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Reduction by stages and the Raïs-type formula for the index of Lie algebra extensions

0 Introduction

The index of a Lie algebra is defined as the dimension of the stabilizer of a generic element with respect to the coadjoint representation, or equivalently, the codimension of a generic coadjoint orbit. The index is an important characteristic of a Lie algebra, which is used in different applications. The aim of this note is to give a generalization of the so-called Raïs formula for the index of a semidirect product of a Lie algebra and an abelian ideal.

It is well known that the index of a semisimple Lie algebra coincides with its rank. For the nonsemisimple case, one of the most popular in applications related result is the above mentioned Raïs formula calculating the index of a semidirect product $\mathfrak{s} = \mathfrak{g} \times_\rho V$ of a Lie algebra $\mathfrak{g}$ and a vector space V, by means of the representation $\rho^* : \mathfrak{g} \to \mathfrak{gl}(V^*)$ dual to the representation ρ ([Rai78], see also Corollary 1.8). More precisely, the index of $\mathfrak{s}$ is equal to the sum of the codimension of a generic orbit of ρ^* and the index of the stabilizer with respect to ρ^* of a generic element in V^*.

The Raïs formula, first proved by purely algebraic methods, can be also deduced from Poisson geometric results on semidirect products. Namely, the following theorem could be found in [RSTS94, Section 5]. If $\mathfrak{s} = \mathfrak{g} \times_\rho V$ is a semidirect product and $V_\nu := \mathfrak{g}^* \times O_\nu$, where $O_\nu \subset V^*$ is the G-orbit of any element $\nu \in V^*$, then: 1) V_ν is a Poisson submanifold in $(\mathfrak{g} \times_\rho V)^*$; 2) V_ν is Poisson diffeomorphic to T^*G/G_ν (here $G \supset G_\nu$ are the Lie groups corresponding to the Lie algebras $\mathfrak{g}, \mathfrak{g}_\nu$, where $\mathfrak{g}_\nu$ is the stabilizer of ν, T^*G/G_ν is endowed with the Poisson structure being the reduction of the canonical one). Since the action of G_ν on T^*G is free, by the so called bifurcation lemma

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(cf. [OR04, Section 4.5]) the momentum map of this action is a submersion and, consequently, the generic (in V_ν) orbits have codimension in V_ν equal to $\operatorname{ind}\mathfrak{g}_\nu$, the index of $\mathfrak{g}_\nu$. On the other hand, $\operatorname{codim}_{(\mathfrak{g}\times_\rho V)^*} V_\nu = \operatorname{codim}_{V^*} O_\nu$. From this one deduces the following result.

Theorem 0.1 *The subset V_ν is a Poisson submanifold in $(\mathfrak{s}^*, \eta_\mathfrak{s})$, where $\eta_\mathfrak{s}$ is the canonical Lie-Poisson structure on $\mathfrak{s}^*$, and, moreover,*

1. $\operatorname{codim}_{\mathfrak{s}^*} V_\nu = \operatorname{codim}_{V^*} O_\nu$*;*
2. $\operatorname{corank} \eta_\mathfrak{s}|_{V_\nu} = \operatorname{ind}\mathfrak{g}_\nu$ *(here corank of a Poisson structure is defined as in Definition 1.3).*

Taking a generic ν we get the Raïs formula: $\operatorname{ind}\mathfrak{s} = \operatorname{codim}_{V^*} O_\nu + \operatorname{ind}\mathfrak{g}_\nu$.

In this paper we present a version of this result in the situation when instead of $\mathfrak{s}$ we have any Lie algebra $\mathfrak{m}$ with an ideal $\mathfrak{n}$ such that the corresponding to $\mathfrak{n}$ subgroup N of the connected simply-connected Lie group M corresponding to $\mathfrak{m}$ is closed (Theorem 1.5). As a consequence we get a formula for the index generalizing the Raïs formula (Theorem 1.7). Our considerations are very much inspired by the so-called "symplectic reduction by stages" [MMPR98, Per99, OR04]. The last is related to the case when a Lie group M possessing a normal subgroup N is acting on a symplectic manifold. The reduction by N is performed firstly and then that by M/N. In fact we follow this scheme concentrating rather on codimension of symplectic leaves on each of the "stages" than on their intrinsic geometry as was done in the cited papers.

The situation of a general Lie algebra extension is much more involved in comparison to that of semidirect products. The main difference is that weak hamiltonian actions replace hamiltonian ones on the "second stage" of the reduction mentioned, which results in appearance of an additional 2-cocycle in our formulations. Thus we need some extra preparations to formulate our results and postpone this to Section 1. Here we mention only that, given a Lie algebra $\mathfrak{m}$ with an ideal $\mathfrak{n}$, the ad-representation of $\mathfrak{m}$ on itself restricts to the representation ρ of $\mathfrak{m}$ on $\mathfrak{n}$. Theorem 1.7 relates the index of $\mathfrak{m}$ with the dual representation ρ^* in the same spirit as the classical Raïs formula does.

The paper is organized as follows. In Section 1 we introduce appropriate notions and formulate the main result. The Raïs formula is obtained as a corollary of it. Then we illustrate our result by two elementary examples of nonsplit extension and split extension with the nonabelian ideal. In Section 2 we describe the scheme of the reduction by stages and weak hamiltonian actions which appear on the "second stage". Section 3 continues the considerations of the preceding section and is devoted to calculating the coranks of the reduced Poisson bivectors. In Section 4 we use the results of previous sections to prove the main result. In Appendix we briefly recall main notions related to weak hamiltonian actions.

We conclude this introduction by mentioning that another generalization of the Raïs formula was obtained by D. Panyushev in [Pan05]. It consists in relating the index of $\mathbb{N}$-graded Lie algebras with three terms with that of the semidirect product of two of them. This result is applied to calculation of the index of the so-called seaweed subalgebras of simple Lie algebras. Different results related to these subalgebras and to the general problem of calculating the index can be found in [Ela82, DK00, Pan01, Pan03, Dvo03, TY04a, TY04b].

1 Extensions of Lie algebras. Formulation of the main results

All Lie algebras will be finite-dimensional and defined over a field $\mathbb{K}$ equal to $\mathbb{R}$ or $\mathbb{C}$. Depending on the case we shall use the categories of real- or complex-analytic manifolds and denote by $\mathcal{E}(P)$ the corresponding space of functions (real valued analytic or holomorphic) on a manifold P. If a capital Latin letter denotes a Lie group, then the corresponding small Gothic letter sands for the related Lie algebra and vice versa.

Consider an exact sequence of homomorphisms of Lie algebras

$$0 \to \mathfrak{n} \to \mathfrak{m} \to \mathfrak{g} \to 0. \tag{1.1}$$

Since $\mathfrak{n}$ is an ideal we can consider the representation $\rho : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{n}), \rho_m(n) := \mathrm{ad}_m(n), n \in \mathfrak{n}, m \in \mathfrak{m}$. Fix $\nu \in \mathfrak{n}^*$ and write $\mathfrak{m}_\nu$ for $\{x \in \mathfrak{m} \mid \langle \nu, [x,y]\rangle = 0\ \forall y \in \mathfrak{n}\}$, the stabilizer Lie algebra with respect to the dual representation ρ^* of the element ν. Put also $\mathfrak{n}_\nu := \mathfrak{m}_\nu \cap \mathfrak{n}$ and $\mathfrak{g}_\nu := \mathfrak{m}_\nu/\mathfrak{n}_\nu$. Note that $\mathfrak{n}_\nu$ is an ideal in $\mathfrak{m}_\nu$, so $\mathfrak{g}_\nu$ is a Lie algebra.

1.1. Lemma *Introduce a map $\bar{\Gamma} : \bigwedge^2 \mathfrak{m}_\nu \to \mathbb{K}, \bar{\Gamma}(x,y) := \bar{\nu}([x,y])$, where $\bar{\nu} \in \mathfrak{m}_\nu^*$ is some extension of $\nu|_{\mathfrak{n}_\nu} \in \mathfrak{n}_\nu^*$. Then*

1. *for any $x, y \in \mathfrak{m}_\nu, z \in \mathfrak{n}_\nu$ one has $\bar{\Gamma}(x+z,y) = \bar{\Gamma}(x,y)$, i.e. the map $\bar{\Gamma}$ factorizes to a map $\bar{\gamma} : \bigwedge^2 \mathfrak{g}_\nu \to \mathbb{K}$;*
2. *the map $\bar{\gamma}$ is a cocycle on $\mathfrak{g}_\nu$;*
3. *if $\tilde{\nu} \in \mathfrak{m}_\nu^*$ is another extension of $\nu|_{\mathfrak{n}_\nu} \in \mathfrak{n}_\nu^*$ and $\tilde{\gamma}$ the corresponding cocycle, the difference $\bar{\gamma} - \tilde{\gamma}$ is a coboundary.*

Thus we get a correctly defined element $\gamma_\nu := [\bar{\gamma}] \in H^2(\mathfrak{g}_\nu)$.

Proof Note that $[z,y] \in \mathfrak{n}_\nu$, hence $\bar{\Gamma}(x+z,y) = \bar{\Gamma}(x,y) + \langle \nu, [z,y]\rangle = \bar{\Gamma}(x,y) - \langle \mathrm{ad}_y^* \nu, z\rangle$. But since $y \in \mathfrak{m}_\nu$, the last term is zero and item (1) is proven.

The second item is obvious, since $\bar{\Gamma}$ is a cohomologically trivial cocycle on $\mathfrak{m}_\nu$. (Note however, that the induced cocycle $\bar{\gamma}$ is nontrivial in general.)

Now let $\bar{\nu}, \tilde{\nu}$ be two extensions of $\nu|_{\mathfrak{n}_\nu}$. Then $t := \bar{\nu} - \tilde{\nu} \in \mathfrak{n}_\nu^\perp$, where $\mathfrak{n}_\nu^\perp \subset \mathfrak{m}_\nu^*$ is the annihilator of $\mathfrak{n}_\nu \subset \mathfrak{m}_\nu$ and $\bar{\gamma}(x,y) - \tilde{\gamma}(x,y) = (\bar{\nu} - \tilde{\nu})([x,y]) = t([x,y]), x, y \in \mathfrak{g}_\nu$, where we understand t as an element of $\mathfrak{g}_\nu^* \simeq \mathfrak{n}_\nu^\perp$. Thus $\bar{\gamma} - \tilde{\gamma} = \partial t$. $\square$

Remark 1.1 The cocycle $\bar{\Gamma}$ allows another description that may be useful in calculations. Given the extension $0 \to \mathfrak{n}_\nu \to \mathfrak{m}_\nu \xrightarrow{p_\nu} \mathfrak{g}_\nu \to 0$, assume that $s_\nu : \mathfrak{g}_\nu \to \mathfrak{m}_\nu$ is a section of p_ν, i.e. $p_\nu \circ s_\nu = \mathrm{Id}_{\mathfrak{g}_\nu}$. Then we have a direct decomposition $\mathfrak{m}_\nu = \mathfrak{n}_\nu \oplus s_\nu(\mathfrak{g}_\nu)$ and two projections $\pi_1^{s_\nu} : \mathfrak{m}_\nu \to \mathfrak{m}_{\nu,1} := \mathfrak{n}_\nu, \pi_2^{s_\nu} : \mathfrak{m}_\nu \to \mathfrak{m}_{\nu,2} := s_\nu(\mathfrak{g}_\nu)$. We also have $[x,y] = [x_1 + x_2, y_1 + y_2] = [x_1,y_1]_1 + [x_1,y_2]_1 + [x_2,y_1]_1 + [x_2,y_2]_1 + [x_2,y_2]_2$ for $x, y \in \mathfrak{m}_\nu$, where the indices refer to the corresponding components of the decomposition $\mathfrak{m}_\nu = \mathfrak{m}_{\nu,1} \oplus \mathfrak{m}_{\nu,2}$. Any such section s_ν determines the extension $\bar{\nu} \in \mathfrak{m}_\nu^*$ of $\nu|_{\mathfrak{n}_\nu}$ by $\bar{\nu} := \nu \circ \pi_1^{s_\nu}$. Thus we get the formula $\bar{\Gamma}(x,y) = \nu([x_1,y_1]_1 + [x_1,y_2]_1 + [x_2,y_1]_1 + [x_2,y_2]_1)$ for the corresponding cocycle which by Lemma 1.1. 1. reads as $\bar{\Gamma}(x,y) = \nu([x_2,y_2]_1)$.

Definition 1.2 Given a Poisson manifold (P, η) in the real analytic or complex analytic category, we define $\operatorname{corank} \eta$ as the codimension of its generic symplectic leaf.

Given a Lie algebra $\mathfrak{g}$ and a 2-cocycle $\gamma \in C^2(\mathfrak{g})$, we can consider the affine (in particular analytic) Poisson structure $\eta_\gamma := \eta_\mathfrak{g} + \hat{\gamma}$ on $\mathfrak{g}^*$, where $\eta_\mathfrak{g}$ is the canonical Lie–Poisson structure on $\mathfrak{g}^*$ and $\hat{\gamma}$ stands for the constant Poisson structure on $\mathfrak{g}^*$ defined by $\langle \hat{\gamma}, \varphi \wedge \psi \rangle := \gamma(\varphi, \psi), \varphi, \psi \in (\mathfrak{g}^*)^* \simeq \mathfrak{g}$. Note that adding a coboundary $\partial a, a \in \mathfrak{g}^*$, to γ results in shifting of η_γ by a, hence does not change corank of η_γ. This justifies the following definition.

Definition 1.3 Given a Lie algebra $\mathfrak{g}$ and an element $[\gamma] \in H^2(\mathfrak{g}), \gamma \in C^2(\mathfrak{g})$, we define the *index* $\operatorname{ind}(\mathfrak{g}, [\gamma])$ of the pair $(\mathfrak{g}, [\gamma])$ as the corank of the Poisson bivector $\eta_\gamma = \eta_\mathfrak{g} + \hat{\gamma}$.

Now we are able to formulate our first main result.

Theorem 1.4 *Assume a Lie algebra extension (1.1) is given with an additional condition that the corresponding to $\mathfrak{n}$ subgroup N of the connected simply-connected Lie group M corresponding to $\mathfrak{m}$ is closed. Let $\nu \in \mathfrak{n}^*$, the representation $\rho : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{n})$, the Lie algebra $\mathfrak{g}_\nu$ and the cohomology class $\gamma_\nu \in H^2(\mathfrak{g}_\nu)$ be as defined above. Write $\mathfrak{m} \cdot \nu$ for the orbit of the element $\nu \in \mathfrak{n}^*$ under the action ρ^* and put $V_\nu := \pi^{-1}(\mathfrak{m} \cdot \nu)$, where $\pi : \mathfrak{m}^* \to \mathfrak{n}^*$ is the canonical projection onto $\mathfrak{n}^* \simeq \mathfrak{m}^*/\mathfrak{n}^\perp$, $\mathfrak{n}^\perp$ being the annihilator of $\mathfrak{n} \subset \mathfrak{m}$. Then*

1. *V_ν is a Poisson submanifold in $\mathfrak{m}^*$;*
2. $\operatorname{corank} \eta_\mathfrak{m}|_{V_\nu} = \operatorname{ind}(\mathfrak{g}_\nu, \gamma_\nu)$*, where $\eta_\mathfrak{m}$ is the canonical Lie–Poisson structure on $\mathfrak{m}^*$.*

Proof The proof of item 1. is easy. Since N is normal, π is M-equivariant, where the action of M on $\mathfrak{n}^*$ is equal to the integration of the infinitesimal action ρ^* defined above. Thus $V_\nu = \pi^{-1}(M \cdot \nu)$ is M-invariant, i.e. is a union of M-orbits in $\mathfrak{m}^*$. In other words V_ν is a union of symplectic leaves of $(\mathfrak{m}^*, \eta_\mathfrak{m})$, hence is Poisson.

The second item of the theorem will be proven in Section 4. The following definition is needed for the formulation of our second main result, the Raïs-type formula for the index of a Lie algebra being an extension.

Definition 1.5 ([Pan01]) Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ its representation. The *index* $\operatorname{ind} \rho$ of the representation ρ is defined as the codimension of a generic orbit of the dual representation ρ^*.

Theorem 1.6 *We retain the notations and assumptions of Theorem 1.5. Then*

$$\operatorname{ind} \mathfrak{m} = \operatorname{ind} \rho + \operatorname{ind}(\mathfrak{g}_\nu, \gamma_\nu),$$

where $\nu \in \mathfrak{n}^*$ is a generic element.

Proof Take $\nu \in \mathfrak{n}^*$ to be generic. Then $\operatorname{ind}\rho$ is equal to $\operatorname{codim} \mathfrak{m} \cdot \nu$, and since the map $\pi : \mathfrak{m}^* \to \mathfrak{n}^*$ is a surjective submersion, to $\operatorname{codim} V_\nu$. Thus we have $\operatorname{ind} \mathfrak{m} = \operatorname{codim} V_\nu + \operatorname{corank} \eta|_{V_\nu} = \operatorname{ind}\rho + \operatorname{corank} \eta|_{V_\nu}$, where the last term by Theorem 1.5 is equal to $\operatorname{ind}(\mathfrak{g}_\nu, \gamma_\nu)$. $\square$

Corollary 1.7 *(The Raïs formula.) Assume an extension*

$$0 \to \mathfrak{n} \to \mathfrak{m} \xrightarrow{p} \mathfrak{g} \to 0$$

satisfies the following two conditions: 1) it is split, i.e. there exists a homomorphic section $s : \mathfrak{g} \to \mathfrak{m}$ of p; 2) the Lie algebra $\mathfrak{n}$ is abelian. Write ρ^s for the representation $\rho \circ s : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{n})$ and $\mathfrak{g}_\nu^s$ for the stabilizer of a generic element $\nu \in \mathfrak{n}^*$ with respect to the dual representation $(\rho^s)^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{n}^*)$. Then

$$\operatorname{ind} \mathfrak{m} = \operatorname{ind} \rho^s + \operatorname{ind} \mathfrak{g}_\nu^s.$$

Proof We first notice that $\operatorname{ind}\rho^s = \operatorname{ind}\rho$. Indeed, since $\mathfrak{n}$ is abelian, We have the formula

$$\operatorname{ad}^*_{(x_1+x_2)}\nu = \operatorname{ad}^*_{x_2}\nu,$$

where $x = x_1 + x_2 \in \mathfrak{m}$ and the indices refer to the components of the direct decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2, \mathfrak{m}_1 := \mathfrak{n}, \mathfrak{m}_2 := s(\mathfrak{g})$. Thus the tangent spaces at ν to the orbits of the representations $(\rho^s)^*$ and ρ^* coincide.

Secondly, the formula above shows also that $\mathfrak{m}_\nu = \mathfrak{n} \oplus s(\mathfrak{g}_\nu^s)$, hence $\mathfrak{g}_\nu \simeq \mathfrak{g}_\nu^s$. Now it remains to show that the corresponding cocycle γ_ν is trivial. To prove this, note that the subalgebra $p(\mathfrak{m}_\nu) \subset \mathfrak{g}$ is naturally isomorphic to $\mathfrak{g}_\nu$ and that $s|_{p(\mathfrak{m}_\nu)} : p(\mathfrak{m}_\nu) \simeq \mathfrak{g}_\nu \to \mathfrak{m}$ is in fact a section of the canonical projection $p_\nu : \mathfrak{m}_\nu \to \mathfrak{g}_\nu$. By the homomorphicity of s, $[x_2, y_2]_1 = 0$ and $\bar{\Gamma}(x, y) = 0$ (see Remark 1.2). $\square$

Example 1.8 (A nonsplit extension.) The Heisenberg Lie algebra is the Lie algebra of triangular matrices

$$\begin{pmatrix} 0 & u & w \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix}, u, v, w \in \mathbb{K}$$

with the standard commutator. The induced Lie bracket on $\mathfrak{m} = \mathbb{K}^3$ is $[(u, v, w), (u', v', w')] = (0, 0, uv' - vu')$. Consider the ideal $\mathfrak{n} = \{(0, 0, w) \mid w \in \mathbb{K}\} \subset \mathfrak{m}$. The representation ρ of $\mathfrak{m}$ on $\mathfrak{n}$ is trivial, hence so is ρ^*. Thus the stabilizer $\mathfrak{m}_\nu$ of any $\nu \in \mathfrak{n}^*$ coincides with the whole $\mathfrak{m}$ and the factor algebra $\mathfrak{g}_\nu = \mathfrak{m}/\mathfrak{n}$ is the abelian two-dimensional algebra. The cocycle $\bar{\Gamma}$ corresponding to the section $(u, v) \mapsto (u, v, 0)$ (see Remark 1.2) is given by $\bar{\Gamma}((u, v, w), (u', v', w')) = \nu(uv' - vu')$. Thus the induced cocycle $\bar{\gamma}$ on $\mathfrak{m}/\mathfrak{n}$ is nontrivial and $\operatorname{ind}(\mathfrak{g}_\nu, \gamma_\nu) = 0$ as soon as $\nu \neq 0$. Theorem 1.7 gives $\operatorname{ind} \mathfrak{m} = \operatorname{ind} \rho = 1$.

Example 1.9 (A split extension with a nonabelian ideal.) Let $\mathfrak{m} \subset \mathfrak{gl}(3, \mathbb{K})$ be the Borel Lie algebra of the upper triangular matrices, $\mathfrak{n} \subset \mathfrak{m}$ be the upper nilpotent Lie algebra of the strict upper triangular matrices. Using the trace form one obtains a natural identification $\mathfrak{n}^* \simeq \mathfrak{gl}(3, \mathbb{K})/\mathfrak{m}$ which is M-equivariant, where M stands for the Lie group of the nondegenerate upper triangular matrices. The extension $0 \to \mathfrak{n} \to \mathfrak{m} \to \mathfrak{m}/\mathfrak{n} \to 0$ is split, since the Lie subalgebra of diagonal matrices in $\mathfrak{m}$ is complement to $\mathfrak{n}$.

The stabilizer $\mathfrak{m}_\nu$ of an element

$$\nu := \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ c & b & 0 \end{pmatrix} + \mathfrak{m} \in \mathfrak{gl}(3, \mathbb{K})/\mathfrak{m}, c \neq 0,$$

is equal to the 3-dimensional abelian Lie algebra of the matrices of the form

$$\begin{pmatrix} x & b(x-y)/c & z \\ 0 & y & a(x-y)/c \\ 0 & 0 & x \end{pmatrix}, x, y, z \in \mathbb{K}.$$

Thus the generic orbits of the $\mathfrak{m}$-action on $\mathfrak{n}^*$ are of codimension 0. The intersection $\mathfrak{m}_\nu \cap \mathfrak{n}$ corresponds to the matrices above with $x = y = 0$, i.e. is 1-dimensional, hence $\mathfrak{g}_\nu$ is the abelian 2-dimensional Lie algebra. Since all the Lie algebras in the extension $0 \to \mathfrak{n}_\nu \to \mathfrak{m}_\nu \to \mathfrak{g}_\nu \to 0$ are abelian, it is split and the arguments of Remark 1.2 show that the corresponding cocycle γ_ν is trivial. Finally, $\operatorname{ind} \mathfrak{m} = \operatorname{ind} \mathfrak{g}_\nu = 2$.

The first of these examples show that the cocycles γ_ν in general can be nontrivial. The second example and the proof of the corollary above suggest that in the case of split extensions one would expect triviality of cocycles γ_ν. Unfortunately, the author knows neither a proof of this, nor an example of a nontrivial cocycle γ_ν in the split case.

2 Reduction by stages: preliminary results

We start from the following result, which in reference [MMPR98] is called "easy Poisson reduction by stages".

Lemma 2.1 *Let M be a connected Lie group with a connected normal closed subgroup N. Assume M is acting on a connected Poisson manifold (P, η), this action is free and proper and preserves the Poisson structure. Let $\eta' := p'_*(\eta), \eta'' := p''_*(\eta)$ be the reduced Poisson structures on $P' := P/N, P'' := P/M$ correspondingly, with respect to the canonical projections $p' : P \to P', p'' : P \to P''$. Then*

1. *there is a natural action of the Lie group $G := M/N$ on P', which is free and proper and preserves the Poisson structure η';*
2. *the reduced (with respect to this action) Poisson manifold (P''', η'''), where $P''' := P'/G, \eta''' := p'''_*(\eta')$ and $p''' : P' \to P'''$ is the canonical projection, is naturally Poisson diffeomorphic with (P'', η'').*

Proof Since, due to the normality of N, $m(Nx) = N(mx), m \in M, x \in P$, this formula defines an action of M on the set of N-orbits. This action obviously induces an action of $G = M/N$. An easy exercise shows that the last is free and proper. It preserves η' since the action of M preserves η.

Now, the formula $G(Nx) \mapsto Mx, x \in P$, defines a diffeomorphism $\varphi : P''' \to P''$. To show that it is Poisson we shall use the fact that so are p', p'', p''' and that $p'' = \varphi \circ p''' \circ p'$. Indeed, since $p''' \circ p'$ is surjective, the equality $\varphi^*\{f,g\}_{\eta''} = \{\varphi^* f, \varphi^* g\}_{\eta'''}$, is valid for any $f, g \in \mathcal{E}(P''')$, if and only if $(p''' \circ p')^*(\varphi^*\{f,g\}_{\eta''}) = (p''' \circ p')^*(\{\varphi^* f, \varphi^* g\}_{\eta'''})$. But the left hand side of the last equality is $(p'')^*\{f,g\}_{\eta''} = \{(p'')^* f, (p'')^* g\}_\eta$. The right hand side, in turn, is $\{(p''' \circ p')^* \circ \varphi^* f, (p''' \circ p')^* \circ \varphi^* g\}_\eta = \{(p'')^* f, (p'')^* g\}_\eta$. $\square$

Now assume that the action of M on (P, η) is hamiltonian, i.e. there exists a Lie algebra homomorphism $\mathcal{J}_M : (\mathfrak{m}, [,]) \to (\mathcal{E}(P), \{,\})$ ($\{,\}$ being the Poisson brackets) with the property $\eta(\mathcal{J}_M(x)) = \xi_x, x \in \mathfrak{m}$, where ξ_x is the fundamental vector field on P corresponding to x. Then the map $J_M : P \to \mathfrak{m}^*$ defined by $\langle J_M(q), x\rangle = (\mathcal{J}_M(x))(q), q \in P$, which is called the momentum map, is M-equivariant, the action of M on $\mathfrak{m}^*$ being the coadjoint one. Recall that if we omit the condition of homomorphicity of $\mathcal{J}_M$ that is equivalent to equivariance of J_M (and also to the Poisson property of J_M), we get the so-called weak hamiltonian action (see Appendix). It is well-known that if the action of M on P is hamiltonian, the action of the subgroup N is also hamiltonian and the corresponding momentum map $J_N : P \to \mathfrak{n}^*$ is given by $J_N = \pi \circ J_M$, where $\pi : \mathfrak{m}^* \to \mathfrak{n}^*$ is the canonical projection (cf. Section 1).

The natural question arises: is the induced action of G on $(P' = P/N, \eta')$ also hamiltonian? In general this action is not even weakly hamiltonian since its fundamental vector fields are not tangent to the leaves of η'. But if we restrict ourselves to the subgroup of G which preserves a particular symplectic leaf of (P', η'), then the corresponding action will be weak hamiltonian and our next aim is to prove this.

Let us fix a symplectic leaf $S \subset P'$ of η'. It corresponds to the unique coadjoint orbit $O = \mathfrak{n} \cdot \nu$ of some element $\nu \in \mathfrak{n}^*$ and the correspondence is given by $(p')^{-1}(S) = J_N^{-1}(O)$. Let $G_S := \{g \in G \mid gS = S\}$ be the stabilizer of S under the G-action on P'. We claim that G_S is naturally isomorphic to M_ν / N_ν, where M_ν is the stabilizer of ν with respect to the M-action and $N_\nu := M_\nu \cap N$.

To prove this notice first that the stabilizer M_S of S under the M-action on P' coincides with the stabilizer M_O of O under the M-action on $\mathfrak{n}^*$. Indeed, M_S is equal to the stabilizer of $(p')^{-1}(S)$ with respect to the M-action on P which in turn is equal to M_O by the M-equivariance of J_N.

On the other hand, for $m \in M$ we have $mN\nu = N\nu$ if and only if $Nm\nu = N\nu$ (by the normality of N) if and only if $m \in NM_\nu$, i.e. $M_O = NM_\nu = M_\nu N$. Since $G_S = M_S/N$, we have $G_S = M_O/N \simeq M_\nu/N_\nu$.

Note also that $M_\nu N = M_{\nu'} N$ for any other element $\nu' = n^{-1}\nu \in O$, $n \in N$: $NM_{\nu'} = NnM_\nu n^{-1} = NM_\nu n^{-1} = M_\nu N n^{-1} = M_\nu N$.

Lemma 2.2 *The action of $G_S \simeq M_\nu/N_\nu =: G_\nu$ on $(S, \eta'|_S)$ is weakly hamiltonian, i.e. has a momentum map $J_{G_S} : S \to \mathfrak{g}_\nu^*$ which may be nonequivariant. The corresponding nonequivariance 2-cocycle (cf. Appendix) $\gamma \in C^2(\mathfrak{g}_\nu)$ coincides with the cocycle $-\gamma_\nu$, where γ_ν is the cocycle from Section 1.*

Proof We want to show that the action of $G_\nu := M_\nu/N_\nu$ on $(S, \eta'|_S)$ is weakly hamiltonian, i.e. has a momentum map which may be nonequivariant. This already has been done in [MMPR98, Section 5.2]. More precisely, it is shown there that G_ν acts naturally on the reduced space $P_\nu := J_N^{-1}(\nu)/N_\nu$ and this action is weakly hamiltonian. On the other hand it is known [OR04] that the symplectic manifolds $J_N^{-1}(\nu)/N_\nu$ and $S = J_N^{-1}(O)/N$ are canonically symplectically diffeomorphic. So all we need is to show that under this diffeomorphism the action from [MMPR98] transforms to the described above G_ν-action on S. Indeed, let $l_\nu : J_N^{-1}(\nu) \to J_N^{-1}(O)$ be the inclusion map and $p'_\nu : J_N^{-1}(\nu) \to J_N^{-1}(\nu)/N_\nu = P_\nu, p'_O : J_N^{-1}(O) \to J_N^{-1}(O)/N = S$ be the canonical projections. Then [OR04, Section 6.4] the map $L_\nu : P_\nu \to S$ defined by the commutative diagram

$$\begin{array}{ccc} J_N^{-1}(\nu) & \xrightarrow{l_\nu} & J_N^{-1}(O) \\ {\scriptstyle p'_\nu}\big\downarrow & & \big\downarrow{\scriptstyle p'_O} \\ P_\nu & \xrightarrow{L_\nu} & S \end{array}$$

is a symplectic diffeomorphism. The action of G_ν on P_ν is induced by the action of M_ν on $J_N^{-1}(\nu)$ [MMPR98, Section 5.2]. On the other hand, the action of $G_S = G_\nu$ on S is induced by the action of M. Thus by definition L_ν commutes with the actions of G_ν on P_ν and on S.

Now we recall some facts from [MMPR98]. The action of G_ν on P_ν (induced by the action of M_ν on $J_N^{-1}(\nu)$) has a momentum map $J_\nu : P_\nu \to \mathfrak{g}_\nu^*$, which in general is not equivariant. The corresponding $\mathfrak{g}_\nu^*$-valued nonequivariance one-cocycle ϖ on G_ν is determined by $r_\nu^*(\varpi([m]) = \mathrm{Ad}^*_{m^{-1}}\bar\nu - \bar\nu$, where $m \in M_\nu$, $[m]$ is the class of m in M_ν/N_ν, $r_\nu^* : \mathfrak{g}_\nu^* \to \mathfrak{m}_\nu^*$ is dual to the canonical projection $r_\nu : \mathfrak{m}_\nu \to \mathfrak{g}_\nu$ and $\bar\nu \in \mathfrak{m}_\nu^*$ is any extension of $\nu|_{n_\nu} \in \mathfrak{n}_\nu^*$.

It is clear that the corresponding two-cocycle γ on $\mathfrak{g}_\nu$ is in turn given by $\gamma(x, y) = \langle -\mathrm{ad}_x^*\bar\nu, y\rangle$ in which one recognizes the cocycle $-\bar\gamma$, where $\bar\gamma$ is defined in Section 1. Note that the freedom in defining the cocycle $\bar\gamma$ (it is defined up to a coboundary, cf. Section 1) agrees with that in defining nonequivariance cocycle of a momentum map (cf. Appendix). $\square$

Corollary 2.3 *The map $J_{G_S} : S \to (\mathfrak{g}_\nu^*, \eta_{\mathfrak{g}_\nu} - \hat\gamma_\nu)$, where $\eta_{\mathfrak{g}_\nu}$ is the canonical Lie-Poisson structure on $\mathfrak{g}_\nu^*$ and $\hat\gamma_\nu$ is the constant Poisson structure corresponding to the cocycle γ_ν (see Section 1), is Poisson.*

Proof See Appendix. $\square$

3 Reduction by stages: calculating the coranks

Retaining the notations and assumptions of the previous section let $\nu \in \mathfrak{n}^*$, let $\mathfrak{m} \cdot \nu$ stand for the $\mathfrak{m}$-orbit of ν and let $V_\nu := J_N^{-1}(\mathfrak{m} \cdot \nu)$.

Lemma 3.1 *Assume that the set $W_\nu := p'(J_M^{-1}(V_\nu))$ is a submanifold in P' and the set $U_\nu := p'''(W_\nu)$ is a submanifold in P'''. Then*

1. W_ν is a G-invariant regular Poisson submanifold in (P', η');
2. the group G is acting transitively on the space of symplectic leaves of W_ν;
3. U_ν is a Poisson submanifold of (P''', η''') and $\operatorname{corank}_{U_\nu} \eta'''|_{U_\nu} = \operatorname{ind}(\mathfrak{g}_\nu, \gamma_\nu)$ ■

Proof Since the Poisson maps J_N and p' form a dual pair [Wei83] and $\mathfrak{m} \cdot \nu$ is the union of symplectic leaves of $\mathfrak{n}^*$, the set $W_\nu = p'(J_N^{-1}(\mathfrak{m} \cdot \nu))$ is the union of symplectic leaves of η'. The G-invariance of W_ν follows from the M-equivariance of J_N and the transitivity of G on these symplectic leaves follows from the fact that M acts transitively on the space of symplectic leaves of the Poisson manifold $\mathfrak{m} \cdot \nu$. This in turn implies that all the leaves of W_ν have the same dimension. Thus we have proven items 1. and 2.

Since W_ν is a Poisson submanifold of (P', η') the set U_ν is the union of symplectic leaves of $\eta''' = p'''_*(\eta')$, hence is Poisson. To prove the equality on corank we notice that by item 2. the Poisson manifold $(U_\nu, \eta'''|_{U_\nu})$ is in fact the reduction of a single symplectic leaf $S \subset W_\nu$ of η' with respect to its stabilizer $G_S \subset G$. Thus we can use Lemma 2.2, Corollary 2.3 and the fact that the Poisson maps $p'''|_S : (S, \eta'|_S) \to (U_\nu, \eta'''|_{U_\nu})$ and $J_{G_S} : (S, \eta'|_S) \to (\mathfrak{g}_\nu^*, \eta_{\mathfrak{g}_\nu} - \hat{\gamma}_\nu)$ form a dual pair. Taking into account that the action of G_S is free we conclude that J_{G_S} is a submersion (by the so-called bifurcation lemma, cf. [OR04, Section 4.5]). Finally, using the fact that the Poisson manifolds which are submersive images of a dual pair have the same coranks we obtain the needed equality. □

4 Proof of the main result

In this section we shall use the results of the previous two sections to prove item 2 of Theorem 1.5.

Let N, M, G be the connected simply-connected Lie groups corresponding to the Lie algebras from extension (1.1). We shall perform the described above reduction in the situation when $P = T^*M$ (with the canonical Poisson structure η of the cotangent bundle) and the action of M is the cotangent lift of the action of M by the left translations on itself. The action of M on P can be considered as the restriction to a closed subgroup of the action of P, regarded as a Lie group, on itself. By [CB97, Appendix B] the action by the left translations of a closed subgroup on a Lie group is free and proper, so the applicability of Lemma 2.1 is justified.

On the other hand, the action of M on T^*M is hamiltonian, so we can also apply Lemma 2.2 and Corollary 2.3.

It is well-known that, given a dual pair [Wei83] of Poisson surjective submersions

$$\begin{array}{ccc} P & \xrightarrow{p_2} & P_2 \\ {\scriptstyle p_1}\big\downarrow & & \\ P_1 & & \end{array},$$

where the Poisson mainifold P is nondegenerate, there is a one-to-one correspondence between the symplectic leaves of P_1 and P_2 which can be described as follows: if S_1 is a symplectic leaf of P_1 then $p_2(p_1^{-1}(S_1))$ is a symplectic leaf of P_2 and vice versa.

In our situation, when we have the dual pair

$$\begin{array}{ccc} T^*M & \xrightarrow{J_M} & \mathfrak{m}^* \\ {\scriptstyle p''}\downarrow & & \\ \mathfrak{m}^* & & \end{array},$$

where the vertical arrow is the canonical projection $p'' : T^*M \to T^*M/M \simeq \mathfrak{m}^*$, the symplectic leaves are selfdual, i.e. each coadjoint orbit corresponds to itself. Indeed, after the identification of T^*M with $M \times \mathfrak{m}^*$ by means of the left translations the maps in the diagrame above have the form: $p''(g, x) = x, J_M(g, x) = -\mathrm{Ad}^*_g x, g \in M, x \in \mathfrak{m}^*$. Thus, given a coadjoint orbit $O \subset \mathfrak{m}^*$, we have $J_M((p'')^{-1}(O)) = J_M(M \times O) = O$.

From this we conclude that the Poisson submanifold $V_\nu \subset \mathfrak{m}^*$ is also selfdual. In other words, under the identification $P''' \simeq P'' = \mathfrak{m}^*$ of Lemma 2.1 V_ν coincides with U_ν of Lemma 3.1. Thus U_ν and $W_\nu = (p''')^{-1}(U_\nu)$ are submanifolds and we can use this lemma to conclude the proof. □

5 Appendix: weak hamiltonian actions

Assume that an action of a connected Lie group G on a connected non-degenerate Poisson manifold (P, η) is given such that there exists a linear map $\mathcal{J} : \mathfrak{g} \to \mathcal{E}(P)$ with the property $\eta(\mathcal{J}(x)) = \xi_x, x \in \mathfrak{g}$, where ξ_x is the fundamental vector field on P corresponding to x and $\eta(f)$ denotes the hamiltonian vector field cooresponding to a function $f \in \mathcal{E}(P)$. Then the action is called *weakly hamiltonian* and the map $J : P \to \mathfrak{g}^*$ defined by $\langle J(q), x\rangle = (\mathcal{J}(x))(q), q \in P$, is called the *momentum map*. If this last is G-equivariant (the G-action on $\mathfrak{g}^*$ being the coadjoint one) or, equivalently, it is Poisson (with the canonical Lie-Poisson structure $\eta_\mathfrak{g}$ on $\mathfrak{g}^*$), the action is called *hamiltonian*. The obstruction to hamiltonicity is measured by the two-cocycle γ on $\mathfrak{g}$ given by $\gamma(x, y) = \mathcal{J}([x, y]) - \{\mathcal{J}(x), \mathcal{J}(y)\}$, where $\{,\}$ are the Poisson brackets. The map J becomes Poisson when we endow $\mathfrak{g}^*$ with the affine Poisson structure $\eta_\mathfrak{g} + \hat{\gamma}$ (cf. Section 1).

One can also define a $\mathfrak{g}^*$-valued one-cocycle ϖ on G by $\varpi(g) = J(gq) - \mathrm{Ad}^*_{g^{-1}}(J(q))$ (this in fact does not depend on $q \in P$). Again the map J becomes equivariant if one defines the action of G on $\mathfrak{g}^*$ by $g\xi = \mathrm{Ad}^*_{g^{-1}}(\xi) + \varpi(g), g \in G, \xi \in \mathfrak{g}^*$. The two cocycles are related by $\gamma(x, y) = \langle d_e\varpi(x), y\rangle$.

Note that we can add constants to the functions $\mathcal{J}(x)$ because this does not change ξ_x, that is the momentum map J is defined up to a constant element in $\mathfrak{g}^*$. This means that the corresponding cocycles are defined up to a coboundary.

The reader is referred to reference [OR04] for more information on weak hamiltonian actions and momentum maps.

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