Discrete elliptic Toda system, cubic closest packing, and Miwa's discrete BKP equation

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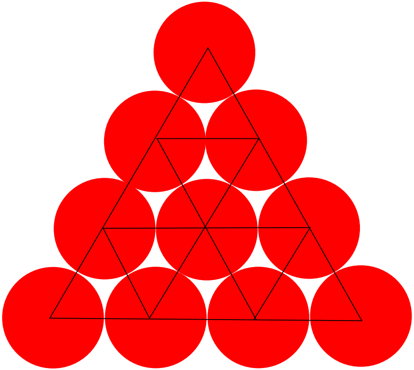
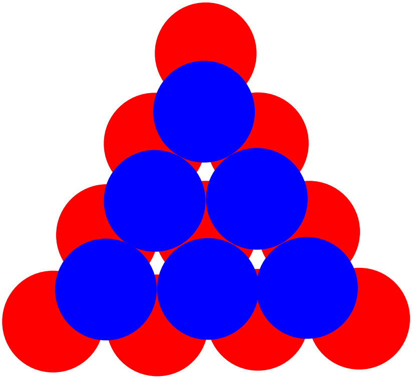
joint work with M. Nieszporski & P. M. Santini
arXiv:0705.0573 [nlin.SI]

Outline

Cubic closest sphere packing lattice



Cubic closest sphere packing lattice



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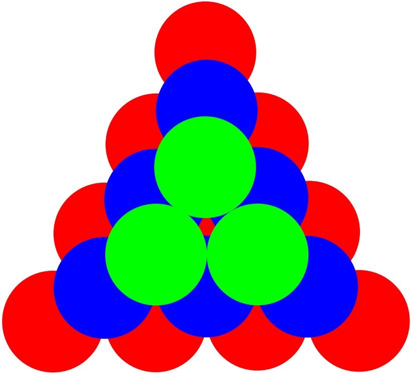
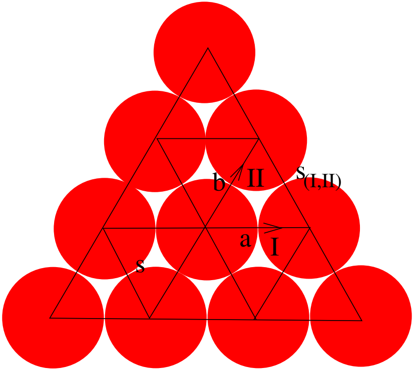


Cubic closest sphere packing lattice

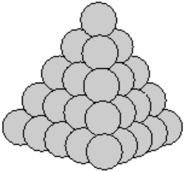


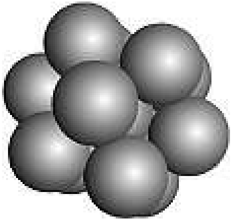
The discrete elliptic Toda system

$$\frac{a^K_{(-II)}}{q^K_{(-II)}} = \frac{a^{K+1}_{(-I,II)}}{r^{K+1}_{(II)}}, \qquad \frac{b^K_{(-I)}}{q^K_{(-I)}} = \frac{b^{K+1}}{r^{K+1}_{(II)}}, \qquad \frac{s^K}{q^K_{(-I,-II)}} = \frac{s^{K+1}_{(-I)}}{r^{K+1}_{(-I)}},$$

where

$$\begin{aligned} q^K &= a^K b^K + a^K s^K_{(I,II)} + b^K s^K_{(I,II)}, \\ r^K &= a^K_{(-I)} b^K_{(-II)} + a^K_{(-I)} s^K + b^K_{(-II)} s^K . \end{aligned}$$





The Laplace transformation – elliptic case

$$L = (\bar{\partial} + B)(\partial + A) + 2V = (\partial + A)(\bar{\partial} + B) + 2U$$

A, B, V, U – functions of $z = x + iy$

$$2H = B_{,z} - A_{,\bar{z}}, \qquad U = V + H.$$

Theorem

When ψ is a solution of the Laplace equation $L\psi = 0$, then $\tilde{\psi} = (\partial + A)\psi$ satisfies new Laplace equation $\tilde{L}\tilde{\psi} = 0$, where

$$\tilde{L} = (\bar{\partial} + \tilde{B})(\partial + \tilde{A}) + 2\tilde{V},$$

$$\tilde{A} = A - (\ln V)_{,z}, \quad \tilde{B} = B, \quad \tilde{H} = H + \frac{1}{2}(\ln V)_{,z\bar{z}}, \quad \tilde{V} = V + \tilde{H}.$$

The Laplace sequence and the Toda system

The Laplace sequence of 2D Laplace–Schrödinger operators

$$L_n \to \tilde{L}_n = L_{n+1}, \qquad V_n \to \tilde{V}_n = V_{n+1}.$$

Laplace and Toda

The potentials V_n of the Laplace sequence satisfy the elliptic Toda field system

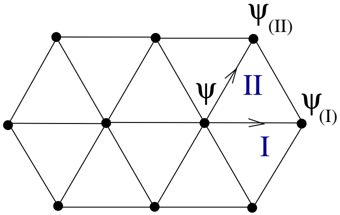
$$\frac{1}{2}(\ln V_n)_{,z\bar{z}} = V_{n+1} - 2V_n + V_{n-1}.$$

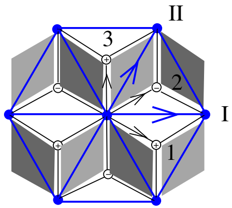
The formal transition to the hyperbolic case: $(z, \bar{z}) \to (x, y)$.

2D discrete elliptic Schrödinger operator

The self-adjoint 7-point operator

$$A\Psi_{(I)}+A_{(-I)}\Psi_{(-I)}+B\Psi_{(II)}+B_{(-II)}\Psi_{(-II)}+S_{(I)}\Psi_{(I,-II)}+S_{(II)}\Psi_{(-I,II)} = F\Psi.$$





The Laplace transformation

S. P. Novikov & I. A. Dynnikov, 1997

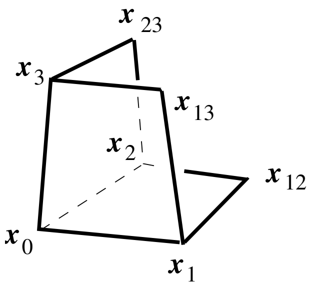
The Darboux-type transformation

M. Nieszporski, P. M. Santini & A. D., 2004

Geometric Integrability Scheme

Given generic points x_0, x_1, x_2 and x_3 in $\mathbb{P}^3$, let x_{ij}, $1 \leq i < j \leq 3$, be generic points of the planes $\langle x_0, x_i, x_j \rangle$.

Then there exists exactly one point x_{123} which belongs simultaneously to the planes $\langle x_3, x_{13}, x_{23} \rangle$, $\langle x_2, x_{12}, x_{23} \rangle$ and $\langle x_1, x_{12}, x_{13} \rangle$.



Definition

A quadrilateral lattice is a map $x : \mathbb{Z}^N \to \mathbb{P}^M$, $2 \leq N \leq M$, whose all elementary quadrilaterals are planar.

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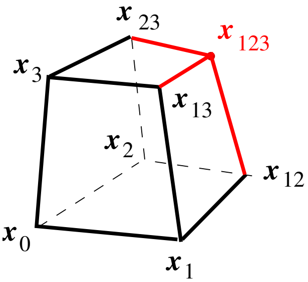

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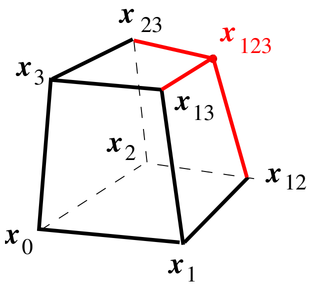

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The Multidimensional Quadrilateral Lattice

- Geometry: $x, x_{(i)}, x_{(j)}, x_{(ij)}$, $1 \leq i < j \leq N$, are coplanar.
- Algebra: the homogeneous coordinates $\psi : \mathbb{Z}^N \to \mathbb{K}_*^{M+1}$, $x = [\psi]$, satisfy the system of discrete Laplace equations

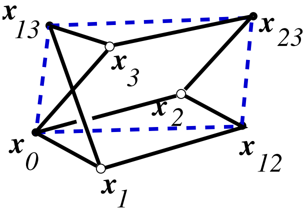
$$\psi_{(ij)} = a^{ij}\psi_{(i)} + a^{ji}\psi_{(j)} + c^{ij}\psi, \quad 1 \leq i < j \leq N$$

  $a^{ij}, c^{ij} : \mathbb{Z}^N \to \mathbb{K}$.
- Compatibility condition (for $N \geq 3$): a nonlinear system (the discrete Darboux equations).

The B-quadrilateral lattice

Under hypotheses of the Geometric Integrability Scheme, assume that x_0, x_{12}, x_{13} and x_{23} are coplanar.

Then the points x_1, x_2, x_3 and x_{123} are coplanar as well.



Definition

A quadrilateral lattice $x : \mathbb{Z}^N \to \mathbb{P}^M$, is called the B-quadrilateral lattice if for any triple of different indices $1 \leq i < j < k \leq N$ the points x, $x_{(ij)}$, $x_{(ik)}$ and $x_{(jk)}$ are coplanar.

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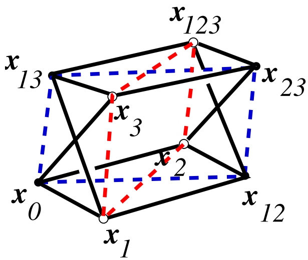

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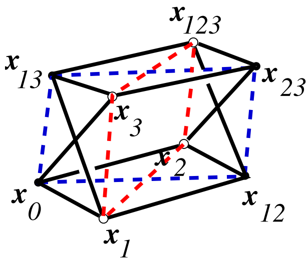

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The Moutard linear system and the discrete BKP eqs

The B-reduction condition is equivalent to the existence of the Moutard gauge

$$\psi_{(ij)} - \psi = f^{ij}(\psi_{(i)} - \psi_{(j)}), \quad 1 \leq i < j \leq N, \quad f^{ij} : \mathbb{Z}^N \to \mathbb{K}.$$

The compatibility condition *E. Date, M. Jimbo & T. Miwa, 1983*

$$1 + f^{jk}_{(i)}(f^{ij} - f^{ik}) = f^{ik}_{(j)} f^{ij} = f^{ij}_{(k)} f^{ik}, \qquad i,j,k \quad \text{distinct}, \quad f^{ji} = -f^{ij},$$

implies existence of the potential τ

$$f^{ij} = \frac{\tau_{(i)}\tau_{(j)}}{\tau \, \tau_{(ij)}},$$

and can be then rewritten in the form *T. Miwa, 1982*

$$\tau \, \tau_{(ijk)} = \tau_{(ij)}\tau_{(k)} - \tau_{(ik)}\tau_{(j)} + \tau_{(jk)}\tau_{(i)}, \quad 1 \leq i < j < k \leq N.$$

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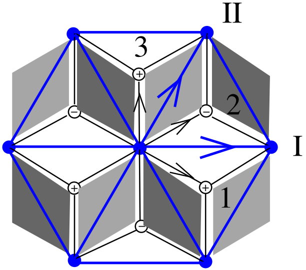
The diagonal staircase section of the $\mathbb{Z}^3$ graph



The black points of the staircase section

$$\mathbb{T} = \{(n_1, n_2, n_3) \in \mathbb{Z}^3,\ n_1 - n_2 + n_3 = 0\};$$

white points of the section

$$\mathbb{T}_\pm = \{(n_1, n_2, n_3) \in \mathbb{Z}^3,\ n_1 - n_2 + n_3 = \pm 1\}.$$

The self-adjoint affine linear problem on $\mathbb{T}$

The following consequence of the discrete Moutard system

$$\frac{1}{f^{12}}(\psi_{(12)}-\psi)+\frac{1}{f^{12}_{(-1-2)}}(\psi_{(-1-2)}-\psi)+\frac{1}{f^{23}}(\psi_{(23)}-\psi)+$$
$$\frac{1}{f^{23}_{(-2-3)}}(\psi_{(-2-3)}-\psi)-f^{13}_{(-1)}(\psi_{(-13)}-\psi)-f^{13}_{(-3)}(\psi_{(1-3)}-\psi)=0,$$

takes the form of the self-adjoint affine 7-point scheme

$$a(\psi_{(I)}-\psi)+a_{(-I)}(\psi_{(-I)}-\psi)+b(\psi_{(II)}-\psi)+b_{(-II)}(\psi_{(-II)}-\psi)+$$
$$+s_{(I)}(\psi_{(I,-II)}-\psi)+s_{(II)}(\psi_{(-I,II)}-\psi)=0,$$

where

$$a=\frac{1}{f^{12}},\quad b=\frac{1}{f^{23}},\quad s=-f^{13}_{(-1-2-3)},\quad (I)=(12),\quad (II)=(23).$$

Generic and affine self-adjoint 7-point linear problems

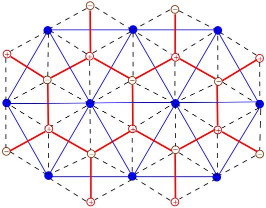
Given a scalar solution ρ of the generic self-adjoint 7-point scheme

$$A\Psi_{(I)}+A_{(-I)}\Psi_{(-I)}+B\Psi_{(II)}+B_{(-II)}\Psi_{(-II)}+S_{(I)}\Psi_{(I,-II)}+S_{(II)}\Psi_{(-I,II)} = F\Psi.$$

then $\psi = \Psi/\rho$ satisfies the affine self-adjoint seven point scheme with the coefficients

$$a = A\rho_{(I)}\rho, \qquad b = B\rho_{(II)}\rho, \qquad s = S\rho_{(-I)}\rho_{(-II)}.$$

The honeycomb sublattice



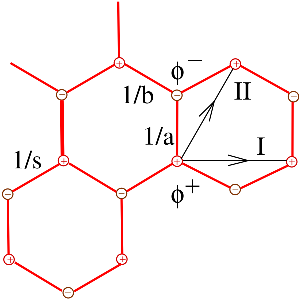
The following consequences of the Moutard system

$$f^{12}(\psi_{(1)} - \psi_{(2)}) + f^{23}(\psi_{(3)} - \psi_{(2)}) - \frac{1}{f^{13}_{(2)}}(\psi_{(123)} - \psi_{(2)}) = 0,$$

$$f^{12}(\psi_{(2)} - \psi_{(1)}) + f^{23}_{(1-3)}(\psi_{(12-3)} - \psi_{(1)}) - \frac{1}{f^{13}_{(-3)}}(\psi_{(-3)} - \psi_{(1)}) = 0,$$

in the sublattice notation with $(\phi^+, \phi^-) = (\psi_{(1)}, \psi_{(2)})$, give the honeycomb linear problem:

The honeycomb lattice



$$\frac{1}{a}(\phi^+ - \phi^-) + \frac{1}{b}(\phi^+_{(-I,II)} - \phi^-) + \frac{1}{s_{(I,II)}}(\phi^+_{(II)} - \phi^-) = 0,$$

$$\frac{1}{a}(\phi^- - \phi^+) + \frac{1}{b_{(I,-II)}}(\phi^-_{(I,-II)} - \phi^+) + \frac{1}{s_{(I)}}(\phi^-_{(-II)} - \phi^+) = 0.$$

Factorized linear problems on the triangular lattice and geometric idea of the Laplace transformation

$$\phi^+ = \frac{1}{r_I}\left(b_{I,-II}s_I + ab_{I,-II}T_{II}^{-1} + as_I T_I T_{II}^{-1}\right)\frac{1}{q}\left(bs_{I,II} + abT_{II} + as_{I,II}T_I T_{II}^{-1}\right)\phi^+,$$
$$\phi^- = \frac{1}{q}\left(bs_{I,II} + abT_{II} + as_{I,II}T_I T_{II}^{-1}\right)\frac{1}{r_I}\left(b_{I,-II}s_I + ab_{I,-II}T_{II}^{-1} + as_I T_I T_{II}^{-1}\right)\phi^-,$$

T_I, T_{II} - shift operators.
Foliation of the $\mathbb{Z}^3$ lattice into black (even ℓ) and white (odd ℓ) triangular lattices

$$\mathbb{T}_\ell = \{(n_1, n_2, n_2) \in \mathbb{Z}^3,\ n_1 - n_2 + n_3 = \ell\}, \qquad \ell \in \mathbb{Z}.$$

Transition between two subsequent lattices of the same colour is the Laplace transformation.

$$\psi^{K+1} = \frac{1}{q^K_{(-II)}}\left(b^K_{(-II)} s^K_{(I)} \psi^K + a^K_{(-I)} b^K_{(-II)} \psi^K_{(-II)} + a^K_{(-I)} s^K_{(I)} \psi^K_{(I,-II)}\right),$$

$$\psi^{K-1} = \frac{1}{r^K_{(II)}}\left(b^K s^K_{(I,II)} \psi^K + a^K_{(-I,II)} b^K \psi^K_{(II)} + a^K_{(-I,II)} s^K_{(I,II)} \psi^K_{(-I,II)}\right),$$

$$\begin{aligned} q^K &= a^K b^K + a^K s^K_{(I,II)} + b^K s^K_{(I,II)}, \\ r^K &= a^K_{(-I)} b^K_{(-II)} + a^K_{(-I)} s^K + b^K_{(-II)} s^K. \end{aligned}$$

Discrete elliptic Toda eqn. is the CCP sub-lattice Miwa eqn.

$$\frac{a^K_{(-II)}}{q^K_{(-II)}} = \frac{a^{K+1}_{(-I,II)}}{r^{K+1}_{(II)}}, \qquad \frac{b^K_{(-I)}}{q^K_{(-I)}} = \frac{b^{K+1}}{r^{K+1}_{(II)}}, \qquad \frac{s^K}{q^K_{(-I,-II)}} = \frac{s^{K+1}_{(-I)}}{r^{K+1}_{(-I)}}.$$

The discrete Moutard transformation

If θ is a scalar solution of the discrete Moutard system

$$\theta_{(ij)} - \theta = f^{ij}(\theta_{(i)} - \theta_{(j)}), \qquad i < j,$$

then the solution $\bar{\psi}$ of equations

$$\bar{\psi}_{(i)} - \psi = \frac{\theta}{\theta_{(i)}}(\bar{\psi} - \psi_{(i)}),$$

satisfies the discrete Moutard system with potentials

$$\bar{f}^{ij} = f^{ij}\frac{\theta_{(i)}\theta_{(j)}}{\theta_{(ij)}\theta}, \qquad \text{or} \quad \bar{\tau} = \theta\tau.$$

J. J. C. Nimmo & W. K. Schief, 1997

The Darboux transformation on $\mathbb{T}$ $\hat{\psi} = \theta_{(-1-2-3)}\bar{\psi}_{(-1-2-3)}$

Given solution ψ of the affine self-adjoint 7-point scheme and given its particular solution θ, then equations

$$\begin{aligned} \hat{\psi}_{(I)} - \hat{\psi} &= -b_{(-II)}\theta_{(-II)}\psi - s\theta_{(-II)}\psi_{(-I)} + (b_{(-II)}\theta + s\theta_{(-I)})\psi_{(-II)}, \\ \hat{\psi}_{(II)} - \hat{\psi} &= \quad a_{(-I)}\theta_{(-I)}\psi + s\theta_{(-I)}\psi_{(-II)} - (a_{(-I)}\theta + s\theta_{(-II)})\psi_{(-I)}, \end{aligned}$$

define the corresponding solution $\hat{\psi}$ of the scheme with the coefficients

$$\hat{a} = -\frac{a_{(-I)}}{\theta_{(-II)}p}, \qquad \hat{b} = -\frac{b_{(-II)}}{\theta_{(-I)}p}, \qquad \hat{s}_{(I,II)} = -\frac{s}{\theta p},$$

where $p = s(a_{(-I)}\theta_{(-I)} + b_{(-II)}\theta_{(-II)}) + a_{(-I)}b_{(-II)}\theta$.

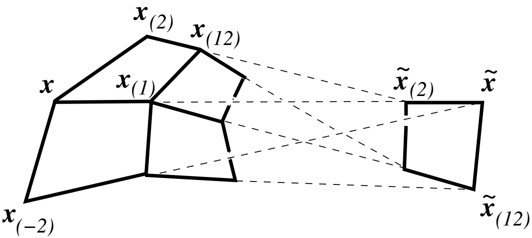
M. Nieszporski, P. M. Santini & A. D., 2004

Further research directions

- Evoutions (discrete or continuous) of the triangular and honeycomb lattices, compatible with the linear problem (see the talk of M. Nieszporski).
- Superposition with other integrable reductions of the quadrilateral lattice in order to obtain $1+1$ dimensional discrete equations on the triangular (honeycomb) lattice.
- Restriction (cut-and-project method) of reductions of the quadrilateral lattice equations to other graphs: Penrose tilings, quasicrystals ...

Thank you for your attention

The Laplace transformation – discrete hyperbolic case



2D quadrilateral lattice
(discrete conjugate net)
R. Sauer, 1937

Geometry

$x : \mathbb{Z}^2 \to \mathbb{P}^M$ with elementary quadrilaterals planar

Algebra

$\psi : \mathbb{Z}^2 \to \mathbb{K}_*^{M+1}$ with $\psi, \psi_{(1)}, \psi_{(2)}, \psi_{(12)}$ linearly dependent

$$\psi_{(12)} + \alpha\psi_{(1)} + \beta\psi_{(2)} + \gamma\psi = 0.$$

A. D., 1997; S. P. Novikov & I. A. Dynnikov, 1997

The Darboux-type transformation on $\mathbb{H}$

$$(\check\phi^+,\check\phi^-) = \left(\theta_{(-2-3)}\bar\psi_{(-2-3)}, \theta_{(-1-3)}\bar\psi_{(-1-3)}\right)$$

Given solution (ϕ^+,ϕ^-) of the honeycomb linear system and given its particular solution (θ^+,θ^-). Then the solution $(\check\phi^+,\check\phi^-)$ of the system

$$\check\phi^+_{(II)} - \check\phi^- = \frac{s}{r}\left(\theta^-_{(-II)}\phi^-_{(-I)} - \theta^-_{(-I)}\phi^-_{(-II)}\right),$$

$$\check\phi^+ - \check\phi^- = \frac{a_{(-I)}}{r}\left(\theta^-_{(-I,-II)}\phi^-_{(-II)} - \theta^-_{(-II)}\phi^-_{(-I,-II)}\right),$$

$$\check\phi^- - \check\phi^+_{(-I,II)} = \frac{b_{(-II)}}{r}\left(\theta^-_{(-I,-II)}\phi^-_{(-I)} - \theta^-_{(-I)}\phi^-_{(-I,-II)}\right),$$

where $r = a_{(-I)}s + b_{(-II)}s + a_{(-I)}b_{(-II)}$, satisfies the honeycomb linear system with the coefficients

$$\check a = -\frac{a_{(-I)}}{r}\theta^-_{(-II)}\theta^-_{(-I,-II)}, \quad \check b = -\frac{b_{(-II)}}{r}\theta^-_{(-I)}\theta^-_{(-I,-II)}, \quad \check s_{(I,II)} = -\frac{s}{r}\theta^-_{(-I)}\theta^-_{(-II)}.$$

The star-triangle relation

The discrete Moutard equations on the staircase section read

$$\psi_{(I)} - \psi = \frac{1}{a}(\phi^+ - \phi^-),$$

$$\psi_{(II)} - \psi = \frac{1}{b}(\phi^- - \phi^+_{(-I,II)}),$$

$$\psi_{(II)} - \psi_{(I)} = \frac{1}{s_{(I,II)}}(\phi^+_{(II)} - \phi^-).$$