetries dimension which is neither maximal nor submaximal.

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In fact by direct computations (using the DifferentialGeometry package in Maple) we show that for the case $n = 1$ (which is the Eguchi-Hanson metric) the quaternionic symmetry algebra is generated by the S^1 action and the symmetries extended from the c-projective symmetries on the submanifold, and so we obtain no 'orthogonal' symmetries in that case.

Let k be the dimension of the algebra of c-projective symmetries on S (for $n = 1$ we take symmetries of a Möbius structure instead) and k' be the dimension of the subalgebra of symmetries that preserve ∇.

- The lower bound for the dimension of quaternionic symmetries on M is equal to $k' + 1$, and this is realizable in the case of the Eguchi-Hanson structure.
- For the considered submaximal structures, the dimension of symmetries is far from the bound and equal to $2k' + 1 = 2k + 1$.
- For the flat quaternionic structure the dimension of symmetries is equal to $2k + 3$.
- Is $2k + 1$ the upper bound in the non-flat case? – Step toward proving uniquenes (or classification) of submaximally symmetric quaternionic structure.

Thank you!