THE B-QUADRILATERAL LATTICE AND ITS REDUCTIONS

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Outline

1. The B-Quadrilateral Lattice
   - Geometric Integrability of the BQL
   - The Miwa (discrete BKP) equation within the QL theory
   - The B-reduction of the fundamental transformation
2. The generalized isothermic lattice
   - Quadrilateral Lattices in quadrics
   - Generalized isothermic lattices

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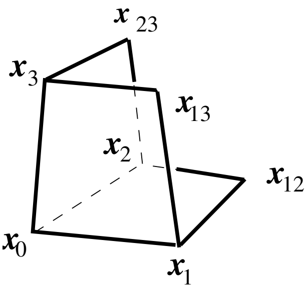
Geometric Integrability Scheme

Given generic points x_0, x_1, x_2 and x_3 in $\mathbb{P}^3$, let x_{ij}, $1 \leq i < j \leq 3$, be generic points of the planes $\langle x_0, x_i, x_j \rangle$, then there exists exactly one point x_{123} which belongs simultaneously to the planes $\langle x_3, x_{13}, x_{23} \rangle$, $\langle x_2, x_{12}, x_{23} \rangle$ and $\langle x_1, x_{12}, x_{13} \rangle$.



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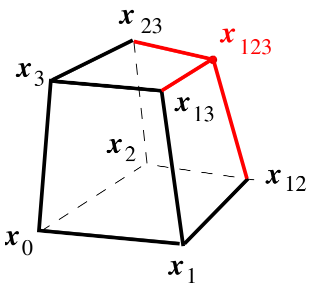

The Multidimensional Quadrilateral Lattice

Definition

A quadrilateral lattice is a map $x : \mathbb{Z}^N \to \mathbb{P}^M$, $3 \le N \le M$, whose all elementary quadrilaterals are planar.

- Geometry: $x, x_{(i)}, x_{(j)}, x_{(ij)}$, $1 \le i < j \le N$, are coplanar.
- Algebra: the homogeneous coordinates $\boldsymbol{x} : \mathbb{Z}^N \to \mathbb{R}^{M+1}_*$, $x = [\boldsymbol{x}]$, satisfy the system of discrete Laplace equations

$$\boldsymbol{x}_{(ij)} = a^{ij}\boldsymbol{x}_{(i)} + a^{ji}\boldsymbol{x}_{(j)} + c^{ij}\boldsymbol{x}, \quad 1 \le i < j \le N$$

$a^{ij}, c^{ij} : \mathbb{Z}^N \to \mathbb{R}$.
- Multidimensional consistency.

Notation: $x_{(i)}(n_1, \dots, n_i, \dots, n_N) = x(n_1, \dots, n_i + 1, \dots, n_N)$.

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The discrete Darboux equations I

Compatibility condition — nonlinear system

$$
\begin{aligned}
a^{ij}_{(k)} c^{ik} + a^{ji}_{(k)} c^{jk} &= a^{ik}_{(j)} c^{ij} + a^{ki}_{(j)} c^{kj}, \\
a^{ij}_{(k)} a^{ik} = a^{ik}_{(j)} a^{ij} &= c^{jk}_{(i)} + a^{jk}_{(i)} a^{ij} + a^{kj}_{(i)} a^{ik},
\end{aligned}
$$

(i, j, k distinct, $c^{ij} = c^{ji}$). *P. M. Santini & A. D., 1997*

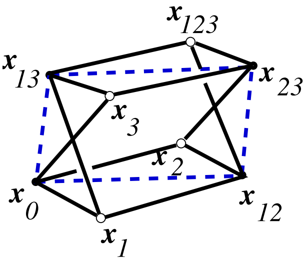
Nonlinear or linear?

Transition from the data $(a^{ij}, a^{ji}, a^{ik}, a^{ki}, a^{jk}, a^{kj}, c^{ij}, c^{ik}, c^{jk})$ to $(a^{ij}_{(k)}, a^{ji}_{(k)}, a^{ik}_{(j)}, a^{ki}_{(j)}, a^{jk}_{(i)}, a^{kj}_{(i)}, c^{ij}_{(k)}, c^{ik}_{(j)}, c^{jk}_{(i)})$ consists on solving a linear system.

A. I. Bobenko & Yu. Suris, 2005

The Möbius theorem (new formulation) 1827

Under hypotheses of the Geometric Integrability Scheme, assume that x, x_{12}, x_{13} and x_{23} are coplanar, then the points x_1, x_2, x_3 and x_{123} are coplanar as well.



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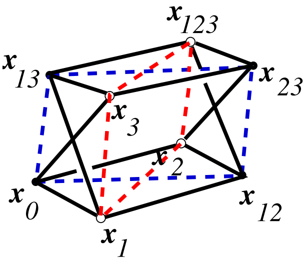

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Beacuse the B-constraint is imposed on the 3D level, one has to check its 4D consistency.

The B-reduction condition

$$\boldsymbol{x} \wedge \boldsymbol{x}_{(ij)} \wedge \boldsymbol{x}_{(ik)} \wedge \boldsymbol{x}_{(jk)} = (a^{ji} a^{kj} a^{ik} + a^{ij} a^{jk} a^{ki}) \, \boldsymbol{x} \wedge \boldsymbol{x}_{(i)} \wedge \boldsymbol{x}_{(j)} \wedge \boldsymbol{x}_{(k)} = 0,$$

is equivalent of the existence of the Moutard gauge

$$\boldsymbol{x}_{(ij)} - \boldsymbol{x} = f^{ij} (\boldsymbol{x}_{(i)} - \boldsymbol{x}_{(j)}), \quad 1 \leq i < j \leq N,$$

$f^{ij} : \mathbb{Z}^N \to \mathbb{R}$.

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The Miwa equation

The compatibility condition *J. C. C. Nimmo & W. K. Schief, 1997*

$$1 + f^{jk}_{(i)}(f^{ij} - f^{ik}) = f^{ik}_{(j)}f^{ij} = f^{ij}_{(k)}f^{ik}, \qquad i, j, k \quad \text{distinct},$$

with $f^{ji} = -f^{ij}$

implies existence of the potential τ

$$f^{ij} = \frac{\tau_{(i)}\tau_{(j)}}{\tau\,\tau_{(ij)}},$$

and can be then rewritten in the form *T. Miwa, 1982*

$$\tau\,\tau_{(ijk)} = \tau_{(ij)}\tau_{(k)} - \tau_{(ik)}\tau_{(j)} + \tau_{(jk)}\tau_{(i)}, \quad 1 \leq i < j < k \leq N.$$

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The discrete Darboux equations II (affine version)

$\vec{\boldsymbol{x}} : \mathbb{Z}^N \to \mathbb{R}^M \subset \mathbb{P}^M$

$$\vec{\boldsymbol{x}}_{(ij)} - \vec{\boldsymbol{x}} = A^{ij}(\vec{\boldsymbol{x}}_{(i)} - \vec{\boldsymbol{x}}) + A^{ji}(\vec{\boldsymbol{x}}_{(j)} - \vec{\boldsymbol{x}}), \quad i < j.$$

$$\Delta_k A^{ij} + A^{ij}_{(k)} A^{ik} = A^{jk}_{(i)} A^{ij} + A^{kj}_{(i)} A^{ik}, \qquad i, j, k \quad \text{distinct},$$

the Lamé coefficients h_i

$$A^{ij} = \frac{h_{i(j)}}{h_i}, \quad i \neq j,$$

the rotation coefficients β_{ij}

$$\Delta_i h_j = h_{i(j)} \beta_{ij}, \quad i \neq j,$$

the normalized tangent vectors $\boldsymbol{X}_i$

$$\Delta_i \vec{\boldsymbol{x}} = h_i \boldsymbol{X}_i.$$

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The τ-function of the quadrilateral lattice

The first order form of the linear system

$$\Delta_j \boldsymbol{X}_i = \beta_{ij} \boldsymbol{X}_j, \quad i \neq j,$$

the discrete Darboux (N-wave) system

$$\Delta_j \beta_{ik} = \beta_{ij(k)} \beta_{jk}, \quad i, j, k \quad \text{distinct.}$$

L. V. Bogdanov & B. G. Konopelchenko, 1995

The τ-function of the quadrilateral lattice

$$\frac{\tilde{\tau}\tilde{\tau}_{(ij)}}{\tilde{\tau}_{(i)}\tilde{\tau}_{(j)}} = 1 - \beta_{ij}\beta_{ji}, \quad i \neq j.$$

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Relation between the τ-functions

In the reduction from QL to BQL

$$h_i = (-1)^{\sum_{k<i} m_k} \frac{\tau}{\tau_{(i)}},$$

the rotation coefficients ($i < j$)

$$\beta_{ij} = -(-1)^{\sum_{i\leq k<j} m_k} \left(\frac{\tau_{(i)}}{\tau} + \frac{\tau_{(ij)}}{\tau_{(j)}} \right) \frac{\tau}{\tau_{(j)}},$$

$$\beta_{ji} = -(-1)^{\sum_{i\leq k<j} m_k} \left(\frac{\tau_{(j)}}{\tau} - \frac{\tau_{(ij)}}{\tau_{(i)}} \right), \frac{\tau}{\tau_{(i)}}.$$

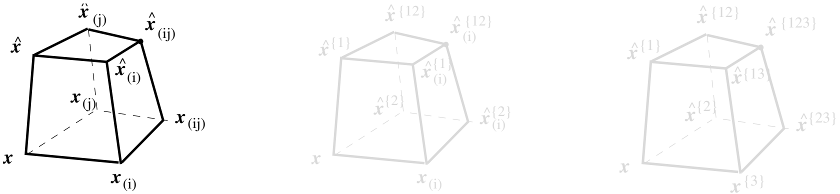
The square of the τ-function of BQL is the corresponding QL τ-function

$$\tilde{\tau} = \tau^2.$$

The fundamental (Jonas) transformation of QL

Definition

A quadrilateral lattice $\hat{x} : \mathbb{Z}^N \to \mathbb{P}^M$ is called the fundamental transform of the quadrilateral lattice $x : \mathbb{Z}^N \to \mathbb{P}^M$ if the quadrilaterals with vertices $x, \hat{x}, x_{(i)}, \hat{x}_{(i)}$ are planar.



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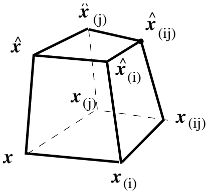

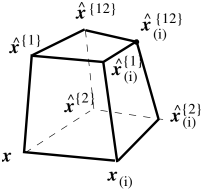

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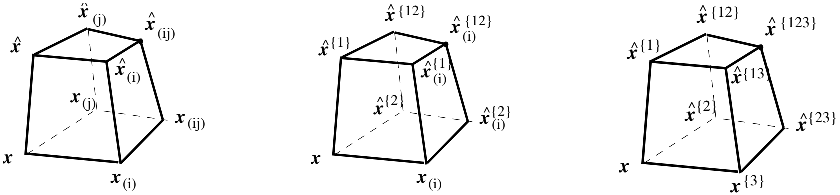

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Data of the vectorial fundamental transformation

A solution $\boldsymbol{Y}_i : \mathbb{Z}^N \to \mathbb{V}$ of the first order linear system

$$\Delta_j \boldsymbol{Y}_i = \beta_{ij} \boldsymbol{Y}_j, \quad i \neq j,$$

a solution $\boldsymbol{Y}_i^* : \mathbb{Z}^N \to \mathbb{V}^*$, of the adjoint linear system

$$\Delta_i \boldsymbol{Y}_j^* = \boldsymbol{Y}_{i(j)}^* \beta_{ij}, \quad i \neq j.$$

These allow to construct the linear operator valued potential $\boldsymbol{\Omega}(\boldsymbol{Y}, \boldsymbol{Y}^*) : \mathbb{Z}^N \to L(\mathbb{V})$,

$$\Delta_i \boldsymbol{\Omega}(\boldsymbol{Y}, \boldsymbol{Y}^*) = \boldsymbol{Y}_i \otimes \boldsymbol{Y}_i^*, \qquad i = 1, \ldots, N;$$

similarly $\Delta_i \boldsymbol{\Omega}(\boldsymbol{X}, \boldsymbol{Y}^*) = \boldsymbol{X}_i \otimes \boldsymbol{Y}_i^*$, $\Delta_i \boldsymbol{\Omega}(\boldsymbol{Y}, h) = \boldsymbol{Y}_i \otimes h_i$, then the corresponding vectorial fundamental transform of x reads

$$\hat{\vec{x}} = \vec{x} - \boldsymbol{\Omega}(\boldsymbol{X}, \boldsymbol{Y}^*) \boldsymbol{\Omega}(\boldsymbol{Y}, \boldsymbol{Y}^*)^{-1} \boldsymbol{\Omega}(\boldsymbol{Y}, h).$$

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The B-reduced fundamental transformation

Lemma

Given B-quadrilateral lattice $x : \mathbb{Z}^N \to \mathbb{P}^M$ and its fundamental transform $\hat{x}$ constructed under additional assumption that for any point x of the lattice and any pair i, j of different directions, the four points x, $x_{(ij)}$, $\hat{x}_{(i)}$ and $\hat{x}_{(j)}$ are coplanar. Then the lattice $\hat{x} : \mathbb{Z}^N \to \mathbb{P}^M$ is B-quadrilateral lattice as well.

Proof: By 4D consistency of the B-quadrilateral lattice condition.

Data of the B-reduced transformation

Given a solution $\boldsymbol{Y}_i : \mathbb{Z}^N \to \mathbb{V}$ of the first order linear system under BQL reduction, denote by $\boldsymbol{\Theta} = \boldsymbol{\Omega}(\boldsymbol{Y}, h)$, then

$$\boldsymbol{Y}_i^* = (-1)^{\sum_{k<i} m_k} \frac{\tau}{\tau_{(i)}} (\boldsymbol{\Theta}^t + \boldsymbol{\Theta}^t_{(i)})$$

is a solution of the corresponding adjoint linear problem, and the potential $\boldsymbol{\Omega}(\boldsymbol{Y}, \boldsymbol{Y}^*)$ allows for the constraint

$$\boldsymbol{\Omega}(\boldsymbol{Y}, \boldsymbol{Y}^*) + \boldsymbol{\Omega}(\boldsymbol{Y}, \boldsymbol{Y}^*)^t = 2\boldsymbol{\Theta} \otimes \boldsymbol{\Theta}^t.$$

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$$\boldsymbol{\Omega}(\boldsymbol{Y}, \boldsymbol{Y}^*) + \boldsymbol{\Omega}(\boldsymbol{Y}, \boldsymbol{Y}^*)^{\mathrm{t}} = 2\boldsymbol{\Theta} \otimes \boldsymbol{\Theta}^{\mathrm{t}}.$$

- Equivalence of the formulas of the B-reduced fundamental transformation with the pfaffian expressions of the Moutard transformation. *J. C. C. Nimmo & W. K. Schief, 1997*
- Algebro-geometric solutions
  - Riemann surfaces equipped with holomorphic involution with exactly two fixed points.
  - Explicit formulas in terms of the Prym-theta functions of the corresponding covering.
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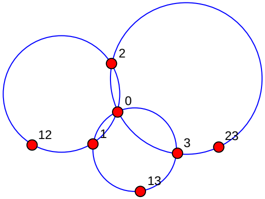
Outline

1. The B-Quadrilateral Lattice
   - Geometric Integrability of the BQL
   - The Miwa (discrete BKP) equation within the QL theory
   - The B-reduction of the fundamental transformation
2. The generalized isothermic lattice
   - Quadrilateral Lattices in quadrics
   - Generalized isothermic lattices

The Miquel configuration

Under hypotheses of the Geometric Integrability Scheme, assume that the seven points x, x_1, x_2, x_3, x_{12}, x_{13}, and x_{23} belong to a quadric surface $\mathcal{Q} \in \mathbb{P}^3$, then the point x_{123} belongs to the quadric $\mathcal{Q}$ as well.



Example: When $\mathcal{Q}$ is the sphere S^2 then we obtain, after the stereographic projection, the Miquel configuration of circles.

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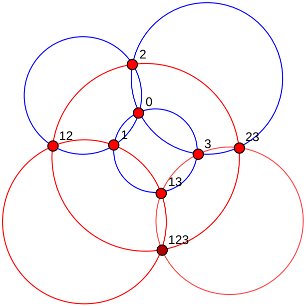

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Quadratic reductions of the quadrilateral lattice

- Integrability of QLs $x : \mathbb{Z}^N \to \mathcal{Q}^M \subset \mathbb{P}^{M+1}$ in hyperquadrics.
- The corresponding (Ribaucour) reduction of the fundamental transformation. *A. D., 1999*

Example: When $\mathcal{Q}^M$ is the (Möbius) sphere S^M then we obtain, after the stereographic projection, the circular lattices in $\mathbb{E}^M$ – discrete analogues of submanifolds in curvature line parametrization.

Important quadrics (or intersections of several of them) in geometry: the Plücker quadrics (space of lines in $\mathbb{P}^3$) and its generalization to other Grassmann manifolds, spaces of constant curvature, the Lie quadric (space of oriented spheres), Segré varieties, Veronese varieties.

Generalized isothermic lattices

General strategy

Superpositions of constraints compatible with the Geometric Integrability Scheme give integrable reductions of the Quadrilateral Lattice.

Definition

A B-quadrilateral lattice in a hyperquadric $x : \mathbb{Z}^N \to \mathcal{Q}^M \subset \mathbb{P}^{M+1}$ is called a generalized isothermic lattice.

The local irreducibility condition: intersections of planes $\langle x, x_{(i)}, x_{(j)}, x_{(ij)} \rangle$ of the elementary quadrilaterals with the hyperquadric $\mathcal{Q}^M$ are conic curves.

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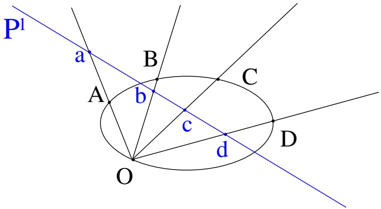
The cross-ratio on conics (J. Steiner)



$$\mathrm{cr}_S(A,B;C,D) = \mathrm{cr}(a,b;c,d) = \frac{(c-a)(d-b)}{(c-b)(d-a)}.$$

Algebraic characterization of generalized isothermic lattices

Theorem

A quadrilateral lattice in a quadric $x : \mathbb{Z}^N \to \mathcal{Q}^M \subset \mathbb{P}^{M+1}$ satisfying the local irreducibility condition is a generalized isothermic lattice if and only if there exist functions $\alpha^i : \mathbb{Z} \to \mathbb{R}$ of single arguments n_i such that

$$\mathrm{cr}_S(x_{(i)}, x_{(j)}; x, x_{(ij)}) = \frac{\alpha^i}{\alpha^j}, \qquad 1 \leq i < j \leq N.$$

Remark: When the conic is a circle then the Steiner's cross-ratio is equal to the (real in this case) cross-ratio computed using the complex structure of the plane.

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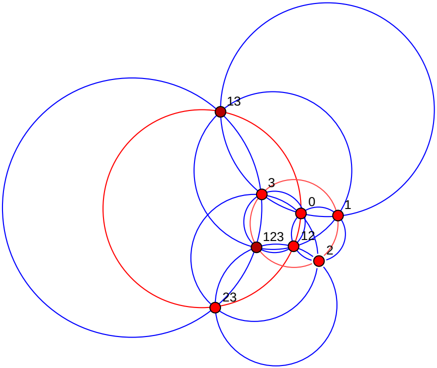
- Discrete isothermic surfaces as circular surfaces satisfying the (complex-quaternionic) cross-ratio condition.
  A. I. Bobenko & U. Pinkall, 1996
- The Darboux transformation of the discrete isothermic surfaces.
  U. Hertrich-Jeromin, T. Hoffmann & U. Pinkall, 1999
  J. Cieśliński, 1999
- Algebraic generalization of circular isothermic lattices to $N > 2$.
  A. I. Bobenko & U. Pinkall, 1999
- Discrete isothermic surfaces and the "discrete Calapso equation". *W. K. Schief, 2001*
- Multidimensional T-nets in quadrics.
  A. I. Bobenko & Yu. Suris, 2005

"Miquel + Möbius = Clifford"

When in the Miquel configuration the points x, x_{12}, x_{13} and x_{23} are cocircular, then the points x_1, x_2, x_3 and x_{123} are cocircular as well.



B. G. Konopelchenko, W. K. Schief, 2003

Summary

- The Quadrilateral Lattice is the basic geometric object in the theory of discrete integrable systems.
- Geometry not only as an interpretation but also as the explanation.

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